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Research Article

Linear Approximation and Asymptotic Expansion of Solutions for a Nonlinear Carrier Wave Equation in an Annular Membrane with Robin-Dirichlet Conditions

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This paper is devoted to the study of a nonlinear Carrier wave equation in an annular membrane associated with Robin-Dirichlet conditions. Existence and uniqueness of a weak solution are proved by using the linearization method for nonlinear terms combined with the Faedo-Galerkin method and the weak compact method. Furthermore, an asymptotic expansion of a weak solution of high order in a small parameter is established.

1. Introduction

In this paper, we consider the following nonlinear Carrier wave equation in the annular membrane:

$$u_{tt} - \mu\left(\|u(t)\|_0^2\right)\left(u_{xx} + \frac{1}{x}u_x\right) = f(x,t,u,u_x,u_t), \quad \rho < x < 1,\ 0 < t < T, \tag{1}$$

associated with Robin-Dirichlet conditions

$$u(\rho,t) = u_x(1,t) + \zeta u(1,t) = 0, \tag{2}$$

and initial conditions

$$u(x,0) = \tilde{u}_0(x), \quad u_t(x,0) = \tilde{u}_1(x), \tag{3}$$

where $\mu, f, \tilde{u}_0, \tilde{u}_1$ are given functions; ρ, ζ are given constants, with $0 < \rho < 1$. In (1), nonlinear term $\mu(\|u(t)\|_0^2)$ depends on integral $\|u(t)\|_0^2 = \int_\rho^1 xu^2(x,t)dx$.

Equation (1) herein is the bidimensional nonlinear wave equation describing nonlinear vibrations of annular membrane $\Omega_1 = \{(x,y) : \rho^2 < x^2 + y^2 < 1\}$. In the vibration processing, the area of the annular membrane and the tension at various points change in time. The condition on boundary $\Gamma_1 = \{(x,y) : x^2 + y^2 = 1\}$, that is, $u_x(1,t) + \zeta u(1,t) = 0$, describes elastic constraints where ζ constant has a mechanical signification. And with the boundary condition on $\Gamma_\rho = \{(x,y) : x^2 + y^2 = \rho^2\}$ requiring $u(\rho,t) = 0$, the annular membrane is fixed.

In [1], Carrier established the equation which models vibrations of an elastic string when changes in tension are not small:

$$\rho u_{tt} - \left(1 + \frac{EA}{LT_0}\int_0^L u^2(y,t)\,dy\right)u_{xx} = 0, \tag{4}$$

where $u(x,t)$ is x-derivative of the deformation, T_0 is the tension in the rest position, E is the Young modulus, A is the cross section of a string, L is the length of a string, and ρ is the density of a material. Clearly, if properties of a material vary with x and t, then there is a hyperbolic equation of the type [2]:

$$u_{tt} - B\left(x, t, \int_0^1 u^2(y,t)\,dy\right) u_{xx} = 0. \tag{5}$$

The Kirchhoff-Carrier equations of form (1) received much attention. We refer the reader to, for example, Cavalcanti et al. [3, 4], Ebihara et al. [5], Miranda and Jutuca [6], Lasiecka and Ong [7], Hosoya and Yamada [8], Larkin [2], Medeiros [9], Menzala [10], Park et al. [11, 12], Rabello et al. [13], and Santos et al. [14], for many interesting results and further references.

The paper consists of four sections. Preliminaries are done in Section 2, with the notations, definitions, list of appropriate spaces, and required lemmas. The main results are presented in Sections 3 and 4.

First, by combining the linearization method for nonlinear terms, the Faedo-Galerkin method, and the weak compact method, we prove that problem (1)–(3) has a unique weak solution.

Next, by using Taylor's expansion of given functions μ, μ_1, f, and f_1 up to high order $N+1$, we establish an asymptotic expansion of solution $u = u_\varepsilon$ of order $N+1$ in small parameter ε for

$$\begin{aligned} u_{tt} - \left(\mu\left(\|u(t)\|_0^2\right) + \varepsilon\mu_1\left(\|u(t)\|_0^2\right)\right)\left(u_{xx} + \frac{1}{x}u_x\right) \\ = f(x,t,u,u_x,u_t) + \varepsilon f_1(x,t,u,u_x,u_t), \end{aligned} \tag{6}$$

$\rho < x < 1$, $0 < t < T$, associated with (1) and (2) with $\mu \in C^{N+1}(\mathbb{R}_+)$, $\mu_1 \in C^N(\mathbb{R}_+)$, $\mu(z) \ge \mu_* > 0$, $\mu_1(z) \ge 0$, for all $z \in \mathbb{R}_+$, $f \in C^{N+1}([\rho,1]\times\mathbb{R}_+\times\mathbb{R}^3)$, $f_1 \in C^N([\rho,1]\times\mathbb{R}_+\times\mathbb{R}^3)$. Our results can be regarded as an extension and improvement of the corresponding results of [15, 16].

2. Preliminaries

First, put $\Omega = (\rho, 1)$, $Q_T = \Omega \times (0,T)$, $T > 0$. We omit the definitions of the usual function spaces and denote them by notations $L^p = L^p(\Omega)$, $H^m = H^m(\Omega)$. Let $(\cdot,\cdot)$ be a scalar product in L^2. Notation $\|\cdot\|$ stands for the norm in L^2 and we denote $\|\cdot\|_X$ the norm in Banach space X. We call X' the dual space of X. We denote $L^p(0,T;X)$, $1 \le p \le \infty$ the Banach space of real functions $u : (0,T) \to X$ to be measurable, such that $\|u\|_{L^p(0,T;X)} < +\infty$, with

$$\|u\|_{L^p(0,T;X)} = \begin{cases} \left(\displaystyle\int_0^T \|u(t)\|_X^p\,dt\right)^{1/p}, & \text{if } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{0<t<T} \|u(t)\|_X, & \text{if } p = \infty. \end{cases} \tag{7}$$

With $f \in C^k([\rho,1]\times\mathbb{R}_+\times\mathbb{R}^3)$, $f = f(x,t,y_1,y_2,y_3)$, we put $D_1 f = \partial f/\partial x$, $D_2 f = \partial f/\partial t$, $D_{i+2} f = \partial f/\partial y_i$ with $i = 1,\dots,3$, and $D^\alpha f = D_1^{\alpha_1}\cdots D_5^{\alpha_5} f$, $\alpha = (\alpha_1,\dots,\alpha_5) \in \mathbb{Z}_+^5$, $|\alpha| = \alpha_1 + \cdots + \alpha_5 = k$, $D^{(0,\dots,0)} f = f$.

On H^1, H^2, we shall use the following norms:

$$\|v\|_{H^1} = \left(\|v\|^2 + \|v_x\|^2\right)^{1/2}, \tag{8}$$

$$\|v\|_{H^2} = \left(\|v\|^2 + \|v_x\|^2 + \|v_{xx}\|^2\right)^{1/2}, \tag{9}$$

respectively.

We remark that L^2, H^1, H^2 are the Hilbert spaces with respect to the corresponding scalar products:

$$\begin{gathered} \langle u, v\rangle = \int_\rho^1 x u(x) v(x)\,dx, \\ \langle u, v\rangle + \langle u_x, v_x\rangle, \quad \langle u, v\rangle + \langle u_x, v_x\rangle + \langle u_{xx}, v_{xx}\rangle. \end{gathered} \tag{10}$$

The norms in L^2, H^1, and H^2 induced by the corresponding scalar products (10) are denoted by $\|\cdot\|_0$, $\|\cdot\|_1$, and $\|\cdot\|_2$.

Consider the following set:

$$V = \left\{v \in H^1 : v(\rho) = 0\right\}. \tag{11}$$

It is obviously that V is a closed subspace of H^1 and on V two norms $\|v\|_{H^1}$ and $\|v_x\|$ are equivalent norms. On the other hand, V is continuously and densely embedded in L^2. Identifying L^2 with $(L^2)'$ (the dual of L^2), we have $V \hookrightarrow L^2 \hookrightarrow V'$. We note more that the notation $\langle\cdot,\cdot\rangle$ is also used for the pairing between V and V'.

We then have the following lemmas.

Lemma 1. *The following inequalities are fulfilled:*

(i) $\sqrt{\rho}\|v\| \le \|v\|_0 \le \|v\|$ *for all* $v \in L^2$.

(ii) $\sqrt{\rho}\|v\|_{H^1} \le \|v\|_1 \le \|v\|_{H^1}$ *for all* $v \in H^1$.

Proof of Lemma 1. It is easy to verify the above inequalities via the following inequalities:

$$\begin{aligned} &\rho\int_\rho^1 v^2(x)\,dx \le \int_\rho^1 x v^2(x)\,dx \le \int_\rho^1 v^2(x)\,dx, \quad \forall v \in L^2, \\ &\rho\int_\rho^1 v_x^2(x)\,dx \le \int_\rho^1 x v_x^2(x)\,dx \le \int_\rho^1 v_x^2(x)\,dx, \quad \forall v \in H^1. \end{aligned} \tag{12}$$

□

Lemma 2. *Embedding* $V \hookrightarrow C^0(\overline{\Omega})$ *is compact and for all* $v \in V$, *we have*

(i) $\|v\|_{C^0(\overline{\Omega})} \le \sqrt{1-\rho}\|v_x\|$,

(ii) $\|v\| \le ((1-\rho)/\sqrt{2})\|v_x\|$,

(iii) $\|v\|_0 \le ((1-\rho)/\sqrt{2\rho})\|v_x\|_0$,

(iv) $\|v_x\|_0^2 + v^2(1) \ge \|v\|_0^2$,

(v) $|v(1)| \le \sqrt{3}\|v\|_1$.

Proof of Lemma 2. Embedding $V \hookrightarrow H^1$ is continuous and embedding $H^1 \hookrightarrow C^0(\overline{\Omega})$ is compact, so embedding $V \hookrightarrow C^0(\overline{\Omega})$ is compact. In what follows, we prove (i)–(v).

(i) For all $v \in V$ and $x \in [\rho, 1]$,

$$\begin{aligned} |v(x)| &= \left|\int_\rho^x v_x(y)\,dy\right| \le \int_\rho^1 |v_x(y)|\,dy \\ &\le \sqrt{1-\rho}\,\|v_x\|. \end{aligned} \tag{13}$$

(ii) For all $v \in V$and $x \in [\rho, 1]$,

$$\begin{aligned} v^2(x) &= \left|\int_\rho^x v_x(y)\,dy\right|^2 \le (x-\rho)\int_\rho^x v_x^2(y)\,dy \\ &\le (x-\rho)\|v_x\|^2. \end{aligned} \tag{14}$$

Integrating over x from ρ to 1, we obtain

$$\begin{aligned} \|v\|^2 &= \int_\rho^1 v^2(x)\,dx \le \int_\rho^1 (x-\rho)\|v_x\|^2\,dx \\ &= \frac{(1-\rho)^2}{2}\|v_x\|^2. \end{aligned} \tag{15}$$

(iii) For all $v \in V$,

$$\|v\|_0 \le \|v\| \le \frac{1-\rho}{\sqrt{2}}\|v_x\| \le \frac{1-\rho}{\sqrt{2\rho}}\|v_x\|_0. \tag{16}$$

(iv) Using integration by part, it leads to

$$\begin{aligned} \|v\|_0^2 &= \int_\rho^1 xv^2(x)\,dx \\ &= \frac{1}{2}\left[x^2v^2(x)\right]_\rho^1 - \int_\rho^1 x^2 v(x)\,v_x(x)\,dx \\ &= \frac{1}{2}v^2(1) - \int_\rho^1 x^2 v(x)\,v_x(x)\,dx \\ &\le \frac{1}{2}v^2(1) + \|v\|_0\|v_x\|_0 \\ &\le \frac{1}{2}v^2(1) + \frac{1}{2}\left(\|v\|_0^2 + \|v_x\|_0^2\right), \end{aligned} \tag{17}$$

for any $v \in V$, so we get (iv).

(v) By $\|v\|_0^2 = (1/2)v^2(1) - \int_\rho^1 x^2 v(x)v_x(x)dx$, we have

$$\begin{aligned} v^2(1) &= 2\|v\|_0^2 + 2\int_\rho^1 x^2 v(x)\,v_x(x)\,dx \\ &\le 2\|v\|_0^2 + 2\|v\|_0\|v_x\|_0 \le 2\|v\|_0^2 + \|v\|_0^2 + \|v_x\|_0^2 \\ &\le 3\|v\|_1^2, \end{aligned} \tag{18}$$

implying (v).

Lemma 2 is proved. □

Remark 3. On L^2, two norms $v \mapsto \|v\|$ and $v \mapsto \|v\|_0$ are equivalent. So are two norms $v \mapsto \|v\|_{H^1}$ and $v \mapsto \|v\|_1$ on H^1, and five norms $v \mapsto \|v\|_{H^1}$, $v \mapsto \|v\|_1$, $v \mapsto \|v_x\|$, $v \mapsto \|v_x\|_0$, and $v \mapsto \sqrt{\|v_x\|_0^2 + v^2(1)}$ on V.

Now, we define the following bilinear form:

$$a(u,v) = \zeta u(1)\,v(1) + \int_\rho^1 xu_x(x)\,v_x(x)\,dx, \qquad \forall u, v \in V, \tag{19}$$

where $\zeta \ge 0$ is a constant.

Lemma 4. *Symmetric bilinear form $a(\cdot,\cdot)$ defined by (19) is continuous on $V \times V$ and coercive on V, that is,*

(i) $|a(u,v)| \le C_1\|u\|_1\|v\|_1$,

(ii) $a(v,v) \ge C_0\|v\|_1^2$,

for all u, $v \in V$, where $C_0 = (1/2)\min\{1, 2\rho/(1-\rho)^2\}$ and $C_1 = 1 + 3\zeta$.

Proof of Lemma 4. (i) By $\sqrt{1-\rho}\|v_x\| \ge \|v\|_{C^0(\overline{\Omega})} \ge |v(1)|$, and $\sqrt{\rho}\|v_x\| \le \|v_x\|_0$ for all $v \in V$, we have

$$\begin{aligned} |a(u,v)| &\le \zeta\,|u(1)|\,|v(1)| + \int_\rho^1 |xu_x(x)\,v_x(x)|\,dx \\ &\le 3\zeta\|u\|_1\|v\|_1 + \|u_x\|_0\|v_x\|_0 \\ &\le (3\zeta + 1)\|u\|_1\|v\|_1. \end{aligned} \tag{20}$$

(ii) By inequality $\|v_x\|_0^2 \ge (2\rho/(1-\rho)^2)\|v\|_0^2$, we get

$$\begin{aligned} a(v,v) &= \zeta v^2(1) + \int_\rho^1 xv_x^2(x)\,dx = \zeta v^2(1) + \|v_x\|_0^2 \\ &\ge \|v_x\|_0^2 = \frac{1}{2}\|v_x\|_0^2 + \frac{1}{2}\|v_x\|_0^2 \\ &\ge \frac{1}{2}\|v_x\|_0^2 + \frac{1}{2}\frac{2\rho}{(1-\rho)^2}\|v\|_0^2 \\ &\ge \frac{1}{2}\min\left\{1, \frac{2\rho}{(1-\rho)^2}\right\}\|v\|_1^2. \end{aligned} \tag{21}$$

Lemma 4 is proved. □

Lemma 5. *There exists Hilbert orthonormal base $\{w_j\}$ of space L^2 consisting of eigenfunctions w_j corresponding to eigenvalues λ_j such that*

(i) $0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_j \le \lambda_{j+1} \le \cdots$, $\lim_{j\to+\infty}\lambda_j = +\infty$,

(ii) $a(w_j, v) = \lambda_j\langle w_j, v\rangle$ *for all* $v \in V$, $j = 1, 2, \ldots$.

Furthermore, sequence $\{w_j/\sqrt{\lambda_j}\}$ is also the Hilbert orthonormal base of V with respect to scalar product $a(\cdot,\cdot)$.

On the other hand, we also have w_j satisfying the following boundary value problem:

$$Aw_j \equiv -\left(w_{jxx} + \frac{1}{x}w_{jx}\right) = -\frac{1}{x}\frac{\partial}{\partial x}\left(xw_{jx}\right) = \lambda_j w_j, \quad \text{in } \Omega, \tag{22}$$

$$w_j(\rho) = w_{jx}(1) + \zeta w_j(1) = 0, \quad w_j \in C^\infty([\rho, 1]).$$

Proof. The proof of Lemma 5 can be found in [17, p. 87, Theorem 7.7], with $H = L^2$, and $a(\cdot,\cdot)$ as defined by (19). □

We also note that operator $A : V \to V'$ in (22) is uniquely defined by Lax-Milgram's lemma; that is,

$$a(u, v) = \langle Au, v\rangle \quad \forall u, v \in V. \tag{23}$$

Lemma 6. *On $V \cap H^2$, three norms $v \mapsto \|v\|_{H^2}$, $v \mapsto \|v\|_2 = \sqrt{\|v\|_0^2 + \|v_x\|_0^2 + \|v_{xx}\|_0^2}$, and $v \mapsto \|v\|_{2*} = \sqrt{\|v_x\|_0^2 + \|Av\|_0^2}$ are equivalent.*

Proof of Lemma 6. (i) It is easy to see that, on $V \cap H^2$, two norms $v \mapsto \|v\|_{H^2}$, $v \mapsto \|v\|_2 = \sqrt{\|v\|_0^2 + \|v_x\|_0^2 + \|v_{xx}\|_0^2}$ are equivalent, because

$$\sqrt{\rho}\,\|v\|_{H^2} \le \|v\|_2 \le \|v\|_{H^2} \quad \forall v \in H^2. \tag{24}$$

(ii) For all $x \in [\rho, 1]$, and $v \in V \cap H^2$, we have

$$\begin{aligned} x\,|Au(x)|^2 &= x\frac{1}{x^2}\left[\frac{\partial}{\partial x}(xu_x)\right]^2 \\ &= xu_{xx}^2 + 2u_x u_{xx} + \frac{1}{x}u_x^2. \end{aligned} \tag{25}$$

(a) Proof $\|u\|_2 \le \text{const}\|u\|_{2*}$.
It follows from (25) that

$$xu_{xx}^2 \le x\,|Au(x)|^2 + 2\,|u_x u_{xx}| + \frac{1}{x}u_x^2. \tag{26}$$

Hence,

$$\begin{aligned} \|u_{xx}\|_0^2 &\le \|Au\|_0^2 + \frac{2}{\rho}\|u_x\|_0\|u_{xx}\|_0 + \frac{1}{\rho^2}\|u_x\|_0^2 \\ &\le \|Au\|_0^2 + \frac{1}{\rho}\left(\frac{2}{\rho}\|u_x\|_0^2 + \frac{\rho}{2}\|u_{xx}\|_0^2\right) \\ &\quad + \frac{1}{\rho^2}\|u_x\|_0^2 \\ &= \|Au\|_0^2 + \frac{2}{\rho^2}\|u_x\|_0^2 + \frac{1}{2}\|u_{xx}\|_0^2 + \frac{1}{\rho^2}\|u_x\|_0^2. \end{aligned} \tag{27}$$

This implies

$$\begin{aligned} \|u_{xx}\|_0^2 &\le 2\,\|Au\|_0^2 + \frac{6}{\rho^2}\|u_x\|_0^2 \\ &\le 2\left(1 + \frac{3}{\rho^2}\right)\left(\|Au\|_0^2 + \|u_x\|_0^2\right) \\ &\le 2\left(1 + \frac{3}{\rho^2}\right)\|u\|_{2*}^2. \end{aligned} \tag{28}$$

By $\|v\|_0 \le ((1-\rho)/\sqrt{2\rho})\|v_x\|_0$, for all $v \in V$, we have

$$\begin{aligned} \|u\|_2^2 &= \|u\|_0^2 + \|u_x\|_0^2 + \|u_{xx}\|_0^2 \\ &\le \frac{(1-\rho)^2}{2\rho}\|u_x\|_0^2 + \|u_x\|_0^2 + \|u_{xx}\|_0^2 \\ &\le \left(1 + \frac{(1-\rho)^2}{2\rho}\right)\|u\|_{2*}^2 + 2\left(1 + \frac{3}{\rho^2}\right)\|u\|_{2*}^2 \\ &= \left(\frac{(1-\rho)^2}{2\rho} + 3 + \frac{6}{\rho^2}\right)\|u\|_{2*}^2. \end{aligned} \tag{29}$$

(b) Proof $\|u\|_{2*} \le \text{const}\|u\|_2$.
It follows from (25) that

$$\begin{aligned} x\,|Au(x)|^2 &= x\frac{1}{x^2}\left[\frac{\partial}{\partial x}(xu_x)\right]^2 \\ &= xu_{xx}^2 + 2u_x u_{xx} + \frac{1}{x}u_x^2. \end{aligned} \tag{30}$$

Hence,

$$x\,|Au(x)|^2 \le xu_{xx}^2 + 2\,|u_x u_{xx}| + \frac{1}{x}u_x^2. \tag{31}$$

Thus,

$$\begin{aligned} \|Au\|_0^2 &\le \|u_{xx}\|_0^2 + \frac{2}{\rho}\|u_x\|_0\|u_{xx}\|_0 + \frac{1}{\rho^2}\|u_x\|_0^2 \\ &\le \|u_{xx}\|_0^2 + \frac{1}{\rho}\left(\|u_x\|_0^2 + \|u_{xx}\|_0^2\right) + \frac{1}{\rho^2}\|u_x\|_0^2 \\ &= \left(1 + \frac{1}{\rho}\right)\left[\|u_{xx}\|_0^2 + \frac{1}{\rho}\|u_x\|_0^2\right] \\ &\le \left(1 + \frac{1}{\rho}\right)\frac{1}{\rho}\left[\|u_{xx}\|_0^2 + \|u_x\|_0^2\right] \\ &\le \left(1 + \frac{1}{\rho}\right)\frac{1}{\rho}\|u\|_2^2. \end{aligned} \tag{32}$$

This implies

$$\begin{aligned} \|u\|_{2*}^2 &= \|u_x\|_0^2 + \|Au\|_0^2 \le \|u\|_2^2 + \left(1 + \frac{1}{\rho}\right)\frac{1}{\rho}\|u\|_2^2 \\ &= \left(1 + \frac{1}{\rho} + \frac{1}{\rho^2}\right)\|u\|_2^2. \end{aligned} \tag{33}$$

Lemma 6 is proved. □

Remark 7. The weak formulation of initial-boundary value problem (1)–(3) can be given in the following manner: find $u \in \overline{W} = \{u \in L^\infty(0, T; V \cap H^2) : u_t \in L^\infty(0, T; V),\ u_{tt} \in L^\infty(0, T; L^2)\}$, such that u satisfies the following variational equation:

$$\begin{aligned} &\langle u_{tt}(t), v\rangle + \mu\left(\|u(t)\|_0^2\right) a(u(t), v) \\ &\quad = \langle f(x, t, u, u_x, u_t), v\rangle, \end{aligned} \tag{34}$$

for all $v \in V$, a.e., $t \in (0,T)$, together with the initial conditions:

$$u(0) = \widetilde{u}_0, \qquad u_t(0) = \widetilde{u}_1, \tag{35}$$

where $a(\cdot,\cdot)$ is the symmetric bilinear form on V defined by (19).

3. The Existence and Uniqueness Theorem

Now, we shall consider problem (1)–(3) with constant $\zeta \geq 0$ and make the following assumptions:

(H_1) $\widetilde{u}_0 \in V \cap H^2$, $\widetilde{u}_1 \in V$;

(H_2) $\mu \in C^1(\mathbb{R}_+)$, with $\mu(z) \geq \mu_* > 0$, $\forall z \in \mathbb{R}_+$;

(H_3) $f \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^3)$ such that $f(\rho, t, 0, y_2, 0) = 0$, $\forall (t, y_2) \in \mathbb{R}_+ \times \mathbb{R}$ and $D_i f \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^3)$, $i = 1, 3, 4, 5$.

Considering $T^* > 0$ fixed and letting $T \in (0, T^*]$ and $M > 0$, we put

$$\begin{aligned}
\widetilde{K}_M(\mu) &= \sup_{0 \leq z \leq M^2} \left(\mu(z) + \left|\mu'(z)\right|\right), \\
K_M(f) &= \|f\|_{C^0(A_*(M))} + \|D_1 f\|_{C^0(A_*(M))} \\
&\quad + \sum_{3 \leq i \leq 5} \|D_i f\|_{C^0(A_*(M))}, \\
\|f\|_{C^0(A_*(M))} &= \sup\left\{\left|f(x, t, y_1, y_2, y_3)\right| : (x, t, y_1, y_2, y_3) \in A_*(M)\right\},
\end{aligned} \tag{36}$$

where

$$A_*(M) = \left\{(x, t, y_1, y_2, y_3) \in \overline{\Omega} \times [0, T^*] \times \mathbb{R}^3 : |y_i| \leq \sqrt{\frac{1-\rho}{\rho}} M,\ i = 1, 2, 3\right\}. \tag{37}$$

Also for each $M > 0$ and $T \in (0, T^*]$, we set

$$\begin{aligned}
W(M,T) &= \left\{u \in L^\infty\left(0, T; V \cap H^2\right) : u_t \in L^\infty(0, T; V),\ u_{tt} \in L^2(Q_T),\ \|u\|_{L^\infty(0,T;V \cap H^2)} \leq M,\ \|u_t\|_{L^\infty(0,T;V)} \leq M,\ \|u_{tt}\|_{L^2(Q_T)} \leq M\right\}, \\
W_1(M,T) &= \left\{u \in W(M,T) : u_{tt} \in L^\infty\left(0, T; L^2\right)\right\}.
\end{aligned} \tag{38}$$

We choose first term $u_0 \equiv 0$, suppose that

$$u_{m-1} \in W_1(M,T), \tag{39}$$

and associate the following variational problem with problem (1)–(3): find $u_m \in W_1(M,T)$ $(m \geq 1)$, so that

$$\begin{aligned}
&\left\langle \ddot{u}_m(t), v\right\rangle + \mu_m(t)\, a\left(u_m(t), v\right) = \left\langle F_m(t), v\right\rangle, \quad \forall v \in V, \\
&u_m(0) = \widetilde{u}_0, \\
&\dot{u}_m(0) = \widetilde{u}_1,
\end{aligned} \tag{40}$$

where

$$\begin{aligned}
\mu_m(t) &= \mu\left(\left\|u_{m-1}(t)\right\|_0^2\right), \\
F_m(t) &= f\left(x, t, u_{m-1}(t), \nabla u_{m-1}(t), \dot{u}_{m-1}(t)\right).
\end{aligned} \tag{41}$$

Then, we have the following result.

Theorem 8. *Let assumptions (H_1)–(H_3) hold. Then, there exist positive constants M, T such that the problem (40), (41) has solution $u_m \in W_1(M,T)$.*

Proof of Theorem 8. It consists of three steps.

Step 1 (the Faedo-Galerkin approximation (introduced by Lions [18])). Consider basis $\{w_j\}$ for V as in Lemma 5. Put

$$u_m^{(k)}(t) = \sum_{j=1}^{k} c_{mj}^{(k)}(t)\, w_j, \tag{42}$$

where coefficients $c_{mj}^{(k)}$ satisfy the system of linear differential equations:

$$\begin{aligned}
&\left\langle \ddot{u}_m^{(k)}(t), w_j\right\rangle + \mu_m(t)\, a\left(u_m^{(k)}(t), w_j\right) = \left\langle F_m(t), w_j\right\rangle, \quad j = 1, \ldots, k, \\
&u_m^{(k)}(0) = u_{0k}, \\
&\dot{u}_m^{(k)}(0) = u_{1k},
\end{aligned} \tag{43}$$

with

$$\begin{aligned}
u_{0k} &= \sum_{j=1}^{k} \alpha_j^{(k)} w_j \longrightarrow \widetilde{u}_0 \quad \text{strongly in } V \cap H^2, \\
u_{1k} &= \sum_{j=1}^{k} \beta_j^{(k)} w_j \longrightarrow \widetilde{u}_1 \quad \text{strongly in } V.
\end{aligned} \tag{44}$$

The system of (43) can be rewritten in form

$$\begin{aligned}
&\ddot{c}_{mj}^{(k)}(t) + \lambda_j \mu_m(t)\, c_{mj}^{(k)}(t) = F_{mj}(t), \quad 1 \leq j \leq k, \\
&c_{mj}^{(k)}(0) = \alpha_j^{(k)}, \\
&\dot{c}_{mj}^{(k)}(0) = \beta_j^{(k)},
\end{aligned} \tag{45}$$

in which

$$F_{mj}(t) = \left\langle F_m(t), w_j\right\rangle, \quad 1 \leq j \leq k. \tag{46}$$

Note that, by (39), it is not difficult to prove that system (45), (46) has a unique solution $c_{mj}^{(k)}(t)$, $1 \le j \le k$ on interval $[0, T]$, so let us omit the details.

Step 2 (a priori estimates). We put

$$S_m^{(k)}(t) = X_m^{(k)}(t) + Y_m^{(k)}(t) + \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 ds, \tag{47}$$

where

$$\begin{aligned} X_m^{(k)}(t) &= \left\|\dot{u}_m^{(k)}(t)\right\|_0^2 + \mu_m(t)\, a\left(u_m^{(k)}(t), u_m^{(k)}(t)\right), \\ Y_m^{(k)}(t) &= a\left(\dot{u}_m^{(k)}(t), \dot{u}_m^{(k)}(t)\right) + \mu_m(t)\left\|Au_m^{(k)}(t)\right\|_0^2. \end{aligned} \tag{48}$$

Then, it follows from (43), (47), and (48) that

$$\begin{aligned} S_m^{(k)}(t) &= S_m^{(k)}(0) \\ &\quad + \int_0^t \mu_m'(s)\left[a\left(u_m^{(k)}(s), u_m^{(k)}(s)\right) + \left\|Au_m^{(k)}(s)\right\|_0^2\right] ds \\ &\quad + 2\int_0^t \left\langle F_m(s), \dot{u}_m^{(k)}(s)\right\rangle ds \\ &\quad + 2\int_0^t a\left(F_m(s), \dot{u}_m^{(k)}(s)\right) ds + \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 ds \\ &\equiv S_m^{(k)}(0) + \sum_{j=1}^4 I_j. \end{aligned} \tag{49}$$

We shall estimate terms I_j on the right-hand side of (49) as follows.

First Term I_1. By the following inequalities,

$$\begin{aligned} \left|\mu_m'(t)\right| &= 2\left|\mu'\left(\left\|u_{m-1}(t)\right\|_0^2\right)\left\langle u_{m-1}(t), u_{m-1}'(t)\right\rangle\right| \\ &\le 2\left|\mu'\left(\left\|u_{m-1}(t)\right\|_0^2\right)\right|\left\|u_{m-1}(t)\right\|_0\left\|u_{m-1}'(t)\right\|_0 \\ &\le 2M^2\widetilde{K}_M(\mu), \\ S_m^{(k)}(t) &\ge \mu_m(t)\left[a\left(u_m^{(k)}(t), u_m^{(k)}(t)\right) + \left\|Au_m^{(k)}(t)\right\|_0^2\right] \\ &\ge \mu_*\left[a\left(u_m^{(k)}(t), u_m^{(k)}(t)\right) + \left\|Au_m^{(k)}(t)\right\|_0^2\right], \end{aligned} \tag{50}$$

we have

$$\begin{aligned} I_1 &= \int_0^t \mu_m'(s)\left[a\left(u_m^{(k)}(s), u_m^{(k)}(s)\right) + \left\|Au_m^{(k)}(s)\right\|_0^2\right] ds \\ &\le \frac{2}{\mu_*}M^2\widetilde{K}_M(\mu)\int_0^t S_m^{(k)}(s)\, ds. \end{aligned} \tag{51}$$

Second Term I_2. By the Cauchy-Schwartz inequality, it gives

$$\begin{aligned} \left|I_2\right| &= 2\left|\int_0^t \left\langle F_m(s), \dot{u}_m^{(k)}(s)\right\rangle ds\right| \\ &\le 2\int_0^t \left\|F_m(s)\right\|_0\left\|\dot{u}_m^{(k)}(s)\right\|_0 ds \\ &\le 2\sqrt{\frac{1-\rho^2}{2}}K_M(f)\int_0^t \left\|\dot{u}_m^{(k)}(s)\right\|_0 ds \\ &\le \frac{1-\rho^2}{2}TK_M^2(f) + \int_0^t S_m^{(k)}(s)\, ds. \end{aligned} \tag{52}$$

Third Term I_3. Similarly, we have

$$\begin{aligned} \left|I_3\right| &= 2\left|\int_0^t a\left(F_m(s), \dot{u}_m^{(k)}(s)\right) ds\right| \\ &\le 2\int_0^t \sqrt{a\left(F_m(s), F_m(s)\right)}\sqrt{a\left(\dot{u}_m^{(k)}(s), \dot{u}_m^{(k)}(s)\right)} ds \\ &\le 2\int_0^t \sqrt{a\left(F_m(s), F_m(s)\right)}\sqrt{S_m^{(k)}(s)} ds \\ &\le \int_0^t a\left(F_m(s), F_m(s)\right) ds + \int_0^t S_m^{(k)}(s)\, ds. \end{aligned} \tag{53}$$

Note that

$$\begin{aligned} a(v, v) &\le C_1\|v\|_1^2 = C_1\left(\|v\|_0^2 + \|v_x\|_0^2\right) \\ &\le C_1\left(\frac{(1-\rho)^2}{2\rho}\|v_x\|_0^2 + \|v_x\|_0^2\right) \\ &= C_1\frac{\left(1+\rho^2\right)}{2\rho}\|v_x\|_0^2 \quad \forall v \in V, \end{aligned} \tag{54}$$

so

$$a\left(F_m(s), F_m(s)\right) \le C_1\frac{\left(1+\rho^2\right)}{2\rho}\left\|F_{mx}(s)\right\|_0^2. \tag{55}$$

We also have

$$\begin{aligned} F_{mx}(t) &= D_1 f\left[u_{m-1}\right] + D_3 f\left[u_{m-1}\right]\nabla u_{m-1}(t) \\ &\quad + D_4 f\left[u_{m-1}\right]\Delta u_{m-1}(t) \\ &\quad + D_5 f\left[u_{m-1}\right]\nabla\dot{u}_{m-1}(t), \end{aligned} \tag{56}$$

where $D_i f[u_{m-1}] = D_i f(x, t, u_{m-1}(t), \nabla u_{m-1}(t), \dot{u}_{m-1}(t))$, $i = 1, \ldots, 5$.

It implies from (39), (56) that

$$\begin{aligned} \left\|F_{mx}(t)\right\|_0 &\le K_M(f)\left[\sqrt{\frac{1-\rho^2}{2}} + \left\|\nabla u_{m-1}(t)\right\|_0\right. \\ &\quad \left. + \left\|\Delta u_{m-1}(t)\right\|_0 + \left\|\nabla\dot{u}_{m-1}(t)\right\|_0\right] \le \left(\sqrt{\frac{1-\rho^2}{2}}\right. \\ &\quad \left. + 3M\right)K_M(f). \end{aligned} \tag{57}$$

Combining (53), (55), and (57), we obtain

$$\begin{aligned}|I_3| &\leq C_1 \frac{(1+\rho^2)}{2\rho} \int_0^t \|F_{mx}(s)\|_0^2 ds + \int_0^t S_m^{(k)}(s) ds \\ &\leq C_1 \frac{(1+\rho^2)}{2\rho} \left(\sqrt{\frac{1-\rho^2}{2}} + 3M\right)^2 TK_M^2(f) \\ &\quad + \int_0^t S_m^{(k)}(s) ds.\end{aligned} \tag{58}$$

Fourth Term I_4. Equation $(43)_1$ can be rewritten as follows:

$$\begin{aligned}&\left\langle \ddot{u}_m^{(k)}(t), w_j \right\rangle + \mu_m(t) \left\langle Au_m^{(k)}(t), w_j \right\rangle \\ &\quad = \left\langle F_m(t), w_j \right\rangle, \quad j = 1, \ldots, k.\end{aligned} \tag{59}$$

Hence, it follows after replacing w_j with $\ddot{u}_m^{(k)}(t)$, that

$$\begin{aligned}&\left\|\ddot{u}_m^{(k)}(t)\right\|_0^2 \\ &\quad = -\mu_m(t) \left\langle Au_m^{(k)}(t), \ddot{u}_m^{(k)}(t) \right\rangle + \left\langle F_m(t), \ddot{u}_m^{(k)}(t) \right\rangle \\ &\quad \leq \left[\mu_m(t) \left\|Au_m^{(k)}(t)\right\|_0 + \|F_m(t)\|_0\right] \left\|\ddot{u}_m^{(k)}(s)\right\|_0 \\ &\quad \leq \left[\mu_m(t) \left\|Au_m^{(k)}(t)\right\|_0 + \|F_m(t)\|_0\right]^2 \\ &\quad \leq 2\mu_m^2(t) \left\|Au_m^{(k)}(t)\right\|_0^2 + 2\|F_m(t)\|_0^2 \\ &\quad \leq 2\mu_m(t) S_m^{(k)}(t) + 2\|F_m(t)\|_0^2 \\ &\quad \leq 2\widetilde{K}_M(\mu) S_m^{(k)}(t) + 2\frac{1-\rho^2}{2} K_M^2(f).\end{aligned} \tag{60}$$

Integrate into t to get

$$\begin{aligned}I_4 &= \int_0^t \left\|\ddot{u}_m^{(k)}(s)\right\|_0^2 ds \\ &\leq (1-\rho^2) TK_M^2(f) + 2\widetilde{K}_M(\mu) \int_0^t S_m^{(k)}(s) ds.\end{aligned} \tag{61}$$

It follows from (49), (51), (52), (58), and (61) that

$$S_m^{(k)}(t) \leq S_m^{(k)}(0) + TD_1(M) + D_2(M) \int_0^t S_m^{(k)}(s) ds, \tag{62}$$

where

$$\begin{aligned}D_1(M) &= \left[\frac{3}{2}(1-\rho^2)\right. \\ &\quad \left. + C_1 \frac{(1+\rho^2)}{2\rho} \left(\sqrt{\frac{1-\rho^2}{2}} + 3M\right)^2\right] K_M^2(f), \\ D_2(M) &= 2\left[1 + \left(1 + \frac{1}{\mu_*} M^2\right) \widetilde{K}_M(\mu)\right].\end{aligned} \tag{63}$$

By means of the convergences in (44), we can deduce the existence of constant $M > 0$ independent of k and m such that

$$\begin{aligned}S_m^{(k)}(0) &= \|u_{1k}\|_0^2 + a(u_{1k}, u_{1k}) \\ &\quad + \mu\left(\|\widetilde{u}_0\|_0^2\right)\left[a(u_{0k}, u_{0k}) + \|Au_{0k}\|_0^2\right] \\ &\leq \frac{1}{2} M^2,\end{aligned} \tag{64}$$

for all $m, k \in \mathbb{N}$.

Therefore, from (63) and (64), we can choose $T \in (0, T^*]$, such that

$$\left(\frac{1}{2}M^2 + TD_1(M)\right) \exp(TD_2(M)) \leq M^2, \tag{65}$$

$$\begin{aligned}k_T &= \left(1 + \frac{1}{\sqrt{\mu_* C_0}}\right) \sqrt{T} \\ &\quad \cdot \exp\left[T\left(1 + \frac{1}{\mu_*} M^2 \widetilde{K}_M(\mu)\right)\right] \\ &\quad \cdot \left[4M^4 \widetilde{K}_M^2(\mu) + K_M^2(f)\left(1 + \frac{1-\rho}{\sqrt{2\rho}}\right)^2\right]^{1/2} < 1.\end{aligned} \tag{66}$$

Finally, it follows from (62), (64), and (65) that

$$\begin{aligned}S_m^{(k)}(t) &\leq M^2 \exp(-TD_2(M)) \\ &\quad + D_2(M) \int_0^t S_m^{(k)}(s) ds.\end{aligned} \tag{67}$$

By using Gronwall's Lemma, (67) yields

$$S_m^{(k)}(t) \leq M^2 \exp(-TD_2(M)) \exp(tD_2(M)) \leq M^2, \tag{68}$$

for all $t \in [0, T]$, for all m and k. Therefore, we have

$$u_m^{(k)} \in W(M, T), \quad \forall m, k. \tag{69}$$

Step 3 (limiting process). From (69), there exists a subsequence of $\{u_m^{(k)}\}$, still so denoted, such that

$$\begin{aligned}&u_m^{(k)} \longrightarrow u_m \quad \text{in } L^\infty(0, T; V \cap H^2) \text{ weakly}^*, \\ &\dot{u}_m^{(k)} \longrightarrow u_m' \quad \text{in } L^\infty(0, T; V) \text{ weakly}^*, \\ &\ddot{u}_m^{(k)} \longrightarrow u_m'' \quad \text{in } L^2(Q_T) \text{ weakly}, \\ &\qquad u_m \in W(M, T).\end{aligned} \tag{70}$$

Passing to limit in (43), we have u_m satisfying (40), (41) in $L^2(0, T)$. On the other hand, it follows from $(40)_1$ and $(70)_4$ that $u_m'' = -\mu_m(t) Au_m + F_m \in L^\infty(0, T; L^2)$, and hence $u_m \in W_1(M, T)$ and the proof of Theorem 8 is complete. □

We will use the result obtained in Theorem 8 and the compact imbedding theorems to prove the existence and uniqueness of a weak solution of problem (1)–(3). Hence, we get the main result in this section.

Theorem 9. *Let (H_1)–(H_3) hold. Then, there exist positive constants M, T satisfying (64)–(66) such that problem (1)–(3) has unique weak solution $u \in W_1(M,T)$. Furthermore, the linear recurrent sequence $\{u_m\}$ defined by (40), (41) converges to solution u strongly in space $W_1(T) = \{v \in L^\infty(0,T;V) : v' \in L^\infty(0,T;L^2)\}$, with estimate*

$$\left\|u_m - u\right\|_{W_1(T)} \leq \frac{M}{1-k_T} k_T^m, \quad \forall m \in \mathbb{N}. \tag{71}$$

Proof of Theorem 9.
(a) The Existence. First, we note that $W_1(T)$ is a Banach space with respect to norm $\|v\|_{W_1(T)} = \|v\|_{L^\infty(0,T;V)} + \|v'\|_{L^\infty(0,T;L^2)}$ (see Lions [18]).

We shall prove that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Let $w_m = u_{m+1} - u_m$. Then, w_m satisfies the variational problem:

$$\begin{gathered}\left\langle w_m''(t), v\right\rangle + \mu_{m+1}(t)\, a\left(w_m(t), v\right) \\ + \left[\mu_{m+1}(t) - \mu_m(t)\right]\left\langle Au_m(t), v\right\rangle \\ = \left\langle F_{m+1}(t) - F_m(t), v\right\rangle, \quad \forall v \in V, \\ w_m(0) = w_m'(0) = 0.\end{gathered} \tag{72}$$

Taking $v = w_m'$ in $(72)_1$, after integrating into t, we get

$$\begin{aligned}Z_m(t) &= \int_0^t \mu_{m+1}'(s)\, a\left(w_m(s), w_m(s)\right) ds \\ &\quad - 2\int_0^t \left[\mu_{m+1}(s) - \mu_m(s)\right]\left\langle Au_m(s), w_m'(s)\right\rangle ds \\ &\quad + 2\int_0^t \left\langle F_{m+1}(s) - F_m(s), w_m'(s)\right\rangle ds \\ &\equiv J_1 + J_2 + J_3,\end{aligned} \tag{73}$$

where

$$\begin{aligned}Z_m(t) &= \left\|w_m'(t)\right\|_0^2 + \mu_{m+1}(t)\, a\left(w_m(t), w_m(t)\right) \\ &\geq \left\|w_m'(t)\right\|_0^2 + \mu_* a\left(w_m(t), w_m(t)\right) \\ &\geq \left\|w_m'(t)\right\|_0^2 + \mu_* C_0 \left\|w_m(t)\right\|_1^2.\end{aligned} \tag{74}$$

All integrals on the right-hand side of (73) will be estimated as below.

First Integral J_1. By (50) and (74), we have

$$\begin{aligned}\left|J_1\right| &\leq \int_0^t \left|\mu_{m+1}'(s)\right| a\left(w_m(s), w_m(s)\right) ds \\ &\leq \frac{1}{\mu_*} 2M^2 \widetilde{K}_M(\mu) \int_0^t Z_m(s)\, ds.\end{aligned} \tag{75}$$

Second Integral J_2. By (H_2), it is clear to see that

$$\begin{aligned}\left|\mu_{m+1}(t) - \mu_m(t)\right| &= \left|\mu\left(\left\|u_m(t)\right\|_0^2\right) - \mu\left(\left\|u_{m-1}(t)\right\|_0^2\right)\right| \\ &\leq \widetilde{K}_M(\mu)\left|\left\|u_m(t)\right\|_0^2 - \left\|u_{m-1}(t)\right\|_0^2\right| \\ &\leq 2M\widetilde{K}_M(\mu)\left\|w_{m-1}(t)\right\|_0 \\ &\leq 2M\widetilde{K}_M(\mu)\left\|w_{m-1}\right\|_{W_1(T)}.\end{aligned} \tag{76}$$

Hence,

$$\begin{aligned}\left|J_2\right| &\leq 2\left|\int_0^t \left[\mu_{m+1}(s) - \mu_m(s)\right]\left\langle Au_m(s), w_m'(s)\right\rangle ds\right| \\ &\leq 4M\widetilde{K}_M(\mu)\left\|w_{m-1}\right\|_{W_1(T)} \\ &\quad \cdot \int_0^t \left\|Au_m(s)\right\|_0 \left\|w_m'(s)\right\|_0 ds \leq 4M^2\widetilde{K}_M(\mu) \\ &\quad \cdot \left\|w_{m-1}\right\|_{W_1(T)} \int_0^t \left\|w_m'(s)\right\|_0 ds \leq 4TM^4\widetilde{K}_M^2(\mu) \\ &\quad \cdot \left\|w_{m-1}\right\|_{W_1(T)}^2 + \int_0^t Z_m(s)\, ds.\end{aligned} \tag{77}$$

Second Integral J_3. By (H_3), it yields

$$\begin{aligned}&\left\|F_{m+1}(t) - F_m(t)\right\|_0 \leq K_M(f)\left(\left\|w_{m-1}(t)\right\|_0\right. \\ &\quad \left. + \left\|\nabla w_{m-1}(t)\right\|_0 + \left\|w_{m-1}'(t)\right\|_0\right) \leq K_M(f) \\ &\quad \cdot \left(\frac{1-\rho}{\sqrt{2\rho}}\left\|\nabla w_{m-1}(t)\right\|_0 + \left\|\nabla w_{m-1}(t)\right\|_0\right. \\ &\quad \left. + \left\|w_{m-1}'(t)\right\|_0\right) \leq K_M(f)\left(1 + \frac{1-\rho}{\sqrt{2\rho}}\right) \\ &\quad \cdot \left(\left\|\nabla w_{m-1}(t)\right\|_0 + \left\|w_{m-1}'(t)\right\|_0\right) \leq K_M(f)\left(1\right. \\ &\quad \left. + \frac{1-\rho}{\sqrt{2\rho}}\right)\left\|w_{m-1}\right\|_{W_1(T)}.\end{aligned} \tag{78}$$

Hence,

$$\begin{aligned}\left|J_3\right| &\leq 2\left|\int_0^t \left\langle F_{m+1}(s) - F_m(s), w_m'(s)\right\rangle ds\right| \\ &\leq 2\int_0^t \left\|F_{m+1}(s) - F_m(s)\right\|_0 \left\|w_m'(s)\right\|_0 ds \\ &\leq 2K_M(f)\left(1 + \frac{1-\rho}{\sqrt{2\rho}}\right)\left\|w_{m-1}\right\|_{W_1(T)} \\ &\quad \cdot \int_0^t \left\|w_m'(s)\right\|_0 ds \leq TK_M^2(f)\left(1 + \frac{1-\rho}{\sqrt{2\rho}}\right)^2 \\ &\quad \cdot \left\|w_{m-1}\right\|_{W_1(T)}^2 + \int_0^t Z_m(s)\, ds.\end{aligned} \tag{79}$$

Combining (73), (75), (77), and (79), we obtain

$$Z_m(t) \le T\left[4M^4\widetilde{K}_M^2(\mu) + K_M^2(f)\left(1+\frac{1-\rho}{\sqrt{2\rho}}\right)^2\right] \cdot \|w_{m-1}\|_{W_1(T)}^2 + 2\left(1+\frac{1}{\mu_*}M^2\widetilde{K}_M(\mu)\right) \cdot \int_0^t Z_m(s)\,ds. \quad (80)$$

Using Gronwall's lemma, we deduce from (80) that

$$\|w_m\|_{W_1(T)} \le k_T\|w_{m-1}\|_{W_1(T)} \quad \forall m \in \mathbb{N}, \quad (81)$$

which implies that

$$\|u_m - u_{m+p}\|_{W_1(T)} \le \|u_0 - u_1\|_{W_1(T)}(1-k_T)^{-1}k_T^m \le \frac{M}{1-k_T}k_T^m \quad \forall m, p \in \mathbb{N}. \quad (82)$$

It follows that $\{u_m\}$ is a Cauchy sequence in $W_1(T)$. Then, there exists $u \in W_1(T)$ such that

$$u_m \longrightarrow u \quad \text{strongly in } W_1(T). \quad (83)$$

Note that $u_m \in W_1(M,T)$, and then there exists subsequence $\{u_{m_j}\}$ of $\{u_m\}$ such that

$$\begin{aligned} &u_{m_j} \longrightarrow u \quad \text{in } L^\infty(0,T;V\cap H^2) \text{ weakly}^*,\\ &u'_{m_j} \longrightarrow u' \quad \text{in } L^\infty(0,T;V) \text{ weakly}^*,\\ &u''_{m_j} \longrightarrow u'' \quad \text{in } L^2(Q_T) \text{ weakly},\\ &u \in W(M,T). \end{aligned} \quad (84)$$

We also note that

$$\|F_m(t) - f(x,t,u,u_x,u_t)\|_{L^\infty(0,T;L^2)} \le K_M(f)\left(1+\frac{1-\rho}{\sqrt{2\rho}}\right)\|u_{m-1}-u\|_{W_1(T)}. \quad (85)$$

Hence, from (83) and (85), we obtain

$$F_m(t) \longrightarrow f(x,t,u,u_x,u_t) \quad \text{strongly in } L^\infty(0,T;L^2). \quad (86)$$

On the other hand, we have

$$\left|\mu_m(t) - \mu\left(\|u(t)\|_0^2\right)\right| \le 2M\widetilde{K}_M(\mu)\|w_{m-1}(t)\|_0 \le 2M\widetilde{K}_M(\mu)\|u_{m-1}-u\|_{W_1(T)}. \quad (87)$$

Hence, it follows from (83) and (87) that

$$\mu_m(t) \longrightarrow \mu\left(\|u(t)\|_0^2\right) \quad \text{strongly in } L^\infty(0,T). \quad (88)$$

Finally, passing to limit in (40), (41) as $m = m_j \to \infty$, it implies from (83), $(84)_{1,3}$, (86), and (88) that there exists $u \in W(M,T)$ satisfying

$$\left\langle u_{tt}(t), v\right\rangle + \mu\left(\|u(t)\|_0^2\right)a(u(t),v) = \left\langle f(x,t,u,u_x,u_t), v\right\rangle, \quad (89)$$

for all $v \in V$ and the initial conditions

$$u(0) = \widetilde{u}_0, \quad u'(0) = \widetilde{u}_1. \quad (90)$$

Furthermore, from assumptions (H_2), (H_3) we obtain from $(84)_4$, (86), (88), and (89), that

$$u'' = -\mu\left(\|u(t)\|_0^2\right)Au(t) + f(x,t,u,u_x,u_t) \in L^\infty\left(0,T;L^2\right), \quad (91)$$

and thus we have $u \in W_1(M,T)$. The existence of a weak solution of problem (1)–(3) is proved.

(b) The Uniqueness. Let $u_1, u_2 \in W_1(M,T)$ be two weak solutions of problem (1)–(3). Then, $u = u_1 - u_2$ satisfies the variational problem:

$$\begin{aligned} &\left\langle u''(t), v\right\rangle + \mu_1(t)a(u(t),v) + [\mu_1(t) - \mu_2(t)]\left\langle Au_2(t), v\right\rangle = \left\langle F_1(t) - F_2(t), v\right\rangle, \quad \forall v \in V,\\ &u(0) = u'(0) = 0, \end{aligned} \quad (92)$$

where $F_i(x,t) = f(x,t,u_i,\nabla u_i,u_i')$, $\mu_i(t) = \mu(\|u_i(t)\|_0^2)$, $i = 1, 2$.

We take $w = u'$ in $(92)_1$ and integrate in t to get

$$Z(t) = \int_0^t \mu_1'(s)a(u(s),u(s))\,ds + 2\int_0^t \left\langle F_1(s) - F_2(s), u'(s)\right\rangle ds - 2\int_0^t [\mu_1(s) - \mu_2(s)]\left\langle Au_1(s), u'(s)\right\rangle ds, \quad (93)$$

with $Z(t) = \|u'(t)\|_0^2 + \mu_1(t)a(u(t),u(t))$.

Putting $K_M^* = 2[K_M(f)(1+(1-\rho)/\sqrt{2\rho})(1+1/\sqrt{\mu_*}) + (1/\mu_* + 2(1-\rho)/\sqrt{2\mu_*\rho})M^2\widetilde{K}_M(\mu)]$, it follows from (93) that

$$Z(t) \le K_M^*\int_0^t Z(s)\,ds, \quad \forall t \in [0,T]. \quad (94)$$

Using Gronwall's lemma, it follows that $Z(t) \equiv 0$, that is, $u_1 \equiv u_2$.

Therefore, Theorem 9 is proved. □

4. Asymptotic Expansion of the Solution with respect to a Small Parameter

In this section, let (H_1)–(H_4) hold. We make more the following assumptions:

(H_2') $\mu_1 \in C^1(\mathbb{R}_+)$, with $\mu_1(z) \geq 0$, $\forall z \in \mathbb{R}_+$.

(H_3') $f_1 \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^3)$ such that $f_1(\rho, t, 0, y_2, 0) = 0$, $\forall (t, y_2) \in \mathbb{R}_+ \times \mathbb{R}$ and $D_i f_1 \in C^0(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^3)$, $i = 1, 3, 4, 5$.

Considering the following perturbed problem, where ε is a small parameter and $|\varepsilon| \leq 1$:

$$\begin{aligned} &u_{tt} + \mu_\varepsilon [u] Au = F_\varepsilon [u](x,t), \quad \rho < x < 1,\ 0 < t < T, \\ &u(\rho, t) = u_x(1,t) + \zeta u(1,t) = 0, \\ &u(x,0) = \tilde{u}_0(x), \\ &u_t(x,0) = \tilde{u}_1(x), \end{aligned} \tag{P_ε}$$

with

$$\begin{aligned} Au &= \frac{-1}{x}\frac{\partial}{\partial x}(xu_x) = -\left(u_{xx} + \frac{1}{x}u_x\right), \\ \mu_\varepsilon [u] &= \mu_\varepsilon\left(\|u(t)\|_0^2\right) \\ &= \mu\left(\|u(t)\|_0^2\right) + \varepsilon \mu_1\left(\|u(t)\|_0^2\right), \\ F_\varepsilon [u](x,t) &= f[u](x,t) + \varepsilon f_1 [u](x,t), \\ f[u](x,t) &= f(x,t,u,u_t,u_x), \\ f_1[u](x,t) &= f_1(x,t,u,u_t,u_x). \end{aligned} \tag{95}$$

First, we note that if functions μ, μ_1, f, f_1 satisfy (H_2), (H_2'), (H_3), (H_3'), then a priori estimates of the Galerkin approximation sequence $\{u_m^{(k)}\}$ for problem (1)–(3) corresponding to $\mu = \mu_\varepsilon$, $f = F_\varepsilon[u]$, $|\varepsilon| \leq 1$, leads to $u_m^{(k)} \in W_1(M,T)$, where constants M, T, independent of ε, are chosen as in (63)–(66), in which μ, $\widetilde{K}_M(\mu)$, $K_M(f)$ are replaced with $\mu + \mu_1$, $\widetilde{K}_M(\mu) + \widetilde{K}_M(\mu_1)$, $K_M(f) + K_M(f_1)$, respectively. Hence, limit u_ε in suitable function spaces of sequence $\{u_m^{(k)}\}$ as $k \to +\infty$, after $m \to +\infty$, is a unique weak solution of problem (P_ε) satisfying $u_\varepsilon \in W_1(M,T)$.

We can prove in a manner similar to the proof of Theorem 9 that limit u_0 in suitable function spaces of family $\{u_\varepsilon\}$ as $\varepsilon \to 0$ is a unique weak solution of problem (P_0) (corresponding to $\varepsilon = 0$) satisfying $u_0 \in W_1(M,T)$.

Next, we shall study the asymptotic expansion of solution u_ε with respect to a small parameter ε. For multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N$, and $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$, we put

$$\begin{aligned} &|\alpha| = \alpha_1 + \cdots + \alpha_N, \quad \alpha! = \alpha_1! \cdots \alpha_N!, \\ &\alpha, \beta \in \mathbb{Z}_+^N, \quad \alpha \leq \beta \Longleftrightarrow \alpha_i \leq \beta_i \quad \forall i = 1, \ldots, N, \\ &x^\alpha = x_1^{\alpha_1} \cdots x_N^{\alpha_N}. \end{aligned} \tag{96}$$

We need the following lemma.

Lemma 10. *Let $m, N \in \mathbb{N}$ and $= (x_1, \ldots, x_N) \in \mathbb{R}^N$, $\varepsilon \in \mathbb{R}$. Then,*

$$\left(\sum_{i=1}^{N} x_i \varepsilon^i\right)^m = \sum_{k=m}^{mN} P_k^{(m)}[N,x]\,\varepsilon^k, \tag{97}$$

where coefficients $P_k^{(m)}[N,x]$, $m \leq k \leq mN$, depending on $x = (x_1, \ldots, x_N)$, are defined by the following formulas:

$$\begin{aligned} P_k^{(1)}[N,x] &= x_k, \quad 1 \leq k \leq N, \\ P_k^{(m)}[N,x] &= \sum_{\alpha \in A_k^{(m)}(N)} \frac{m!}{\alpha!} x^\alpha, \quad m \leq k \leq mN,\ m \geq 2, \\ A_k^{(m)}(N) &= \left\{\alpha \in \mathbb{Z}_+^N : |\alpha| = m,\ \sum_{i=1}^{N} i\alpha_i = k\right\}. \end{aligned} \tag{98}$$

Proof. The proof of Lemma 10 is easy; hence, we omit the details. □

Now, we assume that

$(H_2^{(N)})$ $\mu \in C^{N+1}(\mathbb{R}_+)$, $\mu_1 \in C^N(\mathbb{R}_+)$, with $\mu(z) \geq \mu_* > 0$, $\mu_1(z) \geq 0$, $\forall z \in \mathbb{R}_+$;

$(H_3^{(N)})$ $f \in C^{N+1}(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^3)$, $f_1 \in C^N(\overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^3)$ such that $f(\rho, t, 0, y_2, 0) = f_1(\rho, t, 0, y_2, 0) = 0$, $\forall (t, y_2) \in \mathbb{R}_+ \times \mathbb{R}$.

Let u_0 be a unique weak solution of problem (P_0); that is,

$$\begin{aligned} &u_0'' + \mu[u_0] Au_0 = f[u_0] \equiv F_0, \quad \rho < x < 1,\ 0 < t < T, \\ &u_0(\rho, t) = u_{0x}(1,t) + \zeta u_0(1,t) = 0, \\ &u_0(x,0) = \tilde{u}_0(x), \\ &u_0'(x,0) = \tilde{u}_1(x), \\ &u_0 \in W_1(M,T). \end{aligned} \tag{P_0}$$

Let us consider the sequence of weak solutions u_k, $1 \leq k \leq N$, defined by the following problems:

$$\begin{aligned} &u_k'' + \mu[u_0] Au_k = F_k, \quad \rho < x < 1,\ 0 < t < T, \\ &u_k(\rho, t) = u_{kx}(1,t) + \zeta u_k(1,t) = 0, \\ &u_k(x,0) = u_k'(x,0) = 0, \\ &u_k \in W_1(M,T), \end{aligned} \tag{$\widetilde{P}_k$}$$

where F_k, $1 \leq k \leq N$, are defined by the following formulas:

$$\begin{aligned} F_k &= f_1[u_0] + \overline{\Phi}_1[N, f, u_0, \vec{u}] - \left(\mu_1[u_0] \right. \\ &\quad \left. + \widehat{\Phi}_1[N, \mu, u_0, \vec{u}]\right) Au_0, \quad k = 1, \end{aligned} \tag{99a}$$

$$\begin{aligned} F_k &= \overline{\Phi}_k[N, f, u_0, \vec{u}] + \overline{\Phi}_{k-1}[N-1, f_1, u_0, \vec{u}] \\ &\quad - \left(\mu_1[u_0] + \widehat{\Phi}_1[N, \mu, u_0, \vec{u}]\right) Au_{k-1} \end{aligned}$$

$$- \sum_{\substack{i+j=k,\\ 1\le i\le N+1, 1\le j\le N-1}} \left(\widehat{\Phi}_{j+1}\left[N,\mu,u_0,\vec{u}\right] + \widehat{\Phi}_{j}\left[N-1,\mu_1,u_0,\vec{u}\right]\right) A u_{i-1}, \quad 2 \le k \le N, \tag{99b}$$

with $\overline{\Phi}_k[N, f, u_0, \vec{u}]$, $\widehat{\Phi}_k[N, \mu, u_0, \vec{u}]$, $1 \le k \le N$, are defined by the following formulas:

$$\begin{aligned}
\overline{\Phi}_k\left[N,f,u_0,\vec{u}\right] &= \sum_{1\le|\gamma|\le k} \frac{1}{\gamma!} D^{\gamma} f\left[u_0\right] \Psi_k\left[\gamma,N,\vec{u}\right], \quad 1 \le k \le N,\\
\Psi_k\left[\gamma,N,\vec{u}\right] &= \sum_{\substack{(k_1,k_2,k_3)\in\widetilde{A}(\gamma,N),\\ k_1+k_2+k_3=k}} P_{k_1}^{(\gamma_1)}\left[N,\vec{u}\right] P_{k_2}^{(\gamma_2)}\left[N,\vec{u}'\right] P_{k_3}^{(\gamma_3)}\left[N,\nabla\vec{u}\right], \quad 1 \le k \le N\left|\gamma\right|,\\
\vec{u} &= \left(u_1,\ldots,u_N\right),\\
\vec{u}' &= \left(u_1',\ldots,u_N'\right),\\
\nabla\vec{u} &= \left(\nabla u_1,\ldots,\nabla u_N\right),\\
\widetilde{A}\left(\gamma,N\right) &= \left\{\left(k_1,k_2,k_3\right)\in\mathbb{Z}_+^3 : \gamma_i \le k_i \le N\gamma_i,\ \forall i = 1,2,3\right\},\\
\gamma &= \left(\gamma_1,\gamma_2,\gamma_3\right)\in\mathbb{Z}_+^3, \quad 1 \le \left|\gamma\right| \le N,
\end{aligned} \tag{100}$$

$$\begin{aligned}
\widehat{\Phi}_k\left[N,\mu,u_0,\vec{u}\right] &= \sum_{m=1}^{k} \frac{1}{m!} \mu^{(m)}\left[u_0\right] P_k^{(m)}\left[2N,\vec{\sigma}\right], \quad 1 \le k \le N,\\
\vec{\sigma} &= \left(\sigma_1,\ldots,\sigma_{2N}\right),\\
\sigma_k &= \begin{cases} 2\left\langle u_0(t), u_1(t)\right\rangle, & k = 1,\\ 2\left\langle u_0(t), u_k(t)\right\rangle + \sum_{j\le k}\left\langle u_j(t), u_{k-j}(t)\right\rangle, & 2 \le k \le N,\\ \sum_{j\le k}\left\langle u_j(t), u_{k-j}(t)\right\rangle, & N+1 \le k \le 2N. \end{cases}
\end{aligned} \tag{101}$$

Then, we have the following theorem.

Theorem 11. *Let (H_1), $(H_2^{(N)})$, and $(H_3^{(N)})$ hold. Then, there exist constants $M > 0$ and $T > 0$ such that, for every $\varepsilon \in [-1, 1]$, problem (P_ε) has unique weak solution $u_\varepsilon \in W_1(M, T)$ satisfying the asymptotic estimation up to order $N+1$ as follows:*

$$\left\| u_\varepsilon - \sum_{k=0}^{N} u_k \varepsilon^k \right\|_{W_1(T)} \le C_T \left|\varepsilon\right|^{N+1}, \tag{102}$$

where functions u_k, $0 \le k \le N$ are the weak solutions of problems (P_0), $(\widetilde{P}_k)$, $1 \le k \le N$, respectively, and C_T is a constant depending only on N, T, ρ, ζ, f, f_1, μ, μ_1, u_k, $0 \le k \le N$.

In order to prove Theorem 11, we need the following lemmas.

Lemma 12. *Let $\overline{\Phi}_k[N, f, u_0, \vec{u}]$, $1 \le k \le N$, be the functions defined by the formulas (100). Put $h = \sum_{k=0}^{N} u_k \varepsilon^k$, then we have*

$$f\left[h\right] = f\left[u_0\right] + \sum_{k=1}^{N} \overline{\Phi}_k\left[N,f,u_0,\vec{u}\right]\varepsilon^k + \left|\varepsilon\right|^{N+1} \overline{R}_N\left[f,u_0,\vec{u},\varepsilon\right], \tag{103}$$

with $\|\overline{R}_N[f, u_0, \vec{u}, \varepsilon]\|_{L^\infty(0,T;L^2)} \le C$, where C is a constant depending only on N, T, f, u_k, $0 \le k \le N$.

Proof of Lemma 12. In the case of $N = 1$, the proof of (103) is easy; hence, we omit the details, and we only prove with $N \ge 2$. Put $h = u_0 + \sum_{k=1}^{N} u_k \varepsilon^k \equiv u_0 + h_1$. By using Taylor's expansion of function $f[h] = f[u_0+h_1] = f(x, t, u_0+h_1, u_0'+$

$h_1', \nabla u_0 + \nabla h_1)$ around point $[u_0] \equiv (x, t, u_0, u_0', \nabla u_0)$ up to order $N+1$, we obtain

$$f[h] = f\left(x, t, u_0 + h_1, u_0' + h_1', \nabla u_0 + \nabla h_1\right) = f[u_0] + \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma f[u_0]\, h_1^{\gamma_1} \left(h_1'\right)^{\gamma_2} \left(\nabla h_1\right)^{\gamma_3} + R_N[f, u_0, h_1], \tag{104}$$

where

$$R_N[f, u_0, h_1] = \sum_{|\gamma|=N+1} \frac{N+1}{\gamma!} \int_0^1 (1-\theta)^N \cdot D^\gamma f\left(x, t, u_0 + \theta h_1, u_0' + \theta h_1', \nabla u_0 + \theta \nabla h_1\right) \cdot h_1^{\gamma_1} \left(h_1'\right)^{\gamma_2} \left(\nabla h_1\right)^{\gamma_3} d\theta \equiv |\varepsilon|^{N+1} R_N^{(1)}[f, u_0, h_1, \varepsilon], \tag{105}$$

$\gamma = (\gamma_1, \gamma_2, \gamma_3) \in \mathbb{Z}_+^3$, $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3$, $\gamma! = \gamma_1!\gamma_2!\gamma_3!$, $D^\gamma f = D_3^{\gamma_1} D_4^{\gamma_2} D_5^{\gamma_3} f$, $D^\gamma f[u_0] = D^\gamma f(x, t, u_0, u_0', \nabla u_0)$.

By formula (97), we get

$$h_1^{\gamma_1} = \left(\sum_{k=1}^N u_k \varepsilon^k\right)^{\gamma_1} = \sum_{k=\gamma_1}^{N\gamma_1} P_k^{(\gamma_1)}[N, \vec{u}]\, \varepsilon^k, \tag{106}$$

where $\vec{u} = (u_1, \ldots, u_N)$.

Similarly, with $(h_1')^{\gamma_2}$, $(\nabla h_1)^{\gamma_3}$, we also have

$$\left(h_1'\right)^{\gamma_2} = \left(\sum_{k=1}^N u_k' \varepsilon^k\right)^{\gamma_2} = \sum_{k=\gamma_2}^{N\gamma_2} P_k^{(\gamma_2)}\left[N, \vec{u}'\right] \varepsilon^k, \qquad \left(\nabla h_1\right)^{\gamma_3} = \left(\sum_{k=1}^N \nabla u_k \varepsilon^k\right)^{\gamma_3} = \sum_{k=\gamma_3}^{N\gamma_3} P_k^{(\gamma_3)}[N, \nabla\vec{u}]\, \varepsilon^k, \tag{107}$$

where $\vec{u}' = (u_1', \ldots, u_N')$, $\nabla\vec{u} = (\nabla u_1, \ldots, \nabla u_N)$.

Hence, we deduce from (106)-(107) that

$$h_1^{\gamma_1} \left(h_1'\right)^{\gamma_2} \left(\nabla h_1\right)^{\gamma_3} = \sum_{k=|\gamma|}^{N} \Psi_k[\gamma, N, \vec{u}]\, \varepsilon^k + \sum_{k=N+1}^{N|\gamma|} \Psi_k[\gamma, N, \vec{u}]\, \varepsilon^k, \tag{108}$$

where $\Psi_k[\gamma, N, \vec{u}]$, $1 \le k \le N|\gamma|$, are defined by (100).

We deduce from (104), (108) that

$$\begin{aligned} f[h] &= f[u_0] + \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma f[u_0]\, h_1^{\gamma_1} \left(h_1'\right)^{\gamma_2} \left(\nabla h_1\right)^{\gamma_3} + |\varepsilon|^{N+1} R_N^{(1)}[f, u_0, h_1, \varepsilon] \\ &= f[u_0] + \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma f[u_0] \sum_{k=|\gamma|}^{N} \Psi_k[\gamma, N, \vec{u}]\, \varepsilon^k + \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma f[u_0] \sum_{k=N+1}^{N|\gamma|} \Psi_k[\gamma, N, \vec{u}]\, \varepsilon^k + |\varepsilon|^{N+1} R_N^{(1)}[f, u_0, h_1, \varepsilon] \\ &= f[u_0] + \sum_{k=1}^{N} \left(\sum_{1\le|\gamma|\le k} \frac{1}{\gamma!} D^\gamma f[u_0]\, \Psi_k[\gamma, N, \vec{u}] \right) \varepsilon^k + |\varepsilon|^{N+1} \overline{R}_N[f, u_0, \vec{u}, \varepsilon] \\ &= f[u_0] + \sum_{k=1}^{N} \overline{\Phi}_k[N, f, u_0, \vec{u}]\, \varepsilon^k + |\varepsilon|^{N+1} \overline{R}_N[f, u_0, \vec{u}, \varepsilon], \end{aligned} \tag{109}$$

where $\overline{\Phi}_k[N, f, u_0, \vec{u}]$, $1 \le k \le N$, are defined by (100) and

$$|\varepsilon|^{N+1} \overline{R}_N[f, u_0, \vec{u}, \varepsilon] = \sum_{1\le|\gamma|\le N} \frac{1}{\gamma!} D^\gamma f[u_0] \sum_{k=N+1}^{N|\gamma|} \Psi_k[\gamma, N, \vec{u}]\, \varepsilon^k + |\varepsilon|^{N+1} R_N^{(1)}[f, u_0, h_1, \varepsilon]. \tag{110}$$

By the boundedness of functions u_k, u_k', ∇u_k, $1 \le k \le N$ in the function space $L^\infty(0, T; H^1)$, we obtain from (100), (105), (110) that $\|\overline{R}_N[f, u_0, \vec{u}, \varepsilon]\|_{L^\infty(0,T;L^2)} \le C$, where C is a constant depending only on N, T, f, u_k, $0 \le k \le N$. Thus, Lemma 12 is proved. □

Lemma 13. *Let $\widehat{\Phi}_k[N, \mu, u_0, \vec{u}]$, $1 \le k \le N$, be the functions defined by formulas (101). Put $h = \sum_{k=0}^N u_k \varepsilon^k$, and then we have*

$$\mu[h] = \mu[u_0] + \sum_{k=1}^{N} \widehat{\Phi}_k[N, \mu, u_0, \vec{u}]\, \varepsilon^k + |\varepsilon|^{N+1} \widehat{R}_N[\mu, u_0, \vec{u}, \varepsilon], \tag{111}$$

with $\|\widehat{R}_N[\mu, u_0, \vec{u}, \varepsilon]\|_{L^\infty(0,T;L^2)} \le C$, where C is a constant depending only on N, T, μ, u_k, $0 \le k \le N$.

Proof of Lemma 13. In the case of $N = 1$, the proof of (111) is easy; hence we omit the details, and we only prove with $N \ge 2$.

Put $\xi = \|h(t)\|_0^2 - \|u_0(t)\|_0^2$. By using Taylor's expansion of function $\mu[h] = \mu(\|h(t)\|_0^2) = \mu(\|u_0(t)\|_0^2 + \xi)$ around point $\|u_0(t)\|_0^2$ up to order $N+1$, we obtain

$$\begin{aligned}\mu[h] &= \mu\left(\|h(t)\|_0^2\right) = \mu\left(\|u_0(t)\|_0^2 + \xi\right)\\ &= \mu\left(\|u_0(t)\|_0^2\right) + \sum_{m=1}^{N} \frac{1}{m!}\mu^{(m)}[u_0]\,\xi^m \\ &\quad + R_N[\mu, u_0, \xi],\end{aligned} \tag{112}$$

where

$$\begin{aligned} R_N[\mu, u_0, \xi] &= \frac{1}{N!}\int_0^1 (1-\theta)^N \mu^{(N+1)}\left(\|u_0(t)\|_0^2 + \theta\xi\right)\xi^{N+1} d\theta \\ &= |\varepsilon|^{N+1} R_N^{(1)}[\mu, u_0, \xi, \varepsilon], \\ \mu^{(m)}[u_0] &= \frac{d^m\mu}{dz^m}[u_0] = \frac{d^m\mu}{dz^m}\left(\|u_0(t)\|_0^2\right). \end{aligned} \tag{113}$$

On the other hand, we also get

$$\begin{aligned}\xi &= \|u_0(t) + h_1(t)\|_0^2 - \|u_0(t)\|_0^2 \\ &= 2\langle u_0(t), h_1(t)\rangle + \|h_1(t)\|_0^2 = \sum_{1\le k\le 2N} \sigma_k \varepsilon^k,\end{aligned} \tag{114}$$

where σ_k, $1 \le k \le 2N$, are defined by (101).

Using formula (97) again, it follows from (114) that

$$\begin{aligned}\xi^m &= \left(\sum_{1\le k\le 2N}\sigma_k\varepsilon^k\right)^m = \sum_{k=m}^{2mN} P_k^{(m)}[2N, \vec{\sigma}]\,\varepsilon^k \\ &= \sum_{k=m}^{N} P_k^{(m)}[2N, \vec{\sigma}]\,\varepsilon^k + \sum_{k=N+1}^{2mN} P_k^{(m)}[2N, \vec{\sigma}]\,\varepsilon^k,\end{aligned} \tag{115}$$

where $\vec{\sigma} = (\sigma_1, \ldots, \sigma_{2N})$.

We deduce from (112), (115) that

$$\begin{aligned}\mu[h] &= \mu[u_0] + \sum_{m=1}^{N}\frac{1}{m!}\mu^{(m)}[u_0]\,\xi^m \\ &\quad + |\varepsilon|^{N+1} R_N^{(1)}[\mu, u_0, \xi, \varepsilon] \\ &= \mu[u_0] + \sum_{m=1}^{N}\frac{1}{m!}\mu^{(m)}[u_0]\sum_{k=m}^{N} P_k^{(m)}[2N,\vec{\sigma}]\,\varepsilon^k \\ &\quad + \sum_{m=1}^{N}\frac{1}{m!}\mu^{(m)}[u_0]\sum_{k=N+1}^{2mN} P_k^{(m)}[2N,\vec{\sigma}]\,\varepsilon^k \\ &\quad + |\varepsilon|^{N+1} R_N^{(1)}[\mu, u_0, \xi, \varepsilon] \\ &= \mu[u_0] \\ &\quad + \sum_{k=1}^{N}\left(\sum_{m=1}^{k}\frac{1}{m!}\mu^{(m)}[u_0]\,P_k^{(m)}[2N,\vec{\sigma}]\right)\varepsilon^k \\ &\quad + |\varepsilon|^{N+1}\widehat{R}_N[\mu, u_0, \vec{u}, \varepsilon] \\ &= \mu[u_0] + \sum_{k=1}^{N}\widehat{\Phi}_k[N, \mu, u_0, \vec{u}]\,\varepsilon^k \\ &\quad + |\varepsilon|^{N+1}\widehat{R}_N[\mu, u_0, \vec{u}, \varepsilon],\end{aligned} \tag{116}$$

where $\widehat{\Phi}_k[N, \mu, u_0, \vec{u}]$, $1 \le k \le N$, are defined by (101) and

$$\begin{aligned}|\varepsilon|^{N+1}\widehat{R}_N[\mu, u_0, \vec{u}, \varepsilon] &= \sum_{m=1}^{N}\frac{1}{m!}\mu^{(m)}[u_0]\sum_{k=N+1}^{2mN} P_k^{(m)}[2N,\vec{\sigma}]\,\varepsilon^k \\ &\quad + |\varepsilon|^{N+1} R_N^{(1)}[\mu, u_0, \xi, \varepsilon].\end{aligned} \tag{117}$$

By the boundedness of functions $u_k, u_k', \nabla u_k$, $1 \le k \le N$ in function space $L^\infty(0,T;H^1)$, we obtain from (101), (113), (117) that $\|\widehat{R}_N[\mu, u_0, \vec{u}, \varepsilon]\|_{L^\infty(0,T;L^2)} \le C$, where C is a constant depending only on N, T, μ, u_k, $0 \le k \le N$. Thus, Lemma 13 is proved. □

Remark 14. Lemmas 12 and 13 are a generalization of the formula contained in [19, p. 262, formula (4.38)] and it is useful to obtain Lemma 15 below. These lemmas are the key to establish the asymptotic expansion of weak solution u_ε of order $N+1$ in small parameter ε.

Let $u = u_\varepsilon \in W_1(M,T)$ be the unique weak solution of problem (P_ε). Then, $v = u_\varepsilon - \sum_{k=0}^{N} u_k\varepsilon^k \equiv u_\varepsilon - h$ satisfies the problem:

$$\begin{aligned} v'' + \mu_\varepsilon[v+h]\,Av &= F_\varepsilon[v+h] - F_\varepsilon[h] \\ &\quad - \left(\mu_\varepsilon[v+h] - \mu_\varepsilon[h]\right)Ah \\ &\quad + E_\varepsilon(x,t), \\ &\qquad \rho < x < 1,\ 0 < t < T, \\ v(\rho,t) &= v_x(1,t) + \zeta v(1,t) = 0, \\ v(x,0) &= v'(x,0) = 0,\end{aligned} \tag{118}$$

where

$$\begin{aligned}E_\varepsilon(x,t) &= f[h] - f[u_0] + \varepsilon f_1[h] \\ &\quad - \left(\mu[h] - \mu[u_0] + \varepsilon\mu_1[h]\right)Ah - \sum_{k=1}^{N} F_k\varepsilon^k.\end{aligned} \tag{119}$$

Lemma 15. *Let* (H_1), $(H_2^{(N)})$, *and* $(H_3^{(N)})$ *hold. Then, there exists constant* C_* *such that*

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)} \le C_*|\varepsilon|^{N+1}, \tag{120}$$

where C_* *is a constant depending only on* N, T, f, f_1, μ, μ_1, u_k, $0 \le k \le N$.

Proof of Lemma 15. We only prove with $N \geq 2$.

By using formula (103) for function $f_1[h]$, we obtain

$$\begin{aligned} f_1[h] &= f_1[u_0] + \sum_{k=1}^{N-1} \overline{\Phi}_k[N-1, f_1, u_0, \vec{u}]\,\varepsilon^k \\ &\quad + |\varepsilon|^N\, \overline{R}_{N-1}[f_1, u_0, \vec{u}, \varepsilon], \end{aligned} \tag{121}$$

where $\|\overline{R}_{N-1}[f_1, u_0, \vec{u}, \varepsilon]\|_{L^\infty(0,T;L^2)} \leq C$, with constant C depending only on N, T, f_1, g_1, u_k, $0 \leq k \leq N$.

By (121), we rewrite $\varepsilon f_1[h]$ as follows

$$\begin{aligned} \varepsilon f_1[h] &= \varepsilon f_1[u_0] + \sum_{k=2}^{N} \overline{\Phi}_{k-1}[N-1, f_1, u_0, \vec{u}]\,\varepsilon^k \\ &\quad + \varepsilon\,|\varepsilon|^N\, \overline{R}_{N-1}[f_1, u_0, \vec{u}, \varepsilon]. \end{aligned} \tag{122}$$

Hence, we deduce from (103) and (122) that

$$\begin{aligned} f[h] - f[u_0] + \varepsilon f_1[h] &= \big(f_1[u_0] \\ &\quad + \overline{\Phi}_1[N, f, u_0, \vec{u}]\big)\,\varepsilon \\ &\quad + \sum_{k=2}^{N}\big(\overline{\Phi}_k[N, f, u_0, \vec{u}] + \overline{\Phi}_{k-1}[N-1, f_1, u_0, \vec{u}]\big) \\ &\quad \cdot \varepsilon^k + |\varepsilon|^{N+1}\,\overline{R}_N[f, f_1, u_0, \vec{u}, \varepsilon], \end{aligned} \tag{123}$$

where $\overline{R}_N[f, f_1, u_0, \vec{u}, \varepsilon] = \overline{R}_N[f, u_0, \vec{u}, \varepsilon] + (\varepsilon/|\varepsilon|)\overline{R}_{N-1}[f_1, u_0, \vec{u}, \varepsilon]$ is bounded in function space $L^\infty(0, T; L^2)$ by a constant depending only on N, T, f, f_1, u_k, $0 \leq k \leq N$.

On the other hand, we put $\eta_1 = \mu_1[u_0] + \widehat{\Phi}_1[N, \mu, u_0, \vec{u}]$, $\eta_k = \widehat{\Phi}_k[N, \mu, u_0, \vec{u}] + \widehat{\Phi}_{k-1}[N-1, \mu_1, u_0, \vec{u}]$, $2 \leq k \leq N$, and we deduce from (111) that

$$\begin{aligned} &-(\mu[h] - \mu[u_0] + \varepsilon\mu_1[h])\,Ah = -Ah\Bigg[\big(\mu_1[u_0] \\ &\quad + \widehat{\Phi}_1[N, \mu, u_0, \vec{u}]\big)\,\varepsilon \\ &\quad + \sum_{k=2}^{N}\big(\widehat{\Phi}_k[N, \mu, u_0, \vec{u}] + \widehat{\Phi}_{k-1}[N-1, \mu_1, u_0, \vec{u}]\big) \\ &\quad \cdot \varepsilon^k\Bigg] - Ah\big(|\varepsilon|^{N+1}\,\widehat{R}_N[\mu, u_0, \vec{u}, \varepsilon] + \varepsilon\,|\varepsilon|^N \\ &\quad \cdot \widehat{R}_{N-1}[\mu_1, u_0, \vec{u}, \varepsilon]\big) = -Ah\Bigg[\big(\mu_1[u_0] \\ &\quad + \widehat{\Phi}_1[N, \mu, u_0, \vec{u}]\big)\,\varepsilon \\ &\quad + \sum_{k=2}^{N}\big(\widehat{\Phi}_k[N, \mu, u_0, \vec{u}] + \widehat{\Phi}_{k-1}[N-1, \mu_1, u_0, \vec{u}]\big) \\ &\quad \cdot \varepsilon^k\Bigg] + |\varepsilon|^{N+1}\,\widetilde{R}_N[\mu, \mu_1, u_0, \vec{u}, \varepsilon] \equiv -\left(\sum_{i=0}^{N} Au_i\varepsilon^i\right) \\ &\quad \cdot \left(\eta_1\varepsilon + \sum_{j=2}^{N}\eta_j\varepsilon^j\right) + |\varepsilon|^{N+1}\,\widetilde{R}_N[\mu, \mu_1, u_0, \vec{u}, \varepsilon] \\ &= -\left(\sum_{i=0}^{N} Au_i\varepsilon^i\right)\eta_1\varepsilon - \left(\sum_{i=0}^{N} Au_i\varepsilon^{i+1}\right)\left(\sum_{j=2}^{N}\eta_j\varepsilon^{j-1}\right) \\ &\quad + |\varepsilon|^{N+1}\,\widetilde{R}_N[\mu, \mu_1, u_0, \vec{u}, \varepsilon] = -\left(\sum_{i=1}^{N} Au_{i-1}\varepsilon^i\right)\eta_1 \\ &\quad - \eta_1 Au_N\varepsilon^{N+1} - \left(\sum_{i=1}^{N+1} Au_{i-1}\varepsilon^i\right)\left(\sum_{j=1}^{N-1}\eta_{j+1}\varepsilon^j\right) \\ &\quad + |\varepsilon|^{N+1}\,\widetilde{R}_N[\mu, \mu_1, u_0, \vec{u}, \varepsilon] = -\left(\sum_{k=1}^{N} Au_{k-1}\varepsilon^k\right)\eta_1 \\ &\quad - \sum_{i=1}^{N+1}\sum_{j=1}^{N-1}\eta_{j+1}Au_{i-1}\varepsilon^{i+j} - \eta_1 Au_N\varepsilon^{N+1} + |\varepsilon|^{N+1} \\ &\quad \cdot \widetilde{R}_N[\mu, \mu_1, u_0, \vec{u}, \varepsilon] = -\eta_1 Au_0\varepsilon - \left(\sum_{k=2}^{N} Au_{k-1}\varepsilon^k\right) \\ &\quad \cdot \eta_1 - \sum_{k=2}^{2N}\left(\sum_{\substack{i+j=k,\\ 1\leq i\leq N+1, 1\leq j\leq N-1}} \eta_{j+1}Au_{i-1}\right)\varepsilon^k \\ &\quad - \eta_1 Au_N\varepsilon^{N+1} + |\varepsilon|^{N+1}\,\widetilde{R}_N[\mu, \mu_1, u_0, \vec{u}, \varepsilon] \\ &= -\eta_1 Au_0\varepsilon - \left(\sum_{k=2}^{N}\eta_1 Au_{k-1}\varepsilon^k\right) \\ &\quad - \sum_{k=2}^{N}\left(\sum_{\substack{i+j=k,\\ 1\leq i\leq N+1, 1\leq j\leq N-1}} \eta_{j+1}Au_{i-1}\right)\varepsilon^k \\ &\quad - \sum_{k=N+1}^{2N}\left(\sum_{\substack{i+j=k,\\ 1\leq i\leq N+1, 1\leq j\leq N-1}} \eta_{j+1}Au_{i-1}\right)\varepsilon^k \\ &\quad - \eta_1 Au_N\varepsilon^{N+1} + |\varepsilon|^{N+1}\,\widetilde{R}_N[\mu, \mu_1, u_0, \vec{u}, \varepsilon] \\ &= -\eta_1 Au_0\varepsilon - \sum_{k=2}^{N}\Bigg(\eta_1 Au_{k-1} \\ &\quad + \sum_{\substack{i+j=k,\\ 1\leq i\leq N+1, 1\leq j\leq N-1}} \eta_{j+1}Au_{i-1}\Bigg)\varepsilon^k + |\varepsilon|^{N+1} \\ &\quad \cdot \widetilde{R}_N^{(1)}[\mu, \mu_1, u_0, \vec{u}, \varepsilon], \end{aligned} \tag{124}$$

where

$$\begin{aligned}
&|\varepsilon|^{N+1} \widetilde{R}_N [\mu, \mu_1, u_0, \vec{u}, \varepsilon] \\
&\quad = -Ah\left(|\varepsilon|^{N+1} \widehat{R}_N [\mu, u_0, \vec{u}, \varepsilon] \right.\\
&\quad \left. + \varepsilon |\varepsilon|^N \widehat{R}_{N-1} [\mu_1, u_0, \vec{u}, \varepsilon]\right), \\
&|\varepsilon|^{N+1} \widetilde{R}_N^{(1)} [\mu, \mu_1, u_0, \vec{u}, \varepsilon] \\
&\quad = -\sum_{k=N+1}^{2N} \left(\sum_{\substack{i+j=k,\\ 1\le i\le N+1, 1\le j\le N-1}} \eta_{j+1} A u_{i-1} \right) \varepsilon^k \\
&\quad - \eta_1 A u_N \varepsilon^{N+1} + |\varepsilon|^{N+1} \widetilde{R}_N [\mu, \mu_1, u_0, \vec{u}, \varepsilon].
\end{aligned} \tag{125}$$

Combining (99a)-(99b), (119), (123), and (125) leads to

$$\begin{aligned}
E_\varepsilon (x,t) &= f[h] - f[u_0] + \varepsilon f_1 [h] - (\mu [h] - \mu [u_0] \\
&+ \varepsilon \mu_1 [h]) Ah - \sum_{k=1}^{N} F_k \varepsilon^k = \left(f_1 [u_0]\right. \\
&\left.+ \overline{\Phi}_1 [N, f, u_0, \vec{u}]\right) \varepsilon + \sum_{k=2}^{N} \left(\overline{\Phi}_k [N, f, u_0, \vec{u}]\right. \\
&\left.+ \overline{\Phi}_{k-1} [N-1, f_1, u_0, \vec{u}]\right) \varepsilon^k + |\varepsilon|^{N+1} \overline{R}_N [f, f_1, u_0, \\
&\vec{u}, \varepsilon] - \eta_1 A u_0 \varepsilon - \sum_{k=2}^{N} \left(\eta_1 A u_{k-1} \right.\\
&\left. + \sum_{\substack{i+j=k,\\ 1\le i\le N+1, 1\le j\le N-1}} \eta_{j+1} A u_{i-1} \right) \varepsilon^k + |\varepsilon|^{N+1} \widetilde{R}_N^{(1)} [\mu, \\
&\mu_1, u_0, \vec{u}, \varepsilon] - \sum_{k=1}^{N} F_k \varepsilon^k = \left(f_1 [u_0] + \overline{\Phi}_1 [N, f, u_0, \vec{u}]\right. \\
&\left. - \eta_1 A u_0 - F_1\right) \varepsilon + \sum_{k=2}^{N} \left(\overline{\Phi}_k [N, f, u_0, \vec{u}] \right.\\
&+ \overline{\Phi}_{k-1} [N-1, f_1, u_0, \vec{u}] - \eta_1 A u_{k-1} \\
&\left. - \sum_{\substack{i+j=k,\\ 1\le i\le N+1, 1\le j\le N-1}} \eta_{j+1} A u_{i-1} - F_k \right) \varepsilon^k + |\varepsilon|^{N+1} \\
&\cdot \left(\overline{R}_N [f, f_1, u_0, \vec{u}, \varepsilon] + \widetilde{R}_N^{(1)} [\mu, \mu_1, u_0, \vec{u}, \varepsilon]\right) \\
&= |\varepsilon|^{N+1} \left(\overline{R}_N [f, f_1, u_0, \vec{u}, \varepsilon] \right.\\
&\left. + \widetilde{R}_N^{(1)} [\mu, \mu_1, u_0, \vec{u}, \varepsilon]\right).
\end{aligned} \tag{126}$$

By the boundedness of functions u_k, u_k', ∇u_k, $1 \le k \le N$ in function space $L^\infty(0, T; H^1)$, we obtain from (125), (123), and (126) that

$$\|E_\varepsilon\|_{L^\infty(0,T;L^2)} \le C_* |\varepsilon|^{N+1}, \tag{127}$$

where C_* is a constant depending only on $N, T, \rho, \zeta, f, f_1, \mu, \mu_1, u_k, 0 \le k \le N$.

The proof of Lemma 15 is complete. □

Proof of Theorem 11. Consider sequence $\{v_m\}$ defined by

$$\begin{aligned}
&v_0 \equiv 0, \\
&v_m'' + \mu_\varepsilon [v_{m-1} + h] A v_m \\
&\quad = F_\varepsilon [v_{m-1} + h] - F_\varepsilon [h] \\
&\quad - (\mu_\varepsilon [v_{m-1} + h] - \mu_\varepsilon [h]) Ah + E_\varepsilon (x,t), \\
&\qquad \rho < x < 1, \ 0 < t < T, \\
&v_m (\rho, t) = v_{mx} (1,t) + \zeta v_m (1,t) = 0, \\
&v_m (x, 0) = v_m' (x, 0) = 0, \quad m \ge 1.
\end{aligned} \tag{128}$$

By multiplying two sides of $(128)_1$ with v_m' and after integrating in t, we have

$$\begin{aligned}
&Z_m (t) \\
&\quad = \int_0^t \overline{\mu}_m' (s)\, a (v_m (s), v_m (s))\, ds \\
&\quad + 2 \int_0^t \left\langle E_\varepsilon (s), v_m' (s)\right\rangle ds \\
&\quad + 2 \int_0^t \left\langle F_\varepsilon [v_{m-1} + h] - F_\varepsilon [h], v_m' (s)\right\rangle ds \\
&\quad - 2 \int_0^t (\mu_\varepsilon [v_{m-1} + h] - \mu_\varepsilon [h]) \left\langle Ah, v_m' (s)\right\rangle ds,
\end{aligned} \tag{129}$$

where

$$\begin{aligned}
\overline{\mu}_m (t) &= \mu_\varepsilon [v_{m-1} + h] (t) \\
&= \mu \left(\|v_{m-1} (t) + h (t)\|_0^2\right) \\
&\quad + \varepsilon \mu_1 \left(\|v_{m-1} (t) + h (t)\|_0^2\right), \\
Z_m (t) &= \left\|v_m' (t)\right\|_0^2 + \overline{\mu}_m (t)\, a (v_m (t), v_m (t)).
\end{aligned} \tag{130}$$

Note that

$$\begin{aligned}
&\left|\overline{\mu}_m' (t)\right| = 2 \left|\left[\mu' \left(\|v_{m-1} (t) + h (t)\|_0^2\right)\right.\right. \\
&\quad \left. + \varepsilon \mu_1' \left(\|v_{m-1} (t) + h (t)\|_0^2\right)\right] \left\langle v_{m-1} (t) \right.\\
&\quad \left.\left. + h (t), v_{m-1}' (t) + h' (t)\right\rangle\right| \le 2 \left[\widetilde{K}_{M_2} (\mu)\right. \\
&\quad \left. + \widetilde{K}_{M_2} (\mu_1)\right] \|v_{m-1} (t) + h (t)\|_0 \left\|v_{m-1}' (t) + h' (t)\right\|_0 \\
&\quad \le 2 \left[\widetilde{K}_{M_2} (\mu) + \widetilde{K}_{M_2} (\mu_1)\right] M_2^2 \equiv \sigma_2 (M),
\end{aligned} \tag{131}$$

where $M_2 = (N+2)M$, and

$$Z_m(t) \geq \left\|v_m'(t)\right\|_0^2 + \mu_* a(v_m(t), v_m(t)) \geq \left\|v_m'(t)\right\|_0^2 + \mu_* C_0 \left\|v_m(t)\right\|_1^2. \tag{132}$$

Using Lemma 15, (129) gives

$$\begin{aligned} Z_m(t) &\leq TC_*^2 |\varepsilon|^{2N+2} + \int_0^t \left\|v_m'(s)\right\|_0^2 ds + \frac{1}{\mu_*}\sigma_m(M) \cdot \int_0^t Z_m(s)\, ds \\ &+ 2\int_0^t \left\|f[v_{m-1}+h] - f[h]\right\|_0 \left\|v_m'(s)\right\|_0 ds + 2|\varepsilon| \cdot \int_0^t \left\|f_1[v_{m-1}+h] - f_1[h]\right\|_0 \left\|v_m'(s)\right\|_0 ds \\ &+ 2\int_0^t \left|\mu_\varepsilon[v_{m-1}+h] - \mu_\varepsilon[h]\right| \left\|Ah\right\|_0 \left\|v_m'(s)\right\|_0 ds \\ &\leq TC_*^2 |\varepsilon|^{2N+2} + \left(1 + \frac{1}{\mu_*}\sigma_2(M)\right) \int_0^t Z_m(s)\, ds \\ &+ 2\int_0^t \left\|f[v_{m-1}+h] - f[h]\right\|_0 \left\|v_m'(s)\right\|_0 ds \\ &+ 2\int_0^t \left\|f_1[v_{m-1}+h] - f_1[h]\right\| \left\|v_m'(s)\right\|_0 ds \\ &+ 2(N+1) \cdot M \int_0^t \left|\mu[v_{m-1}+h] - \mu[h]\right| \left\|v_m'(s)\right\|_0 ds + 2|\varepsilon| \cdot (N+1) \\ &\cdot M \int_0^t \left|\mu_1[v_{m-1}+h] - \mu_1[h]\right| \left\|v_m'(s)\right\|_0 ds \\ &= TC_*^2 |\varepsilon|^{2N+2} + \left(1 + \frac{1}{\mu_*}\sigma_2(M)\right) \int_0^t Z_m(s)\, ds + \sum_{i=1}^{4} \bar{J}_i. \end{aligned} \tag{133}$$

We estimate the integrals on the right-hand side of (133) as follows.

Estimating $\bar{J}_1$. We note that

$$\begin{aligned} \left\|f[v_{m-1}+h] - f[h]\right\|_0 &\leq K_{M_2}(f)\left(1 + \frac{1-\rho}{\sqrt{2\rho}}\right) \cdot \left[\left\|\nabla v_{m-1}(t)\right\|_0 + \left\|v_{m-1}'(t)\right\|_0\right] \leq K_{M_2}(f) \\ &\cdot \left(1 + \frac{1-\rho}{\sqrt{2\rho}}\right) \left\|v_{m-1}\right\|_{W_1(T)}, \end{aligned} \tag{134}$$

with $M_2 = (N+2)M$.

It follows from (134) that

$$\begin{aligned} \bar{J}_1 &= 2\int_0^t \left\|f[v_{m-1}+h] - f[h]\right\|_0 \left\|v_m'(s)\right\|_0 ds \\ &\leq TK_{M_2}^2(f)\left(1 + \frac{1-\rho}{\sqrt{2\rho}}\right)^2 \left\|v_{m-1}\right\|_{W_1(T)}^2 + \int_0^t \left\|v_m'(s)\right\|_0^2 ds. \end{aligned} \tag{135}$$

Estimating $\bar{J}_2$. Similarly,

$$\begin{aligned} \bar{J}_2 &= 2\int_0^t \left\|f_1[v_{m-1}+h] - f_1[h]\right\|_0 \left\|v_m'(s)\right\|_0 ds \\ &\leq TK_{M_2}^2(f_1)\left(1 + \frac{1-\rho}{\sqrt{2\rho}}\right)^2 \left\|v_{m-1}\right\|_{W_1(T)}^2 + \int_0^t \left\|v_m'(s)\right\|_0^2 ds. \end{aligned} \tag{136}$$

Estimating $\bar{J}_3$. We have

$$\begin{aligned} &\left|\mu[v_{m-1}+h] - \mu[h]\right| \\ &\quad\leq \widetilde{K}_{M_2}(\mu)\left|\left\|v_{m-1}(t) + h(t)\right\|_0^2 - \left\|h(t)\right\|_0^2\right| \\ &\quad\leq (2N+3) M\widetilde{K}_{M_2}(\mu)\left\|v_{m-1}(t)\right\|_0 \\ &\quad\leq (2N+3) M\widetilde{K}_{M_2}(\mu)\frac{1-\rho}{\sqrt{2\rho}}\left\|v_{m-1}\right\|_{W_1(T)}. \end{aligned} \tag{137}$$

Hence,

$$\begin{aligned} \bar{J}_3 &= 2(N+1) \cdot M\int_0^t \left|\mu[v_{m-1}+h] - \mu[h]\right| \left\|v_m'(s)\right\|_0 ds \\ &\leq 2(N+1)(2N+3) M^2\widetilde{K}_{M_2}(\mu) \cdot \frac{1-\rho}{\sqrt{2\rho}}\left\|v_{m-1}\right\|_{W_1(T)} \int_0^t \left\|v_m'(s)\right\|_0 ds \leq (N+1)^2 \\ &\cdot (2N+3)^2 M^4 T\widetilde{K}_{M_2}^2(\mu)\frac{(1-\rho)^2}{\rho}\left\|v_{m-1}\right\|_{W_1(T)}^2 + \int_0^t \left\|v_m'(s)\right\|_0^2 ds. \end{aligned} \tag{138}$$

Estimating $\bar{J}_4$. Similarly,

$$\begin{aligned} \bar{J}_4 &= 2(N+1) \cdot M\int_0^t \left|\mu_1[v_{m-1}+h] - \mu_1[h]\right| \left\|v_m'(s)\right\|_0 ds \\ &\leq (N+1)^2(2N+3)^2 M^4 T\widetilde{K}_{M_2}(\mu_1) \cdot \frac{(1-\rho)^2}{\rho}\left\|v_{m-1}\right\|_{W_1(T)}^2 + \int_0^t \left\|v_m'(s)\right\|_0^2 ds. \end{aligned} \tag{139}$$

Combining (133), (135), (136), (138), (139), it leads to

$$Z_m(t) \leq T\rho_M \|v_{m-1}\|^2_{W_1(T)} + TC_*^2 |\varepsilon|^{2N+2} + \left(5 + \frac{1}{\mu_*}\sigma_2(M)\right) \int_0^t Z_m(s)\, ds, \tag{140}$$

where $\rho_M = [K^2_{M_2}(f)+K^2_{M_2}(f_1)](1+(1-\rho)/\sqrt{2\rho})^2+[\widetilde{K}^2_{M_2}(\mu)+\widetilde{K}^2_{M_2}(\mu_1)](N+1)^2(2N+3)^2M^4((1-\rho)^2/\rho)$.

By using Gronwall's lemma, we deduce from (140) that

$$\|v_m\|_{W_1(T)} \leq \sigma_T \|v_{m-1}\|_{W_1(T)} + \delta_T(\varepsilon), \quad \forall m \geq 1, \tag{141}$$

where $\sigma_T = \eta_T\sqrt{\rho_M}$, $\delta_T(\varepsilon) = \eta_T C_* |\varepsilon|^{N+1}$, $\eta_T = (1 + 1/\sqrt{\mu_* C_0})\sqrt{T\exp[T(5+(1/\mu_*)\sigma_2(M))]}$.

We can assume that

$$\sigma_T < 1, \quad \text{with the suitable constant } T > 0. \tag{142}$$

We require the following lemma, and its proof is immediate, so we omit the details.

Lemma 16. *Let sequence* $\{\zeta_m\}$ *satisfy*

$$\zeta_m \leq \sigma\zeta_{m-1} + \delta \quad \forall m \geq 1,\ \zeta_0 = 0, \tag{143}$$

where $0 \leq \sigma < 1$, $\delta \geq 0$ *are the given constants. Then,*

$$\zeta_m \leq \frac{\delta}{(1-\sigma)} \quad \forall m \geq 1. \tag{144}$$

Applying Lemma 16 with $\zeta_m = \|v_m\|_{W_1(T)}$, $\sigma = \sigma_T < 1$, $\delta = \delta_T(\varepsilon) = C_*\eta_T|\varepsilon|^{N+1}$, it follows from (144) that

$$\|v_m\|_{W_1(T)} \leq \frac{\delta_T(\varepsilon)}{1-\sigma_T} = C_T |\varepsilon|^{N+1}, \tag{145}$$

where $C_T = \eta_T C_*/(1-\eta_T\sqrt{\rho_M})$.

On the other hand, linear recurrent sequence $\{v_m\}$ defined by (128) converges strongly in space $W_1(T)$ to solution v of problem (118). Hence, letting $m \to +\infty$ in (145), we get

$$\|v\|_{W_1(T)} \leq C_T |\varepsilon|^{N+1}. \tag{146}$$

This implies (102). The proof of Theorem 11 is complete. □

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors contributed equally in this article. They read and approved the final manuscript.

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