Research Article

A Pseudospectral Algorithm for Solving Multipantograph Delay Systems on a Semi-Infinite Interval Using Legendre Rational Functions

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A new Legendre rational pseudospectral scheme is proposed and developed for solving numerically systems of linear and nonlinear multipantograph equations on a semi-infinite interval. A Legendre rational collocation method based on Legendre rational-Gauss quadrature points is utilized to reduce the solution of such systems to systems of linear and nonlinear algebraic equations. In addition, accurate approximations are achieved by selecting few Legendre rational-Gauss collocation points. The numerical results obtained by this method have been compared with various exact solutions in order to demonstrate the accuracy and efficiency of the proposed method. Indeed, for relatively limited nodes used, the absolute error in our numerical solutions is sufficiently small.

1. Introduction

Over the last three decades, the scientists have paid much attention to spectral methods due to their high accuracy (see, for instance, [1–6] and the references therein). On the other hand, spectral methods, in the context of numerical schemes for differential equations, generically belong to the family of weighted residual methods (WRMs) (cf. Finlayson [7]). WRMs represent a particular group of approximation techniques, in which the residuals (or errors) are minimized in a certain way and thereby leading to specific methods including Galerkin, Petrov-Galerkin, collocation, and tau formulations. WRMs are traditionally regarded as the foundation and cornerstone of the finite element, spectral, finite volume, boundary element, and some other methods. Many problems in science and engineering arise in unbounded domains (see, e.g., [8–12]). In general, the use of Jacobi rational functions has the advantage of obtaining the solutions in terms of the Jacobi rational parameters (see, e.g., [13–16]). Moreover, the authors of [17, 18] proposed an efficient collocation schemes based on the operational matrices of rational Legendre and Chebyshev functions for solving problems in the half line.

Systems of multipantograph equations model many real-life phenomena in physics and biology. Numerous numerical and analytical schemes have been presented and developed for solving such multipantograph equations, like variational iteration [19], Bernstein collocation [20], spline [21], homotopy perturbation [22], and Taylor collocation [23, 24] methods. This problem has been the focus of many studies; see, for instance, [21, 25–29].

The main aim of this paper is to develop the spectral-Gauss collocation integration process based on Legendre rational functions for solving systems of linear and nonlinear multipantograph equations on the half line. The implementation of this method reduces the problem to systems of linear and nonlinear algebraic equations. The numerical simulation of multipantograph systems on a semi-infinite domain is investigated. Finally, the accuracy and applicability of the proposed method are demonstrated by test problems. Numerical results are given in which the exponential convergence behaviour is exhibited.

This paper is organized as follows. We present few revelent properties of Legendre rational functions and function approximation in the coming section. In Sections 3 and 4, the pseudospectral algorithms are implemented for solving systems of linear and nonlinear multipantograph equations. Two test examples are introduced in Section 5. Finally, some concluding remarks are given in the last section.

2. Legendre Rational Interpolation

In this section, we present Legendre rational functions and Legendre rational approximation that will be used to construct the Legendre rational-Gauss collocation (LR-GC) method.

2.1. Legendre Rational Functions. The well-known Legendre polynomials $L_k(y)$ are defined on the interval $[-1, 1]$ with respect to the weight function $\rho(y) = 1$. In order to use these polynomials on the interval $t \in (0, \infty)$, we recall the Legendre rational functions by introducing the change of variable $y = (t - 1)/(t + 1)$. Let the Legendre rational functions $L_k((t - 1)/(t + 1))$ be denoted by $R_k(t)$. Then $R_k(t)$ can be obtained with the aid of the following recurrence formula:

$$R_0 = 1, \qquad R_1(t) = \frac{t - 1}{t + 1},$$

$$R_{k+1}(t) = \left(\frac{2k + 1}{k + 1}\right)\left(\frac{t - 1}{t + 1}\right)R_k(t) - \left(\frac{k}{k + 1}\right)R_{k-1}(t),$$

$$k \geq 1. \tag{1}$$

According to the properties of the standard Legendre polynomials, we have

$$R_k(0) = (-1)^k, \tag{2}$$

$$\partial_t R_k(t) = \frac{(k + 1)}{2t(t + 1)}\left[(t - 1)R_k(t) - (t + 1)R_{k+1}(t)\right]. \tag{3}$$

2.2. Function Approximation. Let $w(t) = 2/(t + 1)^2$ denote a nonnegative, integrable, real-valued function over the interval $\Lambda = [0, \infty)$. We define

$$L_w^2(\Lambda) = \{v : \Lambda \longrightarrow \mathbb{R} \mid v \text{ is measurable and } \|v\|_w < \infty\}, \tag{4}$$

where

$$\|v\|_w = \left(\int_0^\infty |v(t)|^2 w(t)dt\right)^{1/2} \tag{5}$$

is the norm induced by the inner product of the space $L_w^2(\Lambda)$:

$$\langle u, v\rangle_w = \int_0^\infty u(t)\, v(t)\, w(t)\, dt. \tag{6}$$

Thus, $\{R_k(t)\}_{k\geq 0}$ denotes a system which is mutually orthogonal under (6); that is,

$$\langle R_k, R_l\rangle_w = \frac{2}{2k + 1}\delta_{kl}, \tag{7}$$

where δ_{kl} is the Kronecker delta function. This system is complete in $L_w^2(\Lambda)$. For any function $u \in L_w^2(\Lambda)$ the following expansion holds:

$$u(t) = \sum_{k=0}^\infty a_k R_k(t), \tag{8}$$

with

$$a_k = \frac{\langle u, R_k\rangle_w}{\|R_k\|_w^2}. \tag{9}$$

2.3. Legendre Rational Interpolation Approximation. We denote by $t_{N,j}$, $0 \leq j \leq N$, the nodes of the standard Legendre-Gauss interpolation on the interval $[-1, 1]$, and $\omega_{N,j}$, $0 \leq j \leq N$ are Christoffel numbers. The nodes of the Legendre rational-Gauss interpolation on the interval $[0, \infty)$ are the zeros of $R_{N+1}(t)$, which is denoted by $t_{R,N,j}$, $0 \leq j \leq N$. It is clear that $t_{R,N,j} = (1 + t_{N,j})/(1 - t_{N,j})$, and $\omega_{R,N,j} = \omega_{N,j}$, $0 \leq j \leq N$. We now set $S_N(0, \infty) = \text{span}\{R_0(t), R_1(t), \ldots, R_N(t)\}$. Therefore, making use of the property of the Legendre-Gauss quadrature, we have for any $\phi \in S_{2N+1}[0, \infty)$,

$$\int_0^\infty w(t)\phi(t)\, dt = \int_{-1}^1 \rho(y)\phi\left(\frac{1 + y}{1 - y}\right)dy$$

$$= \sum_{j=0}^N \omega_{R,N,j}\phi\left(t_{R,N,j}\right). \tag{10}$$

The interpolating function of a smooth function u on a semi-infinite interval is denoted by $P_N u$. It is an element of $\mathfrak{R}_N$ and is defined as

$$P_N u(t) = \sum_0^N a_k R_k(t). \tag{11}$$

$P_N u$ is the orthogonal projection of u upon $\mathfrak{R}_N$ with respect to the inner product (6) and the norm (5). Thus by the orthogonality of Legendre rational functions we have [11]

$$\langle P_N u - u, R_i\rangle_w = 0, \quad \forall R_i \in \mathfrak{R}_N. \tag{12}$$

To obtain the order of convergence of Legendre rational approximation, at first we define the space

$$H^r_{w,A}(\Lambda) = \{v : v \text{ is measurable and } \|v\|_{r,w,A} < \infty\}, \quad (13)$$

where the norm is induced by

$$\|v\|_{r,w,A} = \left(\sum_{k=0}^{r}\left\|(t+1)^{r/2+k}\frac{d^k}{dt^k}v\right\|^2_w\right), \quad (14)$$

and A is the Sturm-Liouville operator as follows:

$$Av(t) = -w^{-1}(t)\frac{d}{dt}\left(t\frac{d}{dt}v(t)\right). \quad (15)$$

We have the following theorem for the convergence.

Theorem 1. *For any* $v \in H^r_{w,A}(\Lambda)$ *and* $r \geq 0$,

$$\|P_N v - v\| \leq cN^{-r}\|v\|_{r,w,A}. \quad (16)$$

A complete proof of the theorem and discussion on convergence are given in [11].

3. Linear Multipantograph System

In this section, we propose the Legendre rational-Gauss collocation method to solve the following system of linear multipantograph equations:

$$\sum_{s=1}^{m}\beta_{1,s}u'_s(t) = \sum_{s=1}^{m}a_{1,s}(t)u_s(t)$$
$$+ \sum_{j=1}^{l}\sum_{s=1}^{m}b^j_{1,s}(t)u_j(\lambda_j t) + g_1(t),$$

$$\sum_{s=1}^{m}\beta_{2,s}u'_s(t) = \sum_{s=1}^{m}a_{2,s}(t)u_s(t)$$
$$+ \sum_{j=1}^{l}\sum_{s=1}^{m}b^j_{2,s}(t)u_j(\lambda_j t) + g_2(t) \quad (17)$$

$$0 < t \leq \infty,$$

$$\vdots$$

$$\sum_{s=1}^{m}\beta_{m,s}u'_s(t) = \sum_{s=1}^{m}a_{m,s}(t)u_s(t)$$
$$+ \sum_{j=1}^{l}\sum_{s=1}^{m}b^j_{m,s}(t)u_j(\lambda_j t) + g_m(t),$$

subject to

$$\sum_{r=1}^{m}u_{r,1}(0) = u_1,$$

$$\sum_{r=1}^{m}u_{r,2}(0) = u_2,$$

$$\vdots \quad (18)$$

$$\sum_{r=1}^{m}u_{r,m}(0) = u_m.$$

Let us first introduce some basic notation that will be used in the sequel. We define the discrete inner product and norm as follows:

$$(u,v)_{w,N} = \sum_{j=0}^{N}u\left(t_{R,N,j}\right)v\left(t_{R,N,j}\right)\varpi_{R,N,j},$$
$$(19)$$
$$\|u\|_{w,N} = \sqrt{(u,u)_{w,N}}.$$

Obviously,

$$(u,v)_{w,N} = (u,v)_w, \quad \forall u,v \in S_{2N+1}. \quad (20)$$

The Legendre rational-Gauss collocation method for solving (17) and (18) is to seek $u_{j,N}(t) \in S_N(0,\infty)$, $j = 1,2,\ldots,k$, such that

$$\sum_{s=1}^{m}\beta_{1,s}u'_s\left(t_{q,R,N,\gamma}\right) = \sum_{s=1}^{m}a_{1,s}(t)u_s\left(t_{q,R,N,\gamma}\right)$$

$$+ \sum_{j=1}^{l}\sum_{s=1}^{m}b^j_{1,s}(t)u_j\left(\lambda_j t_{q,R,N,\gamma}\right)$$

$$+ g_1\left(t_{q,R,N,\gamma}\right),$$

$$\sum_{s=1}^{m}\beta_{2,s}u'_s\left(t_{q,R,N,\gamma}\right) = \sum_{s=1}^{m}a_{2,s}(t)u_s\left(t_{q,R,N,\gamma}\right)$$

$$+ \sum_{j=1}^{l}\sum_{s=1}^{m}b^j_{2,s}(t)u_j\left(\lambda_j t_{q,R,N,\gamma}\right)$$

$$+ g_2\left(t_{q,R,N,\gamma}\right),$$

$$\vdots$$

$$\sum_{s=1}^{m}\beta_{m,s}u'_s\left(t_{q,R,N,\gamma}\right) = \sum_{s=1}^{m}a_{m,s}(t)u_s\left(t_{q,R,N,\gamma}\right)$$

$$+ \sum_{j=1}^{l}\sum_{s=1}^{m}b^j_{m,s}(t)u_j\left(\lambda_j t_{q,R,N,\gamma}\right)$$

$$+ g_m\left(t_{q,R,N,\gamma}\right),$$

$$q = 1,2,\ldots,m, \quad \gamma = 0,1,\ldots,N-1,$$

$$\sum_{r=1}^{m} u_{r,1}(0) = u_1,$$

$$\sum_{r=1}^{m} u_{r,2}(0) = u_2,$$

$$\vdots$$

$$\sum_{r=1}^{m} u_{r,m}(0) = u_m.$$

$$(21)$$

We derive the algorithm for solving (17)-(18). To do this, let

$$u_{i,N}(t) = \sum_{h=0}^{N} c_{i,h} R_{i,h}(t). \qquad (22)$$

Now, substitution of (22) into (17) enables us to write

$$\sum_{h=0}^{N} \sum_{s=1}^{m} \beta_{1,s} c_{1,h} D R_{s,h}(t) = \sum_{h=0}^{N} \sum_{s=1}^{m} a_{1,s}(t) c_{1,h} R_{s,h}(t)$$

$$+ \sum_{h=0}^{N} \sum_{j=1}^{l} \sum_{s=1}^{m} b_{1,s}^{j}(t) c_{1,j} R_{s,j}(\lambda_j t)$$

$$+ g_1(t),$$

$$\sum_{h=0}^{N} \sum_{s=1}^{m} \beta_{2,s} c_{2,h} D R_{s,h}(t) = \sum_{h=0}^{N} \sum_{s=1}^{m} a_{2,s}(t) c_{2,h} R_{s,h}(t)$$

$$+ \sum_{h=0}^{N} \sum_{j=1}^{l} \sum_{s=1}^{m} b_{2,s}^{j}(t) c_{2,j} R_{s,j}(\lambda_j t)$$

$$+ g_2(t),$$

$$\vdots$$

$$\sum_{h=0}^{N} \sum_{s=1}^{m} \beta_{m,s} c_{m,h} D R_{s,h}(t) = \sum_{h=0}^{N} \sum_{s=1}^{m} a_{m,s}(t) c_{m,h} R_{s,h}(t)$$

$$+ \sum_{h=0}^{N} \sum_{j=1}^{l} \sum_{s=1}^{m} b_{m,s}^{j}(t) c_{m,j} R_{s,j}(\lambda_j t)$$

$$+ g_m(t).$$

$$(23)$$

Then, by virtue of (3), we deduce that

$$\sum_{h=0}^{N} \sum_{s=1}^{m} \beta_{1,s} c_{1,h} \frac{(h+1)}{2t(t+1)} \left[(t-1) R_{s,h}(t) - (t+1) R_{s,h+1}(t) \right]$$

$$= \sum_{h=0}^{N} \sum_{s=1}^{m} a_{1,s}(t) c_{1,h} R_{s,h}(t) + \sum_{h=0}^{N} \sum_{j=1}^{l} \sum_{s=1}^{m} b_{1,s}^{j}(t) c_{1,j} R_{s,j}(\lambda_j t)$$

$$+ g_1(t),$$

$$\sum_{h=0}^{N} \sum_{s=1}^{m} \beta_{2,s} c_{2,h} \frac{(h+1)}{2t(t+1)} \left[(t-1) R_{s,h}(t) - (t+1) R_{s,h+1}(t) \right]$$

$$= \sum_{h=0}^{N} \sum_{s=1}^{m} a_{2,s}(t) c_{2,h} R_{s,h}(t) + \sum_{h=0}^{N} \sum_{j=1}^{l} \sum_{s=1}^{m} b_{2,s}^{j}(t) c_{2,j} R_{s,j}(\lambda_j t)$$

$$+ g_2(t),$$

$$\vdots$$

$$\sum_{h=0}^{N} \sum_{s=1}^{m} \beta_{m,s} c_{m,h} \frac{(h+1)}{2t(t+1)} \left[(t-1) R_{s,h}(t) - (t+1) R_{s,h+1}(t) \right]$$

$$= \sum_{h=0}^{N} \sum_{s=1}^{m} a_{m,s}(t) c_{m,h} R_{s,h}(t) + \sum_{h=0}^{N} \sum_{j=1}^{l} \sum_{s=1}^{m} b_{m,s}^{j}(t) c_{m,j} R_{s,j}(\lambda_j t)$$

$$+ g_m(t).$$

$$(24)$$

Moreover, the initial condition (18)-with the aid of (2)-yields

$$\sum_{h=0}^{N} \sum_{r=1}^{m} (-1)^h a_{1,h} = u_1,$$

$$\sum_{h=0}^{N} \sum_{r=1}^{m} (-1)^h a_{2,h} = u_2,$$

$$\vdots$$

$$\sum_{h=0}^{N} \sum_{r=1}^{m} (-1)^h a_{m,h} = u_m.$$

$$(25)$$

If we collocate (24) at the (N) Legendre rational roots of $R_{N+1}(t)$, then we get

$$\sum_{h=0}^{N} \sum_{s=1}^{m} \beta_{1,s} c_{1,h} \frac{(h+1)}{2t_{q,R,N,\gamma}(t_{q,R,N,\gamma}+1)}$$

$$\times \left[(t_{q,R,N,\gamma} - 1) R_{s,h}(t_{q,R,N,\gamma}) - (t_{q,R,N,\gamma} + 1) R_{s,h+1}(t_{q,R,N,\gamma}) \right]$$

$$= \sum_{h=0}^{N} \sum_{s=1}^{m} a_{1,s}(t_{q,R,N,\gamma}) c_{1,h} R_{s,h}(t_{q,R,N,\gamma})$$

$$+ \sum_{h=0}^{N} \sum_{j=1}^{l} \sum_{s=1}^{m} b_{1,s}^{j}(t_{q,R,N,\gamma}) c_{1,j} R_{s,j}(\lambda_j t_{q,R,N,\gamma})$$

$$+ g_1(t_{q,R,N,\gamma}),$$

$$\sum_{h=0}^{N}\sum_{s=1}^{m}\beta_{2,s}c_{2,h}\frac{(h+1)}{2t_{q,R,N,\gamma}\left(t_{q,R,N,\gamma}+1\right)}$$

$$\times\left[\left(t_{q,R,N,\gamma}-1\right)R_{s,h}\left(t_{q,R,N,\gamma}\right)\right.$$

$$\left.-\left(t_{q,R,N,\gamma}+1\right)R_{s,h+1}\left(t_{q,R,N,\gamma}\right)\right]$$

$$=\sum_{h=0}^{N}\sum_{s=1}^{m}a_{2,s}\left(t_{q,R,N,\gamma}\right)c_{2,h}R_{s,h}\left(t_{q,R,N,\gamma}\right)$$

$$+\sum_{h=0}^{N}\sum_{j=1}^{l}\sum_{s=1}^{m}b_{2,s}^{j}\left(t_{q,R,N,\gamma}\right)c_{2,j}R_{s,j}\left(\lambda_{j}t_{q,R,N,\gamma}\right)$$

$$+g_{2}\left(t_{q,R,N,\gamma}\right),$$

$$\vdots$$

$$\sum_{h=0}^{N}\sum_{s=1}^{m}\beta_{m,s}c_{m,h}\frac{(h+1)}{2t_{q,R,N,\gamma}\left(t_{q,R,N,\gamma}+1\right)}$$

$$\times\left[\left(t_{q,R,N,\gamma}-1\right)R_{s,h}\left(t_{q,R,N,\gamma}\right)\right.$$

$$\left.-\left(t_{q,R,N,\gamma}+1\right)R_{s,h+1}\left(t_{q,R,N,\gamma}\right)\right]$$

$$=\sum_{h=0}^{N}\sum_{s=1}^{m}a_{m,s}\left(t_{q,R,N,\gamma}\right)c_{m,h}R_{s,h}\left(t_{q,R,N,\gamma}\right)$$

$$+\sum_{h=0}^{N}\sum_{j=1}^{l}\sum_{s=1}^{m}b_{m,s}^{j}\left(t_{q,R,N,\gamma}\right)c_{m,j}R_{s,j}\left(\lambda_{j}t_{q,R,N,\gamma}\right)$$

$$+g_{m}\left(t_{q,R,N,\gamma}\right).$$

$$(26)$$

Thus (26) with relation (25) generate $(N+1)$ of a set of algebraic equations which can be solved for the unknown coefficients $c_{i,j}$, $(i=1,\ldots,m;\ j=0,1,2,\ldots,N)$, by using any standard solver technique.

4. Nonlinear Multipantograph System

In this section, we consider the nonlinear multipantograph system of the form

$$\sum_{s=1}^{m}\xi_{1,s}u_{s}'(t)=F\left(t,\sum_{s=1}^{m}\zeta_{1,s}u_{s}(t),\sum_{s=1}^{m}\eta_{1,s}u_{s}(\lambda t)\right),$$

$$\sum_{s=1}^{m}\xi_{2,s}u_{s}'(t)=F\left(t,\sum_{s=1}^{m}\zeta_{2,s}u_{s}(t),\sum_{s=1}^{m}\eta_{2,s}u_{s}(\lambda t)\right),$$

$$t\in(0,\infty)\ \text{ where }\lambda\in(0,1),$$

$$\vdots$$

$$\sum_{s=1}^{m}\xi_{m,s}u_{s}'(t)=F\left(t,\sum_{s=1}^{m}\zeta_{m,s}u_{s}(t),\sum_{s=1}^{m}\eta_{m,s}u_{s}(\lambda t)\right),$$

$$(27)$$

with initial conditions

$$\sum_{r=1}^{m}u_{r,1}(0)=u_{1},$$

$$\sum_{r=1}^{m}u_{r,2}(0)=u_{2},$$

$$\vdots$$

$$\sum_{r=1}^{m}u_{r,m}(0)=u_{m}.$$

$$(28)$$

The Legendre rational-Gauss collocation method for solving (27) and (28) is to seek $u_{i,N}(x)\in S_{N}(0,\infty)$, such that

$$\sum_{s=1}^{m}\xi_{1,s}u_{s,N}'(t)=F\left(t,\sum_{s=1}^{m}\zeta_{1,s}u_{s,N}(t),\sum_{s=1}^{m}\eta_{1,s}u_{s,N}(\lambda t)\right),$$

$$\sum_{s=1}^{m}\xi_{2,s}u_{s,N}'(t)=F\left(t,\sum_{s=1}^{m}\zeta_{2,s}u_{s,N}(t),\sum_{s=1}^{m}\eta_{2,s}u_{s,N}(\lambda t)\right),$$

$$t\in(0,\infty)\ \text{ where }\lambda\in(0,1),\quad(29)$$

$$\vdots$$

$$\sum_{s=1}^{m}\xi_{m,s}u_{s,N}'(t)=F\left(t,\sum_{s=1}^{m}\zeta_{m,s}u_{s,N}(t),\sum_{s=1}^{m}\eta_{m,s}u_{s,N}(\lambda t)\right)$$

is satisfied exactly at the collocation points $t_{R,N,k}$, $k=0,1,\ldots,N-1$. In other words, we have to collocate (29) at the (N) Legendre rational roots $t_{R,N,k}$, which immediately yields

$$\sum_{h=0}^{N}\sum_{s=1}^{m}\xi_{m,s}c_{1,j}DR_{s,h}\left(t_{q,R,N,k}\right)$$

$$=F\left(t_{q,R,N,k},\sum_{h=0}^{N}\sum_{s=1}^{m}\zeta_{1,s}c_{1,h}R_{s,h}\left(t_{q,R,N,k}\right),\right.$$

$$\left.\sum_{h=0}^{N}\sum_{s=1}^{m}\eta_{1,s}c_{1,h}R_{s,h}\left(\lambda t_{q,R,N,k}\right)\right),$$

$$\sum_{h=0}^{N}\sum_{s=1}^{m}\xi_{m,s}c_{2,j}DR_{s,h}\left(t_{q,R,N,k}\right)$$

$$=F\left(t_{q,R,N,k},\sum_{h=0}^{N}\sum_{s=1}^{m}\zeta_{2,s}c_{2,h}R_{s,h}\left(t_{q,R,N,k}\right),\right.$$

$$\left.\sum_{h=0}^{N}\sum_{s=1}^{m}\eta_{2,s}c_{2,h}R_{s,h}\left(\lambda t_{q,R,N,k}\right)\right),$$

$$\vdots$$

$$\sum_{h=0}^{N}\sum_{s=1}^{m}\xi_{m,s}c_{m,j}DR_{s,h}\left(t_{q,R,N,k}\right)$$

$$= F\left(t_{q,R,N,k}, \sum_{h=0}^{N}\sum_{s=1}^{m}\zeta_{m,s}c_{m,h}R_{s,h}\left(t_{q,R,N,k}\right),\right.$$

$$\left.\sum_{h=0}^{N}\sum_{s=1}^{m}\eta_{m,s}c_{m,h}R_{s,h}\left(\lambda t_{q,R,N,k}^{(\theta,\vartheta)}\right)\right),$$

$$(30)$$

with (28) written in the form

$$\sum_{h=0}^{N}\sum_{r=1}^{m}c_{1,j}R_{r,h}\left(0\right) = u_{1,0},$$

$$\sum_{h=0}^{N}\sum_{r=1}^{m}c_{2,j}R_{r,h}\left(0\right) = u_{2,0},$$

$$(31)$$

$$\vdots$$

$$\sum_{h=0}^{N}\sum_{r=1}^{m}c_{m,j}R_{r,h}\left(0\right) = u_{n,0}.$$

This constitute a system of $(N + 1)$ nonlinear algebraic equations in the unknown expansion coefficients $c_{i,j}$, $(i = 1, \ldots, m; \ j = 0, 1, 2, \ldots, N)$, which can be solved by using any standard iteration technique, like Newton's iteration method.

5. Numerical Results

In this section, we present two numerical examples in order to show the accuracy of Legendre rational collocation method for solving multipantograph delay system. The following tables and figures contain the values of the exact solutions $u_i(x)$, $i = 1, 2, \ldots, k$, the approximate solutions $u_{i,N}(x)$, $i = 1, 2, \ldots, k$, and the absolute error functions $e_{i,N}(x) = |u_i(x) - u_{i,N}(x)|$, $i = 1, 2, \ldots, k$, at the selected points of the given interval.

Example 1. Consider the following linear multipantograph delay system:

$$u_1'\left(t\right) + 5u_2'\left(t\right) + tu_1\left(t\right) - 2t^2u_2\left(t\right)$$

$$+ u_1\left(0.5t\right) + tu_2\left(0.8t\right) = g_1\left(t\right),$$

$$0 \leq t \leq 100, \qquad (32)$$

$$- 4u_1'\left(t\right) + u_2'\left(t\right) - u_1\left(t\right) + tu_2\left(t\right)$$

$$+ tu_1\left(0.2t\right) + u_2\left(0.4t\right) = g_2\left(t\right),$$

with the initial conditions

$$u_1\left(0\right) = 0, \qquad u_2'\left(0\right) = 1, \qquad (33)$$
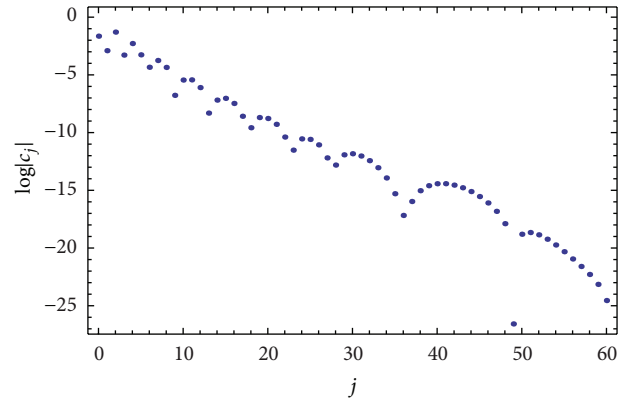

FIGURE 1: Logarithmic graph of absolute coefficients $|c_j|$ of Legendre rational functions at $N = 60$ of Example 1.

TABLE 1: Comparison of the absolute errors $e_{1,N}$ for $u_1(t)$ of (32).

t_i	LR-GC method		
	$N = 20$	$N = 40$	$N = 60$
0.0	$0.000 \times 10^{+00}$	$0.000 \times 10^{+00}$	2.775×10^{-17}
10.0	1.105×10^{-4}	5.638×10^{-8}	3.779×10^{-10}
20.0	2.175×10^{-5}	4.347×10^{-7}	2.609×10^{-9}
30.0	3.087×10^{-3}	6.613×10^{-7}	8.111×10^{-9}
40.0	1.080×10^{-2}	6.894×10^{-7}	1.604×10^{-8}
50.0	2.036×10^{-2}	1.644×10^{-6}	4.074×10^{-8}
60.0	3.008×10^{-2}	6.281×10^{-6}	9.717×10^{-8}
70.0	3.922×10^{-2}	1.594×10^{-5}	1.190×10^{-7}
80.0	4.756×10^{-2}	3.053×10^{-5}	8.903×10^{-8}
90.0	5.506×10^{-2}	4.921×10^{-5}	4.396×10^{-8}
100.0	6.178×10^{-2}	7.100×10^{-5}	3.624×10^{-8}

where

$$g_1\left(t\right) = 2t - 4t^2 + \left(3t^2 - t + 6\right)e^{-t} + 0.5te^{-0.5t} - te^{-0.8t},$$

$$g_2\left(t\right) = 2 + 2t + \left(2t - 3\right)e^{-t} - e^{-0.4t} + 0.2t^2e^{-0.2t}.$$

$$(34)$$

The exact solution of the system is $u_1(t) = te^{-t}$ and $u_2(t) = 2 - e^{-t}$.

Tables 1 and 2 list the results obtained by the Legendre rational-Gauss collocation method in terms of maximum absolute errors with different values of N. The logarithmic graphs of absolute coefficients for Legendre rational functions are shown in Figures 1 and 2. This confirms that the proposed method has reasonable convergence rate.

TABLE 2: Comparison of the absolute errors $e_{2,N}$ for $u_2(t)$ of (32).

t_i	LR-GC method		
	$N = 20$	$N = 40$	$N = 60$
0.0	8.273×10^{-7}	9.056×10^{-12}	4.393×10^{-11}
10.0	1.395×10^{-5}	6.763×10^{-9}	4.703×10^{-12}
20.0	9.412×10^{-6}	8.911×10^{-9}	5.661×10^{-11}
30.0	6.376×10^{-5}	1.177×10^{-8}	1.242×10^{-10}
40.0	1.499×10^{-4}	7.887×10^{-9}	2.575×10^{-10}
50.0	2.121×10^{-4}	1.621×10^{-8}	3.371×10^{-10}
60.0	2.529×10^{-4}	4.544×10^{-8}	6.993×10^{-10}
70.0	2.791×10^{-4}	1.027×10^{-7}	9.146×10^{-10}
80.0	2.957×10^{-4}	1.820×10^{-7}	7.991×10^{-10}
90.0	3.061×10^{-4}	2.729×10^{-7}	5.234×10^{-10}
100.0	3.125×10^{-4}	3.658×10^{-7}	3.478×10^{-10}

Example 2. Consider the following nonlinear multipantograph delay system:

$$u_1'(t) + 2u_2'(t) + u_1(t) + tu_2(t) + tu_2^2(0.5t) = g_1(t),$$

$$0 \leq t \leq 1, \quad (35)$$

$$3u_1'(t) - u_2'(t) + tu_1^2(0.2t) - (t+1)u_2(t) = g_2(t),$$

with the initial conditions

$$u_1(0) = 0, \qquad u_2'(0) = 1, \quad (36)$$

where

$$g_1(t) = -2e^{-t}\sin(t) - e^{-t}\cos(t)$$
$$+ e^{-t}t\cos(t) + e^{-t}t\cos^2(0.5t),$$
$$g_2(t) = 3e^{-t}\cos(t) - 2e^{-t}\sin(t) \quad (37)$$
$$- te^{-t}\sin(t) + e^{-0.4t}t\sin^2(0.2t).$$

The exact solution of the system is $u_1(x) = e^{-t}\sin(t)$ and $u_2(t) = e^{-t}\cos(t)$.

In Tables 3 and 4, we list the absolute errors obtained by the Legedre rational-Gauss collocation method, with different values of N. Figures 3 and 4 are plotted to compare the analytic solution with the approximate solution at $N = 28$.

6. Conclusion

In this paper, a collocation Legendre rational method has been proposed to obtain the approximate solutions of systems of multipantograph delay equations. The derivation of this method is essentially based on Legendre rational functions and Gauss quadrature formula. The main advantage of the developed method is that high accurate solutions were achieved using few numbers of the Legendre rational functions. Additionally, if N is increased, it can be seen that approximate solutions obtained by the method are close to the exact solutions.
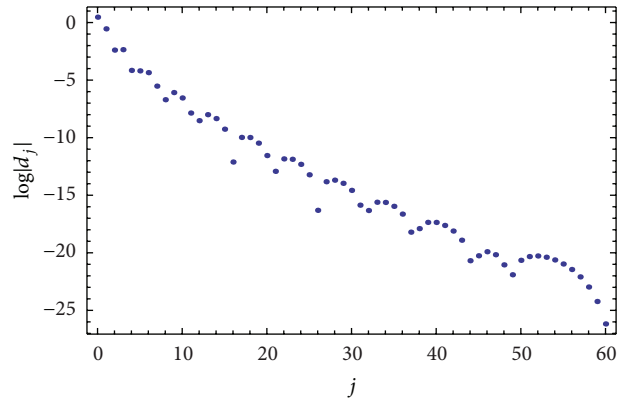

FIGURE 2: Logarithmic graph of absolute coefficients $|d_j|$ of Legendre rational functions at $N = 60$ of Example 1.
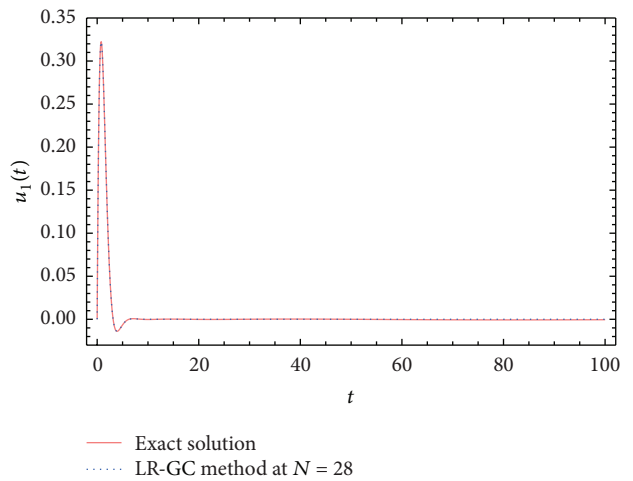

Exact solution
LR-GC method at $N = 28$

FIGURE 3: Graph of exact solution and approximate solution $u_1(t)$ at $N = 28$ for Example 2.

Exact solution
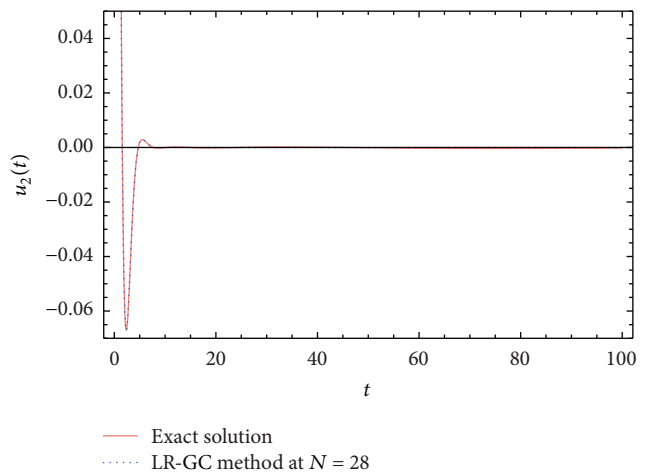LR-GC method at $N = 28$

FIGURE 4: Graph of exact solution and approximate solution $u_2(t)$ at $N = 28$ for Example 2.

TABLE 3: Comparison of the absolute errors $e_{1,N}$ for $u_1(t)$ of (35).

t_i	LR-GC method		
	$N = 10$	$N = 30$	$N = 50$
0.0	5.551×10^{-17}	$0.000 \times 10^{+00}$	$0.000 \times 10^{+00}$
1.0	1.554×10^{-3}	4.799×10^{-6}	8.262×10^{-8}
2.0	2.085×10^{-3}	2.349×10^{-6}	4.957×10^{-8}
3.0	6.583×10^{-3}	1.392×10^{-6}	1.069×10^{-7}
4.0	1.408×10^{-3}	9.788×10^{-9}	1.247×10^{-7}
5.0	5.932×10^{-3}	2.183×10^{-5}	8.508×10^{-7}
6.0	6.690×10^{-3}	6.643×10^{-5}	1.169×10^{-6}
7.0	3.789×10^{-3}	6.919×10^{-6}	1.485×10^{-6}
8.0	1.092×10^{-3}	6.729×10^{-5}	6.698×10^{-7}
9.0	1.546×10^{-3}	2.765×10^{-5}	1.382×10^{-6}
10.0	2.137×10^{-4}	2.345×10^{-5}	1.450×10^{-6}

TABLE 4: Comparison of the absolute errors $e_{2,N}$ for $u_2(t)$ of (35).

t_i	LR-GC method		
	$N = 10$	$N = 30$	$N = 50$
0.0	$0.000 \times 10^{+00}$	1.110×10^{-16}	1.665×10^{-16}
1.0	6.260×10^{-4}	2.819×10^{-6}	5.391×10^{-8}
2.0	6.997×10^{-5}	2.048×10^{-5}	4.171×10^{-8}
3.0	5.800×10^{-5}	1.042×10^{-6}	9.088×10^{-8}
4.0	6.901×10^{-4}	7.814×10^{-6}	1.541×10^{-7}
5.0	2.427×10^{-3}	9.524×10^{-6}	2.014×10^{-7}
6.0	3.012×10^{-3}	1.126×10^{-6}	1.116×10^{-7}
7.0	1.483×10^{-3}	1.283×10^{-5}	2.553×10^{-7}
8.0	8.598×10^{-3}	4.171×10^{-6}	5.499×10^{-7}
9.0	1.529×10^{-2}	6.713×10^{-6}	2.949×10^{-7}
10.0	1.995×10^{-2}	3.503×10^{-5}	3.311×10^{-7}

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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