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Research Article

Bi-Integrable and Tri-Integrable Couplings of a Soliton Hierarchy Associated with SO(3)

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Based on the three-dimensional real special orthogonal Lie algebra SO(3), by zero curvature equation, we present bi-integrable and tri-integrable couplings associated with SO(3) for a hierarchy from the enlarged matrix spectral problems and the enlarged zero curvature equations. Moreover, Hamiltonian structures of the obtained bi-integrable and tri-integrable couplings are constructed by applying the variational identities.

1. Introduction

Among the well-known soliton hierarchies are the KdV hierarchy, the AKNS hierarchy, and the Kaup-Newell hierarchy [1]. The trace identity is used for constructing Hamiltonian structures of soliton equations, which is proposed by Tu [2, 3]. In the case of non-semi-simple Lie algebras, integrable couplings of soliton equations are generated by zero curvature equations [4, 5] and the corresponding Hamiltonian structures are obtained by the variational identity [6–8].

An integrable coupling equation

$$u_t = K(u) = K(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) \quad (1)$$

is a triangular integrable system of the following form [9]:

$$\begin{aligned} u_t &= K(u), \\ v_t &= S(u, v), \end{aligned} \quad (2)$$

where u is a function of variables t and x, $u_x = \partial u/\partial x$, and $u_t = \partial u/\partial t$. If S is nonlinear with respect to the second dependent variable v, the integrable coupling is called nonlinear.

An integrable system of the following form [10]

$$\begin{aligned} u_t &= K(u), \\ u_{1,t} &= S_1(u, u_1), \\ u_{2,t} &= S_2(u, u_1, u_2) \end{aligned} \quad (3)$$

is called a bi-integrable of (1).

Similarly, an integrable system of the following form [10]

$$\begin{aligned} u_t &= K(u), \\ u_{1,t} &= S_1(u, u_1), \\ u_{2,t} &= S_2(u, u_1, u_2), \\ u_{3,t} &= S_3(u, u_1, u_2, u_3) \end{aligned} \quad (4)$$

is called a tri-integrable of (1).

Integrable couplings correspond to non-semi-simple Lie algebras $\overline{g}$, and such Lie algebras can be written as semidirect sums [11]:

$$\overline{g} = g \uplus g_c, \quad g\text{-semisimple}, \quad g_c\text{-solvable}. \quad (5)$$

The notion of semidirect sums $\overline{g} = g \uplus g_c$ means that g and g_c satisfy $[g, g_c] \subseteq g_c$, where $[g, g_c] = \{[A, B] \mid A \in g,\ B \in g_c\}$, with $[\cdot,\cdot]$ denoting the Lie bracket of $\overline{g}$. Obviously, g_c is an ideal of $\overline{g}$. The subscript c indicates a contribution to the construction of coupling systems. We also require the closure property between g and g_c under the matrix multiplication: $gg_c, g_c g \subseteq g_c$, where $gg_c = \{AB \mid A \in g,\ B \in g_c\}$.

Integrable couplings are useful tools for describing and explaining nonlinear phenomena of new evaluation equations. There are very rich mathematical structures behind integrable couplings. In particular, integrable couplings generalize the symmetry problem and describe other integrable properties of integrable equations. In order to enrich multicomponent integrable equations, it has been an important task to explore more integrable properties from multi-integrable couplings. For example, one can find work on the integrable couplings [12, 13]. It is always interesting to explore any new procedure for generating integrable couplings for different soliton hierarchies, even from existing non-semi-simple Lie algebras.

Recently, seeking new integrable systems including soliton hierarchies and integrable couplings forms a pretty important and interesting area of research in mathematical physics. To generate integrable couplings, bi-integrable couplings and tri-integrable couplings of soliton hierarchies, Ma proposed a new way to generate integrable couplings through a few classes of matrix Lie algebras consisting of block matrices [10]. Recently, bi-integrable couplings and tri-integrable couplings for the KdV hierarchy and the AKNS hierarchy have been studied considerably [14, 15]. From [16, 17], bi-integrable couplings of a new soliton hierarchy associated with SO(3) and bi-integrable couplings of a new soliton hierarchy associated with SO(4) have been studied.

In this paper, we will construct bi-integrable and tri-integrable couplings associated with SO(3) for a hierarchy from the enlarged matrix spectral problems and the enlarged zero curvature equations. Our work is essentially motivated by [17–19].

2. Bi-Integrable Couplings and Hamiltonian Structures

2.1. Bi-Integrable Couplings Associated with SO(3). So as to generate bi-integrable couplings, we introduce a kind of block matrices:

$$M_2(A, A_1, A_2) = \begin{pmatrix} A & A_1 & A_2 \\ 0 & A + \alpha A_1 & A_1 + \alpha A_2 \\ 0 & 0 & A + \alpha A_1 \end{pmatrix}, \tag{6}$$

where α is an arbitrary nonzero constant and A, A_1, and A_2 are square matrices of the same order. In the following, we define the corresponding non-semi-simple Lie algebra $\overline{g}$ by a semidirect sum:

$$\overline{g}(\lambda) = g \uplus g_c, \tag{7}$$

with

$$\begin{aligned} g &= \left\{M_2(A, 0, 0) \mid A \in \widetilde{SO(3)}\right\}, \\ g_c &= \left\{M_2(0, A_1, A_2) \mid A_1, A_2 \in \widetilde{SO(3)}\right\}, \end{aligned} \tag{8}$$

where the loop algebra $\widetilde{SO(3)}$ is defined by

$$\begin{aligned} \widetilde{SO(3)} = \{A(\lambda) \in SO(3) \mid & \text{ entries of } A(\lambda) \\ & - \text{Laurent series in } \lambda\}. \end{aligned} \tag{9}$$

Obviously, we have the matrix commutator relation:

$$[M_2(A, A_1, A_2), M_2(B, B_1, B_2)] = M_2(C, C_1, C_2), \tag{10}$$

with C, C_1, and C_2 being defined by

$$\begin{aligned} C &= [A, B], \\ C_1 &= [A, B_1] + [A_1, B] + \alpha [A_1, B_1], \\ C_2 &= [A, B_2] + [A_2, B] + [A_1, B_1] + \alpha [A_1, B_2] \\ &\quad + \alpha [A_2, B_1]. \end{aligned} \tag{11}$$

Let us consider the Lie algebra SO(3). It has a basis

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{12}$$

with which the structure equations of SO(3) are $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = e_2$.

The soliton hierarchy introduced in [18] has a spectral problem

$$\phi_x = U\phi = U(u, \lambda)\phi, \quad u = (p, q)^T, \tag{13}$$

with the spectral matrix U being chosen as

$$U = U(u, \lambda) = \lambda e_1 + p e_2 + q e_3 = \begin{pmatrix} 0 & q & \lambda \\ -q & 0 & -p \\ -\lambda & p & 0 \end{pmatrix}. \tag{14}$$

Based on this special non-semi-simple Lie algebra $\overline{g}(\lambda)$, we begin with the corresponding enlarged spectral matrix

$\overline{U}_1 = M_2(U, U_1, U_2)$ and let supplementary spectral matrices be

$$U_1 = U_1(u_1, \lambda) = \begin{pmatrix} 0 & q' & 0 \\ -q' & 0 & -p' \\ 0 & p' & 0 \end{pmatrix}, \quad u_1 = \begin{pmatrix} p' \\ q' \end{pmatrix},$$
$$U_2 = U_2(u_2, \lambda) = \begin{pmatrix} 0 & q'' & 0 \\ -q'' & 0 & -p'' \\ 0 & p'' & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} p'' \\ q'' \end{pmatrix}. \tag{15}$$

For purpose of solving the enlarged stationary zero curvature equation $\overline{V}_{1x} = [\overline{U}_1, \overline{V}_1]$, we take $\overline{V}_1 = M_2(V, V_1, V_2)$, where V is defined as in [18]

$$V = V(u, \lambda) = \begin{pmatrix} 0 & c & a \\ -c & 0 & -b \\ -a & b & 0 \end{pmatrix} = \sum_{i \geq 0} \begin{pmatrix} 0 & c_i & a_i \\ -c_i & 0 & -b_i \\ -a_i & b_i & 0 \end{pmatrix} \lambda^{-i}, \tag{16}$$

and the supplementary spectral matrices V_1 and V_2 read

$$V_1 = V_1(u, u_1, \lambda) = \begin{pmatrix} 0 & c' & a' \\ -c' & 0 & -b' \\ -a' & b' & 0 \end{pmatrix} = \sum_{i \geq 0} \begin{pmatrix} 0 & c_i' & a_i' \\ -c_i' & 0 & -b_i' \\ -a_i' & b_i' & 0 \end{pmatrix} \lambda^{-i},$$
$$V_2 = V_2(u, u_1, u_2, \lambda) = \begin{pmatrix} 0 & c'' & a'' \\ -c'' & 0 & -b'' \\ -a'' & b'' & 0 \end{pmatrix} = \sum_{i \geq 0} \begin{pmatrix} 0 & c_i'' & a_i'' \\ -c_i'' & 0 & -b_i'' \\ -a_i'' & b_i'' & 0 \end{pmatrix} \lambda^{-i}. \tag{17}$$

The enlarged stationary zero curvature equation $\overline{V}_{1x} = [\overline{U}_1, \overline{V}_1]$ is equivalent to

$$\begin{aligned} V_x &= [U, V], \\ V_{1x} &= [U, V_1] + [U_1, V] + \alpha [U_1, V_1], \\ V_{2x} &= [U, V_2] + [U_2, V] + [U_1, V_1] + \alpha [U_1, V_2] \\ &\quad + \alpha [U_2, V_1]. \end{aligned} \tag{18}$$

The above equation system equivalently leads to

$$\begin{aligned} a_x &= pc - qb, \\ b_x &= -\lambda c + qa, \\ c_x &= \lambda b - pa, \\ a_x' &= pc' - qb' - q'b + p'c + \alpha\left(p'c' - q'b'\right), \\ b_x' &= -\lambda c' + qa' + q'a + \alpha q'a', \\ c_x' &= \lambda b' - pa' - p'a - \alpha p'a', \\ a_x'' &= -qb'' + pc'' - q''b + p''c - q'b' + p'c' \\ &\quad + \alpha\left(-q'b'' + p'c''\right) + \alpha\left(-q''b' + p''c'\right), \\ b_x'' &= -\lambda c'' + qa'' + q''a + q'a' + \alpha q'a'' + \alpha q''a', \\ c_x'' &= \lambda b'' - pa'' - p''a - p'a' - \alpha p'a'' - \alpha p''a'. \end{aligned} \tag{19}$$

Now, we define the enlarged Lax matrices $\overline{V}_1^{[m]} = (\lambda^m \overline{V}_1)_+ = M_2(V^{[m]}, V_1^{[m]}, V_2^{[m]})$, $m \geq 0$, where $V^{[m]}$ is defined as $V^{[m]} = (\lambda^m V)_+$, and $V_i^{[m]} = (\lambda^m V_i)_+$, $i = 1, 2$.

Solving the enlarged zero curvature equations $\overline{U}_{1t_m} - \overline{V}_{1x}^{[m]} + [\overline{U}_1, \overline{V}_1^{[m]}] = 0$, $m \geq 0$, we get bi-integrable couplings of the soliton hierarchy in [18]

$$\overline{u}_{t_m} = \begin{pmatrix} p \\ q \\ p' \\ q' \\ p'' \\ q'' \end{pmatrix}_{t_m} = \begin{pmatrix} -c_{m+1} \\ b_{m+1} \\ -c_{m+1}' \\ b_{m+1}' \\ -c_{m+1}'' \\ b_{m+1}'' \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} -2b_{m+1} \\ -2c_{m+1} \\ -2b_{m+1}' \\ -2c_{m+1}' \\ -2b_{m+1}'' \\ -2c_{m+1}'' \end{pmatrix} = J_1 P_{1,m+1}. \tag{20}$$

2.2. Hamiltonian Structures. In this section, for purpose of generating the Hamiltonian structure of hierarchy (20), we will use the corresponding variational identity [20]:

$$\frac{\delta}{\delta \overline{u}} \int \left\{\overline{V}, \overline{U}_\lambda\right\} dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \left\{\overline{V}, \overline{U}_u\right\}, \tag{21}$$

where $\{\cdot,\cdot\}$ is a required bilinear form, which is symmetric, nondegenerate, and invariant under the Lie bracket.

$\forall a = (a_1, a_2, a_3, a_1', a_2', a_3', a_1'', a_2'', a_3'') \in R^9$, $b = (b_1, b_2, b_3, b_1', b_2', b_3', b_1'', b_2'', b_3'') \in R^9$; we define the Lie bracket $[\cdot,\cdot]$ on R^9 as follows:

$$[a,b] = a^T \overline{R}_1(b),$$
$$\overline{R}_1(b) = \begin{pmatrix} R(b) & R_1(b) & R_2(b) \\ 0 & R(b)+\alpha R_1(b) & R_1(b)+\alpha R_2(b) \\ 0 & 0 & R(b)+\alpha R_1(b) \end{pmatrix}, \quad (22)$$

where

$$R(b) = \begin{pmatrix} 0 & -b_3 & b_2 \\ b_3 & 0 & -b_1 \\ -b_2 & b_1 & 0 \end{pmatrix},$$
$$R_1(b) = \begin{pmatrix} 0 & -b_3' & b_2' \\ b_3' & 0 & -b_1' \\ -b_2' & b_1' & 0 \end{pmatrix}, \quad (23)$$
$$R_2(b) = \begin{pmatrix} 0 & -b_3'' & b_2'' \\ b_3'' & 0 & -b_1'' \\ -b_2'' & b_1'' & 0 \end{pmatrix}.$$

Following the properties of the matrix F_1: $F_1(\overline{R}_1(b))^T = -\overline{R}_1(b)F_1$ and $F_1 = F_1^T$, we get

$$F_1 = \begin{pmatrix} \eta_1 & \eta_2 & 2\eta_3 \\ \eta_2 & \alpha\eta_2 + 2\eta_3 & 2\alpha\eta_3 \\ 2\eta_3 & 2\alpha\eta_3 & 0 \end{pmatrix} \otimes \begin{pmatrix} r_0 & 0 & 0 \\ 0 & r_0 & 0 \\ 0 & 0 & r_0 \end{pmatrix}, \quad (24)$$

where η_1, η_2, η_3, r_0 are arbitrary constants. We are easy to have

$$\det(F_1) = -64\left(\alpha^2\eta_1 - \alpha\eta_2 + 2\eta_3\right)^3 \eta_3^6 r_0^9 \neq 0. \quad (25)$$

In order to get the Hamiltonian structure of the Lax integrable system, we define a bilinear form $\{a,b\}$ on R^9 of the following form:

$$\{a,b\} = a^T F_1 b. \quad (26)$$

Now we can compute that

$$\begin{aligned} \{\overline{V}_1, \overline{U}_{1,\lambda}\} &= a\eta_1 + r_0 a'\eta_2 + 2r_0 a''\eta_3, \\ \{\overline{V}_1, \overline{U}_{1,p}\} &= r_0 b\eta_1 + r_0 b'\eta_2 + 2r_0 b''\eta_3, \\ \{\overline{V}_1, \overline{U}_{1,q}\} &= r_0 c\eta_1 + r_0 c'\eta_2 + 2r_0 c''\eta_3, \\ \{\overline{V}_1, \overline{U}_{1,p'}\} &= \left(r_0 b + \alpha r_0 b'\right)\eta_2 \\ &\quad + \left(2r_0 b' + 2\alpha r_0 b''\right)\eta_3, \\ \{\overline{V}_1, \overline{U}_{1,q'}\} &= \left(r_0 c + \alpha r_0 c'\right)\eta_2 + \left(2r_0 c' + 2\alpha r_0 c''\right)\eta_3, \\ \{\overline{V}_1, \overline{U}_{1,p''}\} &= \left(2r_0 b + 2\alpha r_0 b'\right)\eta_3, \\ \{\overline{V}_1, \overline{U}_{1,q''}\} &= \left(2r_0 c + 2\alpha r_0 c'\right)\eta_3 \end{aligned} \quad (27)$$

and furthermore, we use the following formular [20]:

$$\gamma = -\frac{\lambda}{2}\frac{d}{d\lambda}\ln\left|\{\overline{V}, \overline{V}\}\right|, \quad (28)$$

to obtain that $\gamma = 0$. Applying the corresponding variational identity, we obtain the following Hamiltonian structure for the hierarchy of bi-integrable coupling (20):

$$\overline{u}_{t_m} = \overline{K}_{1,m}(\overline{u}) = \overline{J}_1 \frac{\delta \overline{H}_{1,m}}{\delta \overline{u}}; \quad m \geq 0, \quad (29)$$

where the Hamiltonian operator is

$$\overline{J}_1 = \begin{pmatrix} \eta_1 & \eta_2 & 2\eta_3 \\ \eta_2 & \alpha\eta_2 + 2\eta_3 & 2\alpha\eta_3 \\ 2\eta_3 & 2\alpha\eta_3 & 0 \end{pmatrix}^{-1} \otimes \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad (30)$$

and the Hamiltonian functions read

$$\overline{H}_{1,m} = -\int \frac{a_{m+2}\eta_1 + r_0 a'_{m+2}\eta_2 + 2r_0 a''_{m+2}\eta_3}{m+1}dx, \quad m \geq 0. \quad (31)$$

Based on (19), a direct computation yields a recursion relation:

$$P_{1,m+1} = \overline{L}_1 P_{1,m}, \quad (32)$$

where

$$\overline{L}_1 = M_2^T\left(L^1, L_1^1, L_2^1\right) = \begin{pmatrix} L^1 & 0 & 0 \\ L_1^1 & L^1 + \alpha L_1^1 & 0 \\ L_2^1 & L_1^1 + \alpha L_2^1 & L^1 + \alpha L_1^1 \end{pmatrix}, \quad (33)$$

with L^1, L_1^1, and L_2^1 being defined by

$$L^1 = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix}, \tag{34}$$

$$L_1^1 = \begin{pmatrix} l'_{11} & l'_{12} \\ l'_{21} & l'_{22} \end{pmatrix},$$

$$L_2^1 = \begin{pmatrix} l''_{11} & l''_{12} \\ l''_{21} & l''_{22} \end{pmatrix}, \tag{35}$$

$$\begin{aligned}
l_{11} &= -p\partial^{-1}q, \\
l_{12} &= \partial + p\partial^{-1}p, \\
l_{21} &= -\partial - q\partial^{-1}q, \\
l_{22} &= q\partial^{-1}p, \\
l'_{11} &= -p\partial^{-1}q' - \alpha p'\partial^{-1}q' - p'\partial^{-1}q, \\
l'_{12} &= p\partial^{-1}p' + \alpha p'\partial^{-1}p' + p'\partial^{-1}p, \\
l'_{21} &= -q\partial^{-1}q' - \alpha q'\partial^{-1}q' - q'\partial^{-1}q, \\
l'_{22} &= q\partial^{-1}p' + \alpha q'\partial^{-1}p' + q'\partial^{-1}p, \\
l''_{11} &= -p\partial^{-1}q'' - \alpha p'\partial^{-1}q'' - p'\partial^{-1}q' - \alpha p''\partial^{-1}q' \\
&\quad - p''\partial^{-1}q, \\
l''_{12} &= p\partial^{-1}p'' + \alpha p'\partial^{-1}p'' + p'\partial^{-1}p' + \alpha p''\partial^{-1}p' \\
&\quad + p''\partial^{-1}p, \\
l''_{21} &= -q\partial^{-1}q'' - \alpha q'\partial^{-1}q'' - q'\partial^{-1}q' - \alpha q''\partial^{-1}q' \\
&\quad - q''\partial^{-1}q, \\
l''_{22} &= q\partial^{-1}p'' + \alpha q'\partial^{-1}p'' + q'\partial^{-1}p' + \alpha q''\partial^{-1}p' \\
&\quad + q''\partial^{-1}p,
\end{aligned} \tag{36}$$

where $\partial = d/dx$ and $\partial^{-1} = \int (d/dx)dx$.

3. Tri-Integrable Couplings and Hamiltonian Structures

3.1. Tri-Integrable Couplings Associated with SO(3). So as to generate bi-integrable couplings, we introduce a kind of block matrices:

$$M_3(A, A_1, A_2, A_3) = \begin{pmatrix} A & A_1 & A_2 & A_3 \\ 0 & A + \beta A_1 & \beta A_2 & A_1 + \beta A_3 \\ 0 & 0 & A + \beta A_1 + \mu A_2 & \nu A_2 \\ 0 & 0 & 0 & A + \beta A_1 \end{pmatrix}, \tag{37}$$

where β, μ, and ν are arbitrary nonzero constants and A, A_1, A_2, and A_3 are square matrices of the same order. In the following, we define the corresponding non-semi-simple Lie algebra $\overline{g}(\lambda)$ by a semidirect sum:

$$\overline{g}(\lambda) = g \uplus g_c, \tag{38}$$

with

$$\begin{aligned}
g &= \left\{M_3(A, 0, 0, 0) \mid A \in \widetilde{\mathrm{SO}(3)}\right\}, \\
g_c &= \left\{M_3(0, A_1, A_2, A_3) \mid A_1, A_2, A_3 \in \widetilde{\mathrm{SO}(3)}\right\},
\end{aligned} \tag{39}$$

where the loop algebra $\widetilde{\mathrm{SO}(3)}$ is defined by

$$\begin{aligned}
\widetilde{\mathrm{SO}(3)} = \{A(\lambda) \in \mathrm{SO}(3) \mid & \text{ entries of } A(\lambda) \\
& - \text{Laurent series in } \lambda\}.
\end{aligned} \tag{40}$$

Obviously, we have the matrix commutator relation:

$$\begin{aligned}
&[M_3(A, A_1, A_2, A_3), M_3(B, B_1, B_2, B_3)] \\
&\quad = M_3(C, C_1, C_2, C_3),
\end{aligned} \tag{41}$$

with C, C_1, C_2, and C_3 being defined by

$$\begin{aligned}
C &= [A, B], \\
C_1 &= [A, B_1] + [A_1, B] + \beta[A_1, B_1], \\
C_2 &= [A, B_2] + [A_2, B] + \mu[A_2, B_2] + \beta[A_1, B_2] \\
&\quad + \beta[A_2, B_1], \\
C_3 &= [A, B_3] + [A_3, B] + \beta[A_3, B_1] + \beta[A_1, B_3] \\
&\quad + [A_1, B_1] + \nu[A_2, B_2].
\end{aligned} \tag{42}$$

We introduce the following enlarged spectral matrix to construct tri-integrable couplings for SO(3) hierarchy:

$$\overline{U}_2 = \overline{U}_2(\overline{u}, \lambda) = M_3(U, U_1, U_2, U_3) \in \overline{g}(\lambda), \tag{43}$$

with $U = U(u, \lambda)$ being defined as in (14), where U_1 and U_2 are defined by (15), and also the supplementary spectral matrix U_3 reads

$$U_3 = U_3(u_3, \lambda) = \begin{pmatrix} 0 & q''' & 0 \\ -q''' & 0 & -p''' \\ 0 & p''' & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} p''' \\ q''' \end{pmatrix}. \tag{44}$$

As usual, we take a solution of the following form:

$$\overline{V}_2 = \overline{V}_2(\overline{u}, \lambda) = M_3(V, V_1, V_2, V_3) = \begin{pmatrix} V & V_1 & V_2 & V_3 \\ 0 & V + \beta V_1 & \beta V_2 & V_1 + \beta V_3 \\ 0 & 0 & V + \beta V_1 + \mu V_2 & \nu V_2 \\ 0 & 0 & 0 & V + \beta V_1 \end{pmatrix}, \tag{45}$$

where V, V_1, and V_2 are defined by (16) and (17); also V_3 reads

$$V_3 = V_3(u, u_1, u_2, u_3, \lambda) = \begin{pmatrix} 0 & c''' & a''' \\ -c''' & 0 & -b''' \\ -a''' & b''' & 0 \end{pmatrix},$$
$$a''' = \sum_{i\geq 0} a_i''' \lambda^{-i},$$
$$b''' = \sum_{i\geq 0} b_i''' \lambda^{-i},$$
$$c''' = \sum_{i\geq 0} c_i''' \lambda^{-i}, \quad (46)$$
$$f''' = \sum_{i\geq 0} f_i''' \lambda^{-i},$$
$$g''' = \sum_{i\geq 0} g_i''' \lambda^{-i}.$$

It now follows from the enlarged stationary zero curvature equation $\overline{V}_{2x} = [\overline{U}_2, \overline{V}_2]$ that

$$\begin{aligned} V_x &= [U, V], \\ V_{1x} &= [U, V_1] + [U_1, V] + \beta [U_1, V_1], \\ V_{2x} &= [U, V_2] + [U_2, V] + \mu [U_2, V_2] + \beta [U_1, V_2] \\ &\quad + \beta [U_2, V_1], \\ V_{3x} &= [U, V_3] + [U_3, V] + \beta [U_3, V_1] + [U_1, V_1] \\ &\quad + \beta [U_1, V_3] + \nu [U_2, V_2]. \end{aligned} \quad (47)$$

The above equation system is equivalent to

$$\begin{aligned} a_x &= pc - qb, \\ b_x &= -\lambda c + qa, \\ c_x &= \lambda b - pa, \\ a'_x &= -qb' + pc' - q'b + p'c + \beta(-q'b' + p'c'), \\ b'_x &= -\lambda c' + qa' + q'a + \beta q'a', \\ c'_x &= \lambda b' - pa' - p'a - \beta p'a', \\ a''_x &= -qb'' + pc'' - q''b + p''c + \mu(-q''b'' + p''c'') \\ &\quad + \beta(-q'b'' + p'c'') + \beta(-q''b' + p''c'), \\ b''_x &= -\lambda c'' + qa'' + q''a + \mu q''a'' + \beta q'a'' + \beta q''a', \\ c''_x &= \lambda b'' - pa'' - p''a - \mu p''a'' - \beta p'a'' - \beta p''a', \\ a'''_x &= -qb''' + pc''' - q'''b + p'''c \\ &\quad + \beta(-q'''b' + p'''c') + \beta(-q'b''' + p'c''') \\ &\quad - q'b' + p'c' + \nu(-q''b'' + p''c''), \\ b'''_x &= -\lambda c''' + qa''' + q'''a + \beta q'''a' + \beta q'a''' + q'a' \\ &\quad + \nu q''a'', \\ c'''_x &= \lambda b''' - pa''' - p'''a - \beta p'''a' - \beta p'a''' - p'a' \\ &\quad - \nu q''a''. \end{aligned} \quad (48)$$

Now, we define the enlarged Lax matrices $\overline{V}_2^{[m]} = (\lambda^m \overline{V}_2)_+ = M_3(V^{[m]}, V_1^{[m]}, V_2^{[m]}, V_3^{[m]})$, $m \geq 0$, where $V^{[m]}$ is defined as $V^{[m]} = (\lambda^m V)_+$, and $V_i^{[m]} = (\lambda^m V_i)_+$, $i = 1, 2, 3$.

Solving the enlarged zero curvature equations $\overline{U}_{2t_m} - \overline{V}_{2x}^{[m]} + [\overline{U}_2, \overline{V}_2^{[m]}] = 0$, $m \geq 0$, we get tri-integrable couplings of the soliton hierarchy in [18]

$$\overline{u}_{t_m} = \begin{pmatrix} p \\ q \\ p' \\ q' \\ p'' \\ q'' \\ p''' \\ q''' \end{pmatrix}_{t_m} = \begin{pmatrix} -c_{m+1} \\ b_{m+1} \\ -c'_{m+1} \\ b'_{m+1} \\ -c''_{m+1} \\ b''_{m+1} \\ -c'''_{m+1} \\ b'''_{m+1} \end{pmatrix}$$
$$= \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} -2b_{m+1} \\ -2c_{m+1} \\ -2b'_{m+1} \\ -2c'_{m+1} \\ -2b''_{m+1} \\ -2c''_{m+1} \\ -2b'''_{m+1} \\ -2c'''_{m+1} \end{pmatrix} \quad (49)$$
$$= J_2 P_{2,m+1}.$$

3.2. Hamiltonian Structures. In this section, for the purpose of generating the Hamiltonian structure of the hierarchy (49), we will use the corresponding variational identity [20]:

$$\frac{\delta}{\delta \overline{u}} \int \{\overline{V}, \overline{U}_\lambda\}\, dx = \lambda^{-\gamma} \frac{\partial}{\partial \lambda} \lambda^{\gamma} \{\overline{V}, \overline{U}_u\}, \quad (50)$$

where $\{\cdot,\cdot\}$ is a required bilinear form, which is symmetric, nondegenerate, and invariant under the Lie bracket.

$\forall a = (a_1, a_2, a_3, a'_1, a'_2, a'_3, a''_1, a''_2, a''_3, a'''_1, a'''_2, a'''_3) \in R^{12}$, $b = (b_1, b_2, b_3, b'_1, b'_2, b'_3, b''_1, b''_2, b''_3, b'''_1, b'''_2, b'''_3) \in R^{12}$; we define the Lie bracket $[\cdot,\cdot]$ on R^{12} as follows:

$$[a, b] = a^T \overline{R}_2(b), \quad (51)$$

where

$$\overline{R}_2(b)=\begin{pmatrix} R(b) & R_1(b) & R_2(b) & R_3(b) \\ 0 & R(b)+\beta R_1(b) & \beta R_2(b) & R_1(b)+\beta R_3(b) \\ 0 & 0 & R(b)+\beta R_1(b)+\mu R_2(b) & \nu R_2(b) \\ 0 & 0 & 0 & R(b)+\beta R_1(b) \end{pmatrix}, \quad (52)$$

with $R(b)$, $R_1(b)$, and $R_2(b)$ being defined by (23), and

$$R_3(b)=\begin{pmatrix} 0 & -b_3''' & b_2''' \\ b_3''' & 0 & -b_1''' \\ -b_2''' & b_1''' & 0 \end{pmatrix}. \quad (53)$$

Following the properties of the matrix F_2, $F_2(\overline{R}_2(b))^T = \overline{R}_2(b)F_2$ and $F_2 = F_2^T$, we have

$$F_2=\begin{pmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \eta_2 & \beta\eta_2+\eta_4 & \beta\eta_3 & \beta\eta_4 \\ \eta_3 & \beta\eta_3 & \mu\eta_3+\nu\eta_4 & 0 \\ \eta_4 & \beta\eta_4 & 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} r_0 & 0 & 0 \\ 0 & r_0 & 0 \\ 0 & 0 & r_0 \end{pmatrix}, \quad (54)$$

where $\otimes$ is the Kronecker product and η_1, η_2, η_3, η_4, r_0 are arbitrary constants. It is easy to have

$$\det(F_2)=\eta_4^6\left(\beta^2\eta_1-\beta\eta_2+\eta_4\right)^3\left(\mu\eta_3+\nu\eta_4\right)^3 r_0^{12} \neq 0. \quad (55)$$

In order to get the Hamiltonian structure of the Lax integrable system, we define a bilinear form $\{a,b\}$ on R^{12} of the following form:

$$\{a,b\}=a^T F_2 b. \quad (56)$$

Now, we further compute that

$$\begin{aligned}
\left\{\overline{V}_2,\overline{U}_{2,\lambda}\right\} &= -ar_0\eta_1 - a'r_0\eta_2 - a''r_0\eta_3 - a'''r_0\eta_4, \\
\left\{\overline{V}_2,\overline{U}_{2,p}\right\} &= r_0 b\eta_1 + r_0 b'\eta_2 + r_0 b''\eta_3 + r_0 b'''\eta_4, \\
\left\{\overline{V}_2,\overline{U}_{2,q}\right\} &= -r_0 c\eta_1 - r_0 c'\eta_2 - r_0 c''\eta_3 - r_0 c'''\eta_4, \\
\left\{\overline{V}_2,\overline{U}_{2,p'}\right\} &= \left(r_0 b + r_0 b'\beta\right)\eta_2 + r_0 b''\beta\eta_3 + \left(r_0 b' + r_0 b'''\beta\right)\eta_4, \\
\left\{\overline{V}_2,\overline{U}_{2,q'}\right\} &= \left(-r_0 c + \beta r_0 c'\right)\eta_2 + r_0 c''\beta\eta_3 + \left(r_0 c' + \beta r_0 c'''\right)\eta_4, \\
\left\{\overline{V}_2,\overline{U}_{2,p''}\right\} &= \left(r_0 b + \beta r_0 b' + \mu r_0 b''\right)\eta_3 + \nu r_0 b''\eta_4, \\
\left\{\overline{V}_2,\overline{U}_{2,q''}\right\} &= \left(-r_0 c - \beta r_0 c' - \mu r_0 c''\right)\eta_3 - \nu r_0 c''\eta_4, \\
\left\{\overline{V}_2,\overline{U}_{2,p'''}\right\} &= \left(r_0 b + \beta r_0 b'\right)\eta_4, \\
\left\{\overline{V}_2,\overline{U}_{2,q'''}\right\} &= \left(-r_0 c - \beta r_0 c'\right)\eta_4.
\end{aligned} \quad (57)$$

We use formula (28) and find that $\gamma = 0$. Applying the corresponding variational identity, we obtain the following Hamiltonian structure for the hierarchy of tri-integrable couplings (49):

$$\overline{u}_{t_m}=\overline{K}_{2,m}(\overline{u})=\overline{J}_2\frac{\delta\overline{H}_{2,m}}{\delta\overline{u}}, \quad m\geq 0, \quad (58)$$

where the Hamiltonian operator is

$$\overline{J}_2=\begin{pmatrix} \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \eta_2 & \beta\eta_2+\eta_4 & \beta\eta_3 & \beta\eta_4 \\ \eta_3 & \beta\eta_3 & \mu\eta_3+\nu\eta_4 & 0 \\ \eta_4 & \beta\eta_4 & 0 & 0 \end{pmatrix}^{-1} \otimes \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad (59)$$

and the Hamiltonian functions read

$$\overline{H}_{2,m} = \int \frac{r_0 a_{m+2}\eta_1 + r_0 a'_{m+2}\eta_2 + r_0 a''_{m+2}\eta_3 + r_0 a'''_{m+2}\eta_4}{m+1}dx, \quad m\geq 0. \quad (60)$$

Based on (48), a direct computation shows a recursion relation:

$$P_{2,m+1}=\overline{L}_2 P_{2,m}, \quad (61)$$

where the recursion operator $\overline{L}_2$ is given by

$$\overline{L}_2 = M_3^T\left(L^1, L_1^1, L_3^1, L_4^1\right) = \begin{pmatrix} L^1 & 0 & 0 & 0 \\ L_1^1 & L^1+\beta L_1^1 & 0 & 0 \\ L_3^1 & \beta L_3^1 & L^1+\beta L_1^1+\mu L_3^1 & 0 \\ L_4^1 & L_1+\beta L_4^1 & \nu L_3^1 & L^1+\beta L_1^1 \end{pmatrix}, \quad (62)$$

with L^1 and L_1^1 being given as in (34), and

$$\begin{aligned} L_3^1 &= \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}, \\ L_4^1 &= \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}, \\ x_{11} &= -\left(p+\mu p''+\beta p'\right)\partial^{-1}q'' - \beta p''\partial^{-1}q' \\ &\quad - p''\partial^{-1}q, \\ x_{12} &= \left(p+\mu p''+\beta p'\right)\partial^{-1}p'' + \beta p''\partial^{-1}p' + p''\partial^{-1}p, \\ x_{21} &= -\left(q+\mu q''+\beta q'\right)\partial^{-1}q'' - \beta q''\partial^{-1}q' - q''\partial^{-1}q, \\ x_{22} &= \left(q+\mu q''+\beta q'\right)\partial^{-1}p'' + \beta q''\partial^{-1}p' + q''\partial^{-1}p, \\ y_{11} &= -\left(p+\beta p'\right)\partial^{-1}q''' - \nu p''\partial^{-1}q'' \\ &\quad - \left(\beta p'''+p'\right)\partial^{-1}q' - p'''\partial^{-1}q, \\ y_{12} &= \left(p+\beta p'\right)\partial^{-1}p''' + \nu p''\partial^{-1}p'' \\ &\quad + \left(\beta p'''+p'\right)\partial^{-1}p' + p'''\partial^{-1}p, \\ y_{21} &= -\left(q+\beta q'\right)\partial^{-1}q''' - \nu q''\partial^{-1}q'' \\ &\quad - \left(\beta q'''+q'\right)\partial^{-1}q' - q'''\partial^{-1}q, \\ y_{22} &= \left(q+\beta q'\right)\partial^{-1}p''' + \nu q''\partial^{-1}p'' \\ &\quad + \left(\beta q'''+q'\right)\partial^{-1}p' + q'''\partial^{-1}p. \end{aligned} \quad (63)$$

4. Conclusion

In this paper, we take advantage of the non-semi-simple Lie algebras consisting of 3×3, 4×4 block matrices and apply them to the construction of bi-integrable couplings and tri-integrable couplings associated with SO(3), based on the enlarged zero curvature equations. According to the associated variational identities, their Hamiltonian structures can be generated.

We can think about other related issues, for example, how we can get integrable couplings and their Hamiltonian structures when irreducible representations of SO(3) and SO(4) are used to form matrix loop algebras. In addition, we can also consider the relations between the hierarchy of tri-integrable couplings associated with SO(3) and the hierarchy of tri-integrable couplings associated with SO(4).

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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