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Research Article

On the Expected Discounted Penalty Function for the Classical Risk Model with Potentially Delayed Claims and Random Incomes

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We focus on the expected discounted penalty function of a compound Poisson risk model with random incomes and potentially delayed claims. It is assumed that each main claim will produce a byclaim with a certain probability and the occurrence of the byclaim may be delayed depending on associated main claim amount. In addition, the premium number process is assumed as a Poisson process. We derive the integral equation satisfied by the expected discounted penalty function. Given that the premium size is exponentially distributed, the explicit expression for the Laplace transform of the expected discounted penalty function is derived. Finally, for the exponential claim sizes, we present the explicit formula for the expected discounted penalty function.

1. Introduction

In the classical risk theory, assumption of independence among claims is an important condition to the study of risk models. However, in many practical situations, the assumption is often inconsistent with the operation of insurance companies. In reality, claims may be time-correlated for various reasons, and it is important to study risk model which is able to depict this phenomenon. Since the work by Waters and Papatriandafylou [1], many researchers have studied various kinds of dependencies among claim amounts and claim numbers, such as Gerber [2], Shiu [3], Dickson [4], Willmot [5], and Ambagaspitiya [6, 7]. Among others, in the case of the compound binomial model, Yuen and Guo [8] consider a specific dependence structure between the claim sizes and interclaim times. Under their assumption, each claim causes a byclaim but the occurrence of the byclaim may be delayed. Further, based on the same model, Xiao and Guo [9] investigate the joint distribution of the surplus immediately prior to ruin and the deficit at ruin.

Note that the risk model referred above is based on the assumption that the probability of delay of each byclaim is constant and independent of claim amounts. Albrecher and Boxma [10] consider a generalization of the classical risk model to a dependent setting where the distribution of the time between two claim occurrences depends on the previous claim size. Motivated by the idea, Zou and Xie [11] introduce a risk model with an interesting dependence structure between the amount of main claim and the occurrence of byclaim. It is a natural extension for the delayed claims model due to the fact that the bigger the claim amount for main claim (such as car damage) is, the greater odds of the byclaim (such as injury) would be delayed in the actual practice with insurer. Based on the structure, we consider an improved payment mode named potentially delayed claims where the main claim induces a byclaim with a certain probability. The improvement is inspired from a series of examples similar to the case referred above, in which main claim does not induce byclaim with probability 1.

Because the insurance company may have lump sums of income, we apply potentially delayed claims to the compound Poisson risk model in the presence of random incomes. Since Boucherie et al. [12] described the random incomes by adding a compound Poisson process with positive jumps to the classical risk model, many authors have studied similar topics. Boikov [13] studies ruin problem of a risk model with stochastic premium process. Bao [14] considers a risk model, in which the premium is a Poisson process instead of a linear function of time. Labbé et al. [15] consider a risk model where the stochastic incomes follow a compound Poisson process and research the case when the premiums have Erlang distributions in more depth. Hao and Yang [16] analyze the expected discounted penalty function of a compound Poisson risk model with random incomes and delayed claims. Yu [17] also studies the expected discounted penalty function in a Markov regime-switching risk model with random income.

In this paper, we aim at the expected discounted penalty function of an extensive risk model with random incomes and potentially delayed claims. This paper generalizes the model of Hao and Yang [16]. Based on the extensive model, we obtain explicit expression of the expected discounted penalty function, while [16] derives defective renewal equations of it only. When the main claim induces a byclaim with probability 1 and the byclaim is delayed with a constant probability, the results in this paper will reduce to them in [16]. So [16] can be seen as a special case of this paper. In addition, Zou and Xie [11] derive the probability of ruin in the risk model with delayed claims, but this paper obtains the expected discounted penalty function which contains the probability of ruin. If we define the expected discounted penalty function with the same expression as ruin probability and assume that the premium is a linear function of time, we can get the same results as [11].

The rest of this paper is structured as follows. In Section 2, we introduce the compound Poisson risk model with random incomes and potentially delayed claims. In Section 3, we derive an integral equation for the expected discounted penalty function and obtain explicit expression of its Laplace transform when the premium income is exponentially distributed. The defective renewal equation satisfied by the expected discounted penalty function is studied in Section 4. Section 5 obtains explicit result for the expected discounted penalty function with positive initial surplus when the claim amounts from both classes are exponentially distributed. Section 6 concludes the paper.

2. Model

Now, we can show the extensive risk model with random incomes and potentially delayed claims in mathematics. On the one hand, we denote the aggregate premium incomes at time t by $S_X(t)$ which is a compound Poisson process, and $\{N_1(t) : t \geq 0\}$ is the corresponding Poisson income number process with parameter λ_1. The premium incomes amounts $\{X_i\}_{i\geq1}$ are assumed to be independent and identically distributed (i.i.d.) positive random variables with common distribution F_X, probability function f_X, and mean μ_X. So we get $S_X(t) = \sum_{j=1}^{N_1(t)} X_j$. On the other hand, we consider a continuous time model which involves two types of insurance claims, namely, the main claims and the byclaims. Let the aggregate main claims process be a compound Poisson process and let $\{N_2(t) : t \geq 0\}$ be the corresponding Poisson claim number process with parameter λ_2. Its jump times are denoted by $\{V_i\}_{i\geq1}$ with $V_0 = 0$. The main claim amounts $\{Y_i\}_{i\geq1}$ and the byclaim amounts $\{Z_i\}_{i\geq1}$ are assumed to be independent and identically distributed (i.i.d.) positive random variables with common distribution F_Y and F_Z, respectively. Moreover, they are independent and their means are denoted by μ_Y and μ_Z. Then the surplus process of the risk model is defined as

$$U(t) = u + \sum_{j=1}^{N_1(t)} X_j - \sum_{i=1}^{N_2(t)} Y_i - R(t), \tag{1}$$

where $U(0) = u$ is the initial capital and $R(t)$ is the sum of all byclaims Z_i that occurred before time t. We assume that $N_1(t)$ and $N_2(t)$ are mutually independent.

With the assumption of potentially delayed claims, the claim occurrence process is to be of the following type: there will be a main claim Y_i at every epoch V_i of the Poisson process and the main claim Y_i will induce a byclaim Z_i with probability q. If the main claim amount Y_i induces a byclaim Z_i and the main claim amount Y_i is less than a threshold M_i, the byclaim Z_i and its associated main claim Y_i occur simultaneously; otherwise, the occurrence of the byclaim Z_i is delayed to V_{i+1} and main claim Y_{i+1} occurs simultaneously. From Zou and Xie [18], we know that

$$\begin{aligned} E&\left[\sum_{j=1}^{N_1(t)} X_j - \sum_{i=1}^{N_2(t)} Y_i - R(t)\right] \\ &= \lambda_1 t\mu_X - \left[\lambda_2 t\mu_Y + \lambda_2 t q\mu_Z \right. \\ &\qquad \left. - qP(Y_1 \geq M_1)\mu_Z\left(1 - e^{-\lambda_2 t}\right)\right]. \end{aligned} \tag{2}$$

Therefore, we further assume that

$$\lambda_1\mu_X > \lambda_2(\mu_Y + q\mu_Z). \tag{3}$$

This assumption ensures that the safety loading is positive.

Let $T \triangleq \inf\{t \geq 0 : U(t) < 0\}$ be the time of ruin with $T = \infty$ if $U(t) \geq 0$ for all $t \geq 0$. The ruin probability is defined by $\Psi(u) \triangleq P(T < \infty \mid U(0) = u)$, $u \geq 0$. The expected discounted penalty function is of the following form:

$$\phi(u) \triangleq E\left[e^{-\delta T}\omega(U(T-), |U(T)|) I(T < \infty) \mid U(0) = u\right], \quad u \geq 0, \tag{4}$$

where $\delta \geq 0$ is a constant and $\omega(x_1, x_2)$ is a nonnegative measurable function defined on $[0,\infty) \times [0,\infty)$. $I(A)$ is the indicator function of event A. $|U(T)|$ is the deficit at ruin and $U(T-)$ is the surplus immediately prior to ruin.

3. The Expected Discounted Penalty Function of the Exponential Premium Income

To handle the surplus process (1), we consider a slight change in the risk model. Instead of having one main claim

Y_1 and a byclaim Z_1 with probability $P(Y_1 < M_1)$ at the first epoch V_1, another byclaim Z is added at the first epoch V_1; that is, byclaim Z and main claim Y_1 occur at V_1 simultaneously. Hence, the corresponding surplus process $U_1(t)$ of this auxiliary risk model is defined as

$$U_1(t) = u + \sum_{j=1}^{N_1(t)} X_j - \sum_{i=1}^{N_2(t)} Y_i - R(t) - Z, \tag{5}$$

where Z denotes the other byclaim amount, $U_1(0) = u$. Assume that Z and $\{Z_i\}_{i\geq 1}$ are i.i.d. positive random variables.

The expected discounted penalty function for this auxiliary risk model is denoted by $\phi_1(u)$ which is useful to derive $\phi(u)$.

Obviously there will be a main claim Y_1 at the first epoch V_1. Let W_1 be the time for the first premium. The first claim can be or cannot be earlier than the first premium. If it is, there are three situations.

(1) The main claim does not induce a byclaim; then the surplus process gets renewed except for the initial value. The probability of this event is $1 - q$.

(2) The main claim induces a byclaim Z_1 and the main claim size $Y_1 < M_1$; then the byclaim Z_1 also occurs at the first epoch V_1; the surplus process $U(t)$ will renew itself with different initial reserve. The probability of this event is $qP(Y_1 < M_1)$.

(3) The main claim induces a byclaim Z_1 and the main claim size $Y_1 \geq M_1$; then the occurrence of the byclaim Z_1 will be delayed to V_2; that is, the delayed byclaim Z_1 and the main claim Y_2 occur simultaneously. In this case, $U(t)$ will not renew itself but transfer to the auxiliary model described in the paragraph above. The probability of this event is $qP(Y_1 \geq M_1)$.

Conditioning on the time of the first event, we have

$$\begin{aligned}
\phi(u) &= \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2+\delta)t} dt \int_0^\infty \phi(u+x)\, dF_X(x) \\
&\quad + \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\delta)t} dt\, (1-q) \\
&\quad \times \left[\int_0^u \phi(u-y)\, dF_Y(y) + \int_u^\infty \omega(u, y-u)\, dF_Y(y) \right] \\
&\quad + \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\delta)t} dt q \\
&\quad \times \left[\iint_{0<y+z<u} (1 - F_M(y))\, \phi(u-y-z)\, dF_Y(y) \right. \\
&\qquad \times dF_Z(z) \\
&\qquad + \iint_{y+z>u} (1 - F_M(y)) \\
&\qquad \left. \times \omega(u, y+z-u)\, dF_Y(y)\, dF_Z(z) \right] \\
&\quad + \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\delta)t} dt q \\
&\quad \times \left[\int_0^u F_M(y)\, \phi_1(u-y)\, dF_Y(y) \right. \\
&\qquad \left. + \int_u^\infty F_M(y)\, \omega(u, y-u)\, dF_Y(y) \right] \\
&= \frac{\lambda_1}{\lambda_1+\lambda_2+\delta} A(u) + \frac{\lambda_2 (1-q)}{\lambda_1+\lambda_2+\delta} \\
&\quad \times \left[\int_0^u \phi(u-y)\, dF_Y(y) + \omega_1(u) \right] \\
&\quad + \frac{\lambda_2 q}{\lambda_1+\lambda_2+\delta} \\
&\quad \times \left[\iint_{0<y+z<u} (1 - F_M(y))\, \phi(u-y-z) \right. \\
&\qquad \left. \times dF_Y(y)\, dF_Z(z) + \omega_2(u) \right] \\
&\quad + \frac{\lambda_2 q}{\lambda_1+\lambda_2+\delta} \left[\int_0^u F_M(y)\, \phi_1(u-y) \right. \\
&\qquad \left. \times dF_Y(y) + \omega_3(u) \right],
\end{aligned} \tag{6}$$

where

$$\begin{aligned}
A(u) &= \int_0^\infty \phi(u+x)\, dF_X(x), \\
\omega_1(u) &= \int_u^\infty \omega(u, y-u)\, dF_Y(y), \\
\omega_2(u) &= \iint_{y+z>u} (1 - F_M(y)) \\
&\quad \times \omega(u, y+z-u)\, dF_Y(y)\, dF_Z(z), \\
\omega_3(u) &= \int_u^\infty F_M(y)\, \omega(u, y-u)\, dF_Y(y).
\end{aligned} \tag{7}$$

Similarly, for the expected discounted penalty function $\phi_1(u)$ of the auxiliary risk model, we have

$$\begin{aligned}
\phi_1(u) &= \int_0^\infty \lambda_1 e^{-(\lambda_1+\lambda_2+\delta)t} dt \int_0^\infty \phi_1(u+x)\, dF_X(x) \\
&\quad + \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\delta)t} dt\, (1-q) \\
&\quad \times \left[\int_0^u \phi(u-y)\, dF_Y * F_Z(y) \right. \\
&\qquad \left. + \int_u^\infty \omega(u, y-u)\, dF_Y * F_Z(y) \right] \\
&\quad + \int_0^\infty \lambda_2 e^{-(\lambda_1+\lambda_2+\delta)t} dt q
\end{aligned}$$

$$\times\left[\iint_{0<y+z<u}(1-F_M(y))\phi(u-y-z)\times dF_Y(y)\,dF_Z*F_Z(z)+\iint_{y+z>u}(1-F_M(y))\times\omega(u,y+z-u)\,dF_Y(y)\,dF_Z*F_Z(z)\right]$$

$$+\int_0^\infty\lambda_2e^{-(\lambda_1+\lambda_2+\delta)t}dtq\times\left[\iint_{0<y+z<u}F_M(y)\phi_1(u-y-z)\times dF_Y(y)\,dF_Z(z)+\iint_{y+z>u}F_M(y)\times\omega(u,y+z-u)\,dF_Y(y)\,dF_Z(z)\right]$$

$$=\frac{\lambda_1}{\lambda_1+\lambda_2+\delta}A_1(u)+\frac{\lambda_2(1-q)}{\lambda_1+\lambda_2+\delta}\times\left[\int_0^u\phi(u-y)\,dF_Y*F_Z(y)+\omega_4(u)\right]+\frac{\lambda_2q}{\lambda_1+\lambda_2+\delta}\times\left[\iint_{0<y+z<u}(1-F_M(y))\phi(u-y-z)\times dF_Y(y)\,dF_Z*F_Z(z)+\omega_5(u)\right]+\frac{\lambda_2q}{\lambda_1+\lambda_2+\delta}\left[\iint_{0<y+z<u}\times F_M(y)\phi_1(u-y-z)\times dF_Y(y)\,dF_Z(z)+\omega_6(u)\right],\quad(8)$$

where

$$A_1(u)=\int_0^\infty\phi_1(u+x)\,dF_X(x),$$

$$\omega_4(u)=\int_u^\infty\omega(u,y-u)\,dF_Y*F_Z(y),$$

$$\omega_5(u)=\iint_{y+z>u}(1-F_M(y))\times\omega(u,y+z-u)\,dF_Y(y)\,dF_Z*F_Z(z),$$

$$\omega_6(u)=\iint_{y+z>u}F_M(y)\times\omega(u,y+z-u)\,dF_Y(y)\,dF_Z(z).\quad(9)$$

In the following, we will give the Laplace transforms of the $\phi(u)$ and $\phi_1(u)$.

Let $\chi_1(y)=F_M(y)F_Y'(y)$ and $\chi_2(y)=(1-F_M(y))F_Y'(y)$. For $\mathrm{Re}(s)\geq 0$, we define

$$\hat{\chi}_1(s)\triangleq\int_0^\infty e^{-sy}\chi_1(y)\,dy=E[\exp(-sY)I(Y\geq M)]=\int_0^\infty e^{-sy}F_M(y)\,dF_Y(y),$$

$$\hat{\chi}_2(s)\triangleq\int_0^\infty e^{-sy}\chi_2(y)\,dy=E[\exp(-sY)I(Y<M)]=\int_0^\infty e^{-sy}(1-F_M(y))\,dF_Y(y),$$

$$\hat{b}_1(s)\triangleq\int_0^\infty e^{-sy}dF_Y(y),\quad(10)$$

$$\hat{b}_2(s)\triangleq\int_0^\infty e^{-sy}dF_Z(y),$$

$$\hat{b}_3(s)\triangleq\int_0^\infty e^{-sy}dF_Y*F_Z(y),$$

$$\hat{b}_4(s)\triangleq\int_0^\infty e^{-sy}dF_Z*F_Z(y).$$

Taking the Laplace transforms of (6) and (8), we obtain

$$\hat{\phi}(s)=\frac{\lambda_1}{\lambda_1+\lambda_2+\delta}\hat{A}(s)+\frac{\lambda_2(1-q)}{\lambda_1+\lambda_2+\delta}\times\left[\hat{\phi}(s)\hat{b}_1(s)+\hat{\omega}_1(s)\right]+\frac{\lambda_2q}{\lambda_1+\lambda_2+\delta}\times\left[\hat{\phi}(s)\hat{\chi}_2(s)\hat{b}_2(s)+\hat{\phi}_1(s)\hat{\chi}_1(s)+\hat{\omega}_2(s)+\hat{\omega}_3(s)\right],\quad(11)$$

$$\hat{\phi}_1(s)=\frac{\lambda_1}{\lambda_1+\lambda_2+\delta}\hat{A}_1(s)+\frac{\lambda_2(1-q)}{\lambda_1+\lambda_2+\delta}\times\left[\hat{\phi}(s)\hat{b}_3(s)+\hat{\omega}_4(s)\right]+\frac{\lambda_2q}{\lambda_1+\lambda_2+\delta}\times\left[\hat{\phi}(s)\hat{\chi}_2(s)\hat{b}_4(s)+\hat{\phi}_1(s)\hat{\chi}_1(s)\hat{b}_2(s)+\hat{\omega}_5(s)+\hat{\omega}_6(s)\right].\quad(12)$$

Now we introduce the Dickson-Hipp operator T_r studied in Dickson and Hipp [19]. Define

$$T_r h(x) = \int_x^{\infty} e^{-r(y-x)} h(y)\, dy, \tag{13}$$

where $h(x)$ is a real-valued function and r is a complex number. As in Li and Garrido [20], we find $T_r h(0) = \widehat{h}(r)$. For distinct r and s,

$$T_r T_s h(x) = T_s T_r h(x) = \frac{T_r h(x) - T_s h(x)}{s - r}. \tag{14}$$

If $r = s$,

$$T_r T_s h(x) = \int_x^{\infty} (y - x)\, e^{-r(y-x)} h(y)\, dy. \tag{15}$$

Suppose that the premium incomes X_j are exponentially distributed; that is,

$$F_X(x) = 1 - e^{-x/\mu_X}. \tag{16}$$

According to the definition and properties of the Dickson-Hipp operator, we take the Laplace transform of $A(u)$ and $A_1(u)$; then

$$\widehat{A}(s) = \frac{\widehat{\phi}(s) - \widehat{\phi}(1/\mu_X)}{1 - s\mu_X}, \tag{17}$$

$$\widehat{A}_1(s) = \frac{\widehat{\phi}_1(s) - \widehat{\phi}_1(1/\mu_X)}{1 - s\mu_X}. \tag{18}$$

Plugging (17) and (18) into (11) and (12), respectively, and then making some simplifications, we obtain

$$\begin{aligned}
\widehat{\phi}(s) = {} & \Bigg(\frac{(1 - s\mu_X)}{\lambda_1 + \lambda_2 + \delta} \\
& \times \left[\chi_3(s)\, \widehat{\omega}(s) + \frac{\lambda_2 q (1 - s\mu_X)\, \widehat{\chi}_1(s)}{\lambda_1 + \lambda_2 + \delta} \widehat{\omega}^*(s) \right] \\
& - l_1(s) \Bigg) \\
& \times \Bigg(\left(1 - s\mu_X - \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta} \right)^2 \\
& - (1 - s\mu_X) \frac{\lambda_2 \widehat{b}_1(s) \left(1 - q + q\widehat{b}_2(s)\right)}{\lambda_1 + \lambda_2 + \delta} \\
& \times \left(1 - s\mu_X - \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta} \right) \Bigg)^{-1},
\end{aligned} \tag{19}$$

$$\begin{aligned}
& \widehat{\phi}_1(s) \\
& = \Bigg(\frac{(1 - s\mu_X)}{\lambda_1 + \lambda_2 + \delta} \\
& \quad \times \Big[\chi_4(s)\, \widehat{\omega}^*(s) \\
& \quad + \frac{\lambda_2 (1 - s\mu_X) \left((1 - q)\widehat{b}_3(s) + q\widehat{\chi}_2(s)\widehat{b}_4(s) \right)}{\lambda_1 + \lambda_2 + \delta} \\
& \quad \times \widehat{\omega}(s) \Big] - l_2(s) \Bigg) \\
& \quad \times \Bigg(\left(1 - s\mu_X - \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta} \right)^2 - (1 - s\mu_X) \\
& \quad \times \frac{\lambda_2 \widehat{b}_1(s) \left(1 - q + q\widehat{b}_2(s)\right)}{\lambda_1 + \lambda_2 + \delta} \\
& \quad \times \left(1 - s\mu_X - \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta} \right) \Bigg)^{-1},
\end{aligned} \tag{20}$$

where

$$\begin{aligned}
\widehat{\omega}(s) &= \lambda_2 \left[(1 - q)\, \widehat{\omega}_1(s) + q \left(\widehat{\omega}_2(s) + \widehat{\omega}_3(s) \right) \right], \\
\widehat{\omega}^*(s) &= \lambda_2 \left[(1 - q)\, \widehat{\omega}_4(s) + q \left(\widehat{\omega}_5(s) + \widehat{\omega}_6(s) \right) \right], \\
\chi_3(s) &= 1 - s\mu_X - \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta} \\
& \quad - (1 - s\mu_X) \frac{\lambda_2 q \widehat{\chi}_1(s)\, \widehat{b}_2(s)}{\lambda_1 + \lambda_2 + \delta}, \\
\chi_4(s) &= 1 - s\mu_X - \frac{\lambda_1}{\lambda_1 + \lambda_2 + \delta} - (1 - s\mu_X) \\
& \quad \times \frac{\lambda_2 \left[(1 - q)\, \widehat{b}_1(s) + q\widehat{\chi}_2(s)\, \widehat{b}_2(s) \right]}{\lambda_1 + \lambda_2 + \delta}, \\
l_1(s) &= \frac{\lambda_1 \chi_3(s)}{\lambda_1 + \lambda_2 + \delta} \widehat{\phi}\left(\frac{1}{\mu_X} \right) \\
& \quad + \frac{\lambda_1 \lambda_2 q (1 - s\mu_X)\, \widehat{\chi}_1(s)}{(\lambda_1 + \lambda_2 + \delta)^2} \widehat{\phi}_1\left(\frac{1}{\mu_X} \right), \\
l_2(s) &= \frac{\lambda_1 \chi_4(s)}{\lambda_1 + \lambda_2 + \delta} \widehat{\phi}_1\left(\frac{1}{\mu_X} \right) \\
& \quad + \frac{\lambda_1 \lambda_2 (1 - s\mu_X) \left[(1 - q)\, \widehat{b}_3(s) + q\widehat{\chi}_2(s)\, \widehat{b}_4(s) \right]}{(\lambda_1 + \lambda_2 + \delta)^2} \\
& \quad \times \widehat{\phi}\left(\frac{1}{\mu_X} \right).
\end{aligned} \tag{21}$$

To solve $\widehat{\phi}(s)$ and $\widehat{\phi}_1(s)$ in (19) and (20), we need to find $\widehat{\phi}(1/\mu_X)$ and $\widehat{\phi}_1(1/\mu_X)$. Here we will first consider the zeros of the denominators of (19) and (20) or equally the zeros of the following equation:

$$\left(1 - s\mu_X - \frac{\lambda_1}{\lambda_1+\lambda_2+\delta}\right) \times \left[1 - s\mu_X - \frac{\lambda_1}{\lambda_1+\lambda_2+\delta} - \frac{\lambda_2 \widehat{b}_1(s)\left(1-q+q\widehat{b}_2(s)\right)}{\lambda_1+\lambda_2+\delta}(1-s\mu_X)\right] = 0. \quad (22)$$

Lemma 1. *For $\delta > 0$, the denominators of (19) and (20) have exactly two distinct positive real roots, say, $\rho_1(\delta)$ and $\rho_2(\delta) = (\lambda_2+\delta)/(\lambda_1+\lambda_2+\delta)\mu_X$. Further, $\rho_1(\delta)$ and $\rho_2(\delta)$ are the only roots on the right half of the complex plane.*

Proof. To prove Lemma 1, it is equal to show that (22) has exactly two roots in the right half complex plan. Firstly, (22) can be rewritten as

$$\left(1 - s\mu_X - \frac{\lambda_1}{\lambda_1+\lambda_2+\delta}\right) l(s) = 0, \quad (23)$$

where

$$l(s) = (1-s\mu_X) \times \left[1 - \frac{\lambda_2\widehat{b}_1(s)\left(1-q+q\widehat{b}_2(s)\right)}{\lambda_1+\lambda_2+\delta}\right] - \frac{\lambda_1}{\lambda_1+\lambda_2+\delta}. \quad (24)$$

For $\delta > 0, s \geq 0$, it is easy to check that $l(0) = \delta/(\lambda_1+\lambda_2+\delta) > 0$ and $\lim_{s\to+\infty} l(s) = -\infty$. And

$$\begin{aligned} l'(s) &= -\mu_X\left[1 - \frac{\lambda_2\widehat{b}_1(s)\left(1-q+q\widehat{b}_2(s)\right)}{\lambda_1+\lambda_2+\delta}\right] - (1-s\mu_X) \times \frac{\lambda_2\left[\widehat{b}_1'(s)\left(1-q+q\widehat{b}_2(s)\right)+\widehat{b}_1(s)q\widehat{b}_2'(s)\right]}{\lambda_1+\lambda_2+\delta} \\ &< -\mu_X\left[1 - \frac{\lambda_2\widehat{b}_1(s)\left(1-q+q\widehat{b}_2(s)\right)}{\lambda_1+\lambda_2+\delta}\right] - \frac{\lambda_2\left[\widehat{b}_1'(s)\left(1-q+q\widehat{b}_2(s)\right)+\widehat{b}_1(s)q\widehat{b}_2'(s)\right]}{\lambda_1+\lambda_2+\delta} \\ &< -\mu_X\left[1 - \frac{\lambda_2\widehat{b}_1(s)\left(1-q+q\widehat{b}_2(s)\right)}{\lambda_1+\lambda_2+\delta}\right] + \frac{\lambda_2\left[\mu_Y\left(1-q+q\widehat{b}_2(s)\right)+\widehat{b}_1(s)q\mu_Z\right]}{\lambda_1+\lambda_2+\delta} \\ &< -\mu_X\frac{\lambda_1+\delta}{\lambda_1+\lambda_2+\delta} + \frac{\lambda_2(\mu_Y+q\mu_Z)}{\lambda_1+\lambda_2+\delta} \\ &< \frac{\lambda_2(\mu_Y+q\mu_Z)-\lambda_1\mu_X}{\lambda_1+\lambda_2+\delta} < 0, \end{aligned} \quad (25)$$

which implies $l(s)$ is a strictly decreasing function of s. So $l(s) = 0$ has exactly one positive real root, say, $\rho_1(\delta)$. Obviously, $\rho_1(\delta)$ is also one positive real root of (22). Note that $(\lambda_2+\delta)/(\lambda_1+\lambda_2+\delta)\mu_X$ is another positive real root of (22), say, $\rho_2(\delta)$ and $l((\lambda_2+\delta)/(\lambda_1+\lambda_2+\delta)\mu_X) \neq 0$. That means $\rho_1(\delta) \neq \rho_2(\delta)$, so we conclude that (22) has exactly two distinct positive real roots, say $\rho_1(\delta)$ and $\rho_2(\delta)$.

Now, we prove that $\rho_1(\delta)$ is the exactly one positive real root of equation $l(s) = 0$ on the right half of the complex plane. When s is on the half-circle, $|z| = r$ $(r > 0)$ and $\mathrm{Re}(z) \geq 0$ on the complex plane, for r sufficiently large,

$$\begin{aligned} &\left|1 - s\mu_X - \frac{\lambda_1}{\lambda_1+\lambda_2+\delta}\right| \\ &\quad > 1 - \frac{\lambda_1}{\lambda_1+\lambda_2+\delta} > \frac{\lambda_2}{\lambda_1+\lambda_2+\delta} \\ &\quad > \frac{\lambda_2\left|\widehat{b}_1(s)\left(1-q+q\widehat{b}_2(s)\right)(1-s\mu_X)\right|}{\lambda_1+\lambda_2+\delta}, \end{aligned} \quad (26)$$

while for s on the imaginary axis, $\mathrm{Re}(z) = 0$, the last inequality is true as well. That is to say, on the boundary of the contour enclosed by the half-circle and the imaginary axis,

$$\begin{aligned} &\left|1 - s\mu_X - \frac{\lambda_1}{\lambda_1+\lambda_2+\delta}\right| \\ &\quad > \frac{\lambda_2\left|\widehat{b}_1(s)\left[1-q+q\widehat{b}_2(s)\right](1-s\mu_X)\right|}{\lambda_1+\lambda_2+\delta}. \end{aligned} \quad (27)$$

Then we conclude, by Rouché's theorem, that on the right half of the complex plane, the number of roots of the equation $l(s) = 0$ equals the number of roots of the equation $1 - s\mu_X - \lambda_1/(\lambda_1+\lambda_2+\delta) = 0$. Furthermore, the latter has exactly one root on the right half of the complex plane. It follows that $l(s) = 0$ has exactly one positive real root, say, $\rho_1(\delta)$, on the right half of the complex plane. It is easy to see that $\rho_2(\delta) = (\lambda_2+\delta)/(\lambda_1+\lambda_2+\delta)\mu_X$ is the exactly one positive real root of equation $1-s\mu_X-\lambda_1/(\lambda_1+\lambda_2+\delta) = 0$ on the right half of the complex plane. It follows from all of the above that (22) has exactly two distinct positive real roots $\rho_1(\delta)$ and $\rho_2(\delta)$ on the right half of the complex plane. Hence, the lemma is proved. □

Remark 2. From Klimenok [21], we know that $\lim_{\delta\to 0^+}\rho_1(\delta) = 0$. Thereafter, we denote them by ρ_1, ρ_2 for simplicity.

Since $\widehat{\phi}(s)$ is finite for all s with $\mathrm{Re}(s) > 0$, we know ρ_i, $i = 1, 2$, should also be zeros of the numerator in (19); that is,

$$\begin{aligned} &\frac{(1-\rho_i\mu_X)}{\lambda_1+\lambda_2+\delta} \times \left[\chi_3(\rho_i)\widehat{\omega}(\rho_i) + \frac{\lambda_2 q(1-\rho_i\mu_X)\widehat{\chi}_1(\rho_i)}{\lambda_1+\lambda_2+\delta}\widehat{\omega}^*(\rho_i)\right] \\ &\quad = l_1(\rho_i), \quad i = 1, 2. \end{aligned} \quad (28)$$

By solving these linear equations, we get

$$\begin{aligned}\widehat{\phi}\left(\frac{1}{\mu_X}\right) &= (1-\rho_1\mu_X)(1-\rho_2\mu_X) \\ &\quad\times \left(\lambda_1\left[(1-\rho_1\mu_X)\widehat{\chi}_1(\rho_1)\chi_3(\rho_2) - (1-\rho_2\mu_X)\widehat{\chi}_1(\rho_2)\chi_3(\rho_1)\right]\right)^{-1} \\ &\quad\cdot\Bigg[\widehat{\chi}_1(\rho_1)\chi_3(\rho_2)\widehat{\omega}(\rho_2) - \widehat{\chi}_1(\rho_2)\chi_3(\rho_1)\widehat{\omega}(\rho_1) \\ &\quad + \frac{\lambda_2 q\widehat{\chi}_1(\rho_1)\widehat{\chi}_1(\rho_2)}{\lambda_1+\lambda_2+\delta} \cdot\left((1-\rho_2\mu_X)\widehat{\omega}^*(\rho_2) - (1-\rho_1\mu_X)\widehat{\omega}^*(\rho_1)\right)\Bigg], \\ \widehat{\phi}_1\left(\frac{1}{\mu_X}\right) &= \left((\lambda_1+\lambda_2+\delta)\chi_3(\rho_1)\widehat{\omega}(\rho_1) + \lambda_2 q(1-\rho_1\mu_X)\widehat{\chi}_1(\rho_1)\widehat{\omega}^*(\rho_1)\right) \\ &\quad\times\left(\lambda_1\lambda_2 q\widehat{\chi}_1(\rho_1)\right)^{-1} - \frac{\lambda_1(\lambda_1+\lambda_2+\delta)\chi_3(\rho_1)}{\lambda_1\lambda_2 q(1-\rho_1\mu_X)\widehat{\chi}_1(\rho_1)}\widehat{\phi}\left(\frac{1}{\mu_X}\right).\end{aligned} \tag{29}$$

Then the explicit expression for the $\widehat{\phi}(s)$ and $\widehat{\phi}_1(s)$ can be obtained by (19) and (20), respectively.

4. The Defective Renewal Equation for the Expected Discounted Penalty Function

In this section, we study the defective renewal equation satisfied by the expected discounted penalty function. Note that (19) can be rewritten as

$$\widehat{\phi}(s) = \frac{\widehat{f}_1(s)+\widehat{f}_2(s)}{\widehat{h}_1(s)-\widehat{h}_2(s)}, \tag{30}$$

where

$$\begin{aligned}\widehat{f}_1(s) &= -\frac{\lambda_1\widehat{\phi}(1/\mu_X)}{\lambda_1+\lambda_2+\delta}\left(1-s\mu_X-\frac{\lambda_1}{\lambda_1+\lambda_2+\delta}\right), \\ \widehat{f}_2(s) &= \frac{(1-s\mu_X)}{\lambda_1+\lambda_2+\delta} \times\left[\chi_3(s)\widehat{\omega}(s) + \frac{\lambda_2 q(1-s\mu_X)\widehat{\chi}_1(s)}{\lambda_1+\lambda_2+\delta}\widehat{\omega}^*(s)\right] \\ &\quad + \left(\lambda_1\lambda_2 q(1-s\mu_X)\widehat{\chi}_1(s) \times\left[\widehat{b}_2(s)\widehat{\phi}\left(\frac{1}{\mu_X}\right) - \widehat{\phi}_1\left(\frac{1}{\mu_X}\right)\right]\right) \\ &\quad\times\left((\lambda_1+\lambda_2+\delta)^2\right)^{-1}, \\ \widehat{h}_1(s) &= \left(1-s\mu_X-\frac{\lambda_1}{\lambda_1+\lambda_2+\delta}\right)^2, \\ \widehat{h}_2(s) &= (1-s\mu_X) \times\frac{\lambda_2\widehat{b}_1(s)\left(1-q+q\widehat{b}_2(s)\right)}{\lambda_1+\lambda_2+\delta} \times\left(1-s\mu_X-\frac{\lambda_1}{\lambda_1+\lambda_2+\delta}\right).\end{aligned} \tag{31}$$

Lemma 3. *The Laplace transform of the expected discounted penalty function $\widehat{\phi}(s)$ satisfies*

$$\widehat{\phi}(s) = \frac{T_sT_{\rho_1}T_{\rho_2}h_2(0)}{\mu_X^2}\widehat{\phi}(s) + \frac{T_sT_{\rho_1}T_{\rho_2}f_2(0)}{\mu_X^2}. \tag{32}$$

Proof. Since $\widehat{\phi}(s)$ is analytic for all s with $\mathrm{Re}(s) \geq 0$, we know ρ_i, $i = 1, 2$, are zeros of the numerator in (30). It means $\widehat{f}_1(\rho_i) = -\widehat{f}_2(\rho_i)$, $i = 1, 2$. Because $\widehat{f}_1(s)$ is a polynomial of degree 1, using Lagrange interpolating theorem, we obtain

$$\begin{aligned}\widehat{f}_1(s) &= \widehat{f}_1(\rho_1)\frac{s-\rho_2}{\rho_1-\rho_2} + \widehat{f}_1(\rho_2)\frac{s-\rho_1}{\rho_2-\rho_1} \\ &= -\widehat{f}_2(\rho_1)\frac{s-\rho_2}{\rho_1-\rho_2} - \widehat{f}_2(\rho_2)\frac{s-\rho_1}{\rho_2-\rho_1}.\end{aligned} \tag{33}$$

It yields

$$\begin{aligned}\widehat{f}_1(s)+\widehat{f}_2(s) &= \left(-\widehat{f}_2(\rho_1)(s-\rho_2) + \widehat{f}_2(\rho_2)(s-\rho_1) + (s-\rho_2)\widehat{f}_2(s) - (s-\rho_1)\widehat{f}_2(s)\right) \times(\rho_1-\rho_2)^{-1} \\ &= \left((s-\rho_2)\left[\widehat{f}_2(s)-\widehat{f}_2(\rho_1)\right] - (s-\rho_1)\left[\widehat{f}_2(s)-\widehat{f}_2(\rho_2)\right]\right) \times(\rho_1-\rho_2)^{-1} \\ &= (s-\rho_1)(s-\rho_2) \times\frac{T_sT_{\rho_2}f_2(0) - T_sT_{\rho_1}f_2(0)}{\rho_1-\rho_2} \\ &= (s-\rho_1)(s-\rho_2)T_sT_{\rho_2}T_{\rho_1}f_2(0).\end{aligned} \tag{34}$$

The denominator of (30) can be dealt with in a similar way. From Lemma 1, we know that $\widehat{h}_1(\rho_i) = \widehat{h}_2(\rho_i)$, $i = 1, 2$.

Because $\widehat{h}_1(s)$ is a polynomial of degree 2, using Lagrange interpolating theorem, we obtain

$$\begin{aligned}\widehat{h}_1(s) &= \widehat{h}_1(0)\frac{(s-\rho_1)(s-\rho_2)}{\rho_1\rho_2} \\ &\quad + s\left(\frac{\widehat{h}_1(\rho_1)}{\rho_1}\frac{s-\rho_2}{\rho_1-\rho_2}+\frac{\widehat{h}_1(\rho_2)}{\rho_2}\frac{s-\rho_1}{\rho_2-\rho_1}\right) \\ &= \widehat{h}_1(0)\frac{(s-\rho_1)(s-\rho_2)}{\rho_1\rho_2} \\ &\quad + s\left(\frac{\widehat{h}_2(\rho_1)}{\rho_1}\frac{s-\rho_2}{\rho_1-\rho_2}+\frac{\widehat{h}_2(\rho_2)}{\rho_2}\frac{s-\rho_1}{\rho_2-\rho_1}\right) \\ &= \widehat{h}_1(0)\frac{(s-\rho_1)(s-\rho_2)}{\rho_1\rho_2} \\ &\quad + \widehat{h}_2(\rho_1)\frac{s-\rho_2}{\rho_1-\rho_2}+\widehat{h}_2(\rho_2)\frac{s-\rho_1}{\rho_2-\rho_1} \\ &\quad + (s-\rho_1)(s-\rho_2) \\ &\quad \times\left(\frac{\widehat{h}_2(\rho_1)}{\rho_1}\frac{1}{\rho_1-\rho_2}+\frac{\widehat{h}_2(\rho_2)}{\rho_2}\frac{1}{\rho_2-\rho_1}\right).\end{aligned} \tag{35}$$

Then using Property 6 of the Dickson-Hipp operator given in Li and Garrido [20], we have

$$\begin{aligned}\widehat{h}_1(s)-\widehat{h}_2(s) &= \widehat{h}_1(0)\frac{(s-\rho_1)(s-\rho_2)}{\rho_1\rho_2} \\ &\quad + \widehat{h}_2(\rho_1)\frac{s-\rho_2}{\rho_1-\rho_2}+\widehat{h}_2(\rho_2)\frac{s-\rho_1}{\rho_2-\rho_1} \\ &\quad + (s-\rho_1)(s-\rho_2) \\ &\quad \times\left(\frac{\widehat{h}_2(\rho_1)}{\rho_1}\frac{1}{\rho_1-\rho_2}+\frac{\widehat{h}_2(\rho_2)}{\rho_2}\frac{1}{\rho_2-\rho_1}\right) \\ &\quad - \widehat{h}_2(s) \\ &= (s-\rho_1)(s-\rho_2) \\ &\quad \times\left(\frac{\widehat{h}_1(0)}{\rho_1\rho_2}+\frac{\widehat{h}_1(\rho_1)}{\rho_1(\rho_1-\rho_2)}\right. \\ &\quad -\frac{\widehat{h}_1(\rho_2)}{\rho_2(\rho_1-\rho_2)}+\frac{\widehat{h}_2(\rho_1)}{(\rho_1-\rho_2)(s-\rho_1)} \\ &\quad \left.-\frac{\widehat{h}_2(\rho_2)}{(\rho_1-\rho_2)(s-\rho_2)}-\frac{\widehat{h}_2(s)}{(s-\rho_1)(s-\rho_2)}\right) \\ &= (s-\rho_1)(s-\rho_2) \\ &\quad \times\left(T_0T_{\rho_1}T_{\rho_2}h_1(0)-T_sT_{\rho_1}T_{\rho_2}h_2(0)\right).\end{aligned} \tag{36}$$

It is easy to check that $T_0T_{\rho_1}T_{\rho_2}h_1(0)=\mu_X^2$ which makes (36) become

$$\widehat{h}_1(s)-\widehat{h}_2(s)=(s-\rho_1)(s-\rho_2)\left(\mu_X^2-T_sT_{\rho_1}T_{\rho_2}h_2(0)\right). \tag{37}$$

Invoking (34) and (37) into (30), we could obtain

$$\widehat{\phi}(s)=\frac{T_sT_{\rho_2}T_{\rho_1}f_2(0)}{\mu_X^2-T_sT_{\rho_1}T_{\rho_2}h_2(0)}, \tag{38}$$

which leads to (32). This completes the proof. □

Now, we are ready to derive the defective renewal equation for $\phi(u)$.

Theorem 4. *$\phi(u)$ satisfies the following integral equation:*

$$\phi(u)=\int_0^u\phi(u-y)\frac{T_{\rho_1}T_{\rho_2}h_2(y)}{\mu_X^2}dy+\frac{T_{\rho_1}T_{\rho_2}f_2(u)}{\mu_X^2}. \tag{39}$$

Proof. Equation (39) follows easily from the inverse Laplace transform in (32). We could like to point out that (39) is also a defective renewal equation. This can be verified by showing that

$$\frac{T_0T_{\rho_1}T_{\rho_2}h_2(0)}{\mu_X^2}<1. \tag{40}$$

For $\delta>0$, putting $s=0$ in (36), it follows that

$$\begin{aligned}\frac{T_0T_{\rho_1}T_{\rho_2}h_2(0)}{\mu_X^2} &= 1-\frac{\widehat{h}_1(0)-\widehat{h}_2(0)}{\mu_X^2\rho_1\rho_2} \\ &= 1-\frac{\delta(\lambda_2+\delta)}{\mu_X^2\rho_1\rho_2(\lambda_1+\lambda_2+\delta)^2}<1.\end{aligned} \tag{41}$$

Now we consider the case $\delta=0$. Setting $s=\rho_1(\delta)$ in the denominators of (19) and (20), we have

$$\begin{aligned}&1-\rho_1(\delta)\mu_X-\frac{\lambda_1}{\lambda_1+\lambda_2+\delta} \\ &\quad -\frac{\lambda_2\widehat{b}_1(\rho_1(\delta))\left[1-q+q\widehat{b}_2(\rho_1(\delta))\right]}{\lambda_1+\lambda_2+\delta} \\ &\quad \times(1-\rho_1(\delta)\mu_X)=0.\end{aligned} \tag{42}$$

Differentiating both sides of this equation with respect to δ and then setting $\delta=0$, we have

$$\rho_1'(0)=\frac{1}{\lambda_1\mu_X-\lambda_2(\mu_Y+q\mu_Z)}>0. \tag{43}$$

Then taking the limit $\delta \to 0$ in (41) and using L'Hôspital's rule, we obtain

$$\frac{T_0 T_{\rho_1} T_{\rho_2} h_2(0)}{\mu_X^2} = 1 - \frac{1}{\mu_X^2 \rho_2(0)(\lambda_1+\lambda_2+\delta)^2} \times \lim_{\delta\to 0^+} \frac{(\lambda_2+\delta)\delta}{\rho_1(0)} = 1 - \frac{\lambda_2}{\mu_X^2 \rho_1'(0)\rho_2(0)(\lambda_1+\lambda_2)^2} < 1. \tag{44}$$

Thus, (39) is a defective renewal equation, and the proof is complete. □

Remark 5. When $q \equiv 1$ and $P(Y_1 \geq M_1) = \theta$, then $\widehat{\chi}_1(s) = \theta\widehat{b}_1(s)$, $\widehat{\chi}_2(s) = (1-\theta)\widehat{b}_1(s)$. In the case, each main claim induces a byclaim, and its associated byclaim occurs simultaneously with probability $1-\theta$, or the occurrence of the byclaim may be delayed with probability θ. Actually, the risk model given by (1) will be the compound Poisson risk model with delayed claims and random incomes studied by Hao and Yang [16]. Then, by some simple calculations, we can find that (39) in Theorem 4 is consistent with (4.6) in [16].

Remark 6. The explicit analytic solution to the defective renewal (39) can be obtained by compound geometric distribution (see Lin and Willmot [22]).

5. Explicit Results for Exponential Claim Size Distributions

We now consider the case where both claim sizes are exponentially distributed, that is, distribution functions $F_Y \sim \text{Exp}(\nu)$ and $F_Z \sim \text{Exp}(\omega)$, where $\nu = 1/\mu_Y$ and $\omega = 1/\mu_Z$. Then we have

$$\widehat{b}_1(s) = \frac{\nu}{\nu+s}, \qquad \widehat{b}_2(s) = \frac{\omega}{\omega+s}, \qquad \widehat{b}_3(s) = \frac{\nu\omega}{(\nu+s)(\omega+s)}, \qquad \widehat{b}_4(s) = \frac{\omega^2}{(\omega+s)^2}. \tag{45}$$

For the special case $F_M \sim \text{Exp}(\mu)$, we obtain

$$\widehat{\chi}_2(s) = \int_0^\infty e^{-sy} e^{-\mu y} dF_Y(y) = \widehat{b}_1(s+\mu), \qquad \widehat{\chi}_1(s) = \widehat{b}_1(s) - \widehat{b}_1(s+\mu). \tag{46}$$

So we have

$$\widehat{\chi}_2(s) = \frac{\nu}{\nu+s+\mu}, \qquad \widehat{\chi}_1(s) = \frac{\nu}{\nu+s} - \frac{\nu}{\nu+s+\mu} = \frac{\nu\mu}{(\nu+s)(\nu+s+\mu)}. \tag{47}$$

Let $\widehat{B}_1(s) \triangleq \widehat{\chi}_1(s)\widehat{b}_2(s)\widehat{\omega}(s)$, $\widehat{B}_2(s) \triangleq \widehat{\chi}_1(s)\widehat{\omega}^*(s)$; then $\widehat{f}_2(s)$ can be written as

$$\begin{aligned}\widehat{f}_2(s) = {} & \frac{(1-s\mu_X)}{\lambda_1+\lambda_2+\delta} \times \left(1 - s\mu_X - \frac{\lambda_1}{\lambda_1+\lambda_2+\delta}\right)\widehat{\omega}(s) \\ & - \frac{\lambda_2 q(1-s\mu_X)^2}{(\lambda_1+\lambda_2+\delta)^2}\left[\widehat{B}_1(s) - \widehat{B}_2(s)\right] \\ & + \frac{\lambda_1\lambda_2 q}{(\lambda_1+\lambda_2+\delta)^2} \times \left[\widehat{\phi}\left(\frac{1}{\mu_X}\right)\frac{\omega\nu\mu(1-s\mu_X)}{(\omega+s)(\nu+s)(\nu+s+\mu)} - \widehat{\phi}_1\left(\frac{1}{\mu_X}\right)\frac{\nu\mu(1-s\mu_X)}{(\nu+s)(\nu+s+\mu)}\right].\end{aligned} \tag{48}$$

So we can derive

$$\begin{aligned} & T_s T_{\rho_2} T_{\rho_1} f_2(0) \\ & = \frac{\mu_X}{\lambda_1+\lambda_2+\delta} \times \left[(1-\mu_X\rho_1) T_s T_{\rho_1}\omega(0) + \mu_X T_s \omega(0)\right] \\ & - \frac{\lambda_1^2\lambda_2 q}{(\lambda_1+\lambda_2+\delta)^4} \times \left[T_s T_{\rho_2} T_{\rho_1} B_1(0) - T_s T_{\rho_2} T_{\rho_1} B_2(0)\right] \\ & - \frac{\lambda_2 q\mu_X\left[2-\mu_X(\rho_1+\rho_2)\right]}{(\lambda_1+\lambda_2+\delta)^2} \times \left[T_s T_{\rho_1} B_1(0) - T_s T_{\rho_1} B_2(0)\right] \\ & - \frac{\lambda_2 q\mu_X^2}{(\lambda_1+\lambda_2+\delta)^2}\left[T_s B_1(0) - T_s B_2(0)\right] \\ & + \frac{\lambda_1\lambda_2 q}{(\lambda_1+\lambda_2+\delta)^2}\widehat{\phi}\left(\frac{1}{\mu_X}\right) \\ & \cdot \left[\frac{\lambda_1}{(\lambda_1+\lambda_2+\delta)} \times \frac{\omega\nu\mu\left[c_0 s^2 + (c_0 a_2 + c_1)s + c_0 a_1 + c_1 a_2 + c_2\right]}{g(s)g(\rho_1)g(\rho_2)} + \mu_X \frac{\omega\nu\mu\left[s^2 + (a_2+\rho_1)s + \rho_1^2 + a_2\rho_1 + a_1\right]}{g(s)g(\rho_1)}\right] \\ & - \frac{\lambda_1\lambda_2 q}{(\lambda_1+\lambda_2+\delta)^2}\widehat{\phi}_1\left(\frac{1}{\mu_X}\right)\end{aligned}$$

$$\cdot \left[\frac{\lambda_1}{(\lambda_1+\lambda_2+\delta)} \cdot \left(\nu\mu\left[(2\nu+\mu+s)(2\nu+\mu+\rho_1+\rho_2) - \nu(\nu+\mu)+\rho_1\rho_2\right]\right) \times \left((\nu+s)(\nu+\mu+s)(\nu+\rho_1) \times (\nu+\mu+\rho_1)(\nu+\rho_2)(\nu+\mu+\rho_2)\right)^{-1} + \mu_X \frac{\nu\mu(2\nu+\mu+\rho_1+s)}{(\nu+s)(\nu+\mu+s)(\nu+\rho_1)(\nu+\mu+\rho_1)} \right], \quad (49)$$

where

$$\begin{aligned} a_0 &= \omega\nu(\nu+\mu), \\ a_1 &= \omega\nu+(\omega+\nu)(\nu+\mu), \\ a_2 &= \omega+2\nu+\mu, \\ c_0 &= a_1+a_2(\rho_1+\rho_2)+\rho_1^2+\rho_1\rho_2+\rho_2^2, \\ c_1 &= -a_0+\rho_1\rho_2(a_2+\rho_1+\rho_2), \\ c_2 &= -a_0(\rho_1+\rho_2)-a_1\rho_1\rho_2+\rho_1^2\rho_2^2, \\ g(x) &= (\omega+x)(\nu+x)(\nu+\mu+x). \end{aligned} \quad (50)$$

From (32) and (37), we know that

$$\widehat{\phi}(s) = \frac{(s-\rho_1)(s-\rho_2)T_sT_{\rho_2}T_{\rho_1}f_2(0)}{\widehat{h}_1(s)-\widehat{h}_2(s)}. \quad (51)$$

It turns out that (51) can be transformed to another expression by multiplying both denominator and numerator by $g(s)$:

$$\widehat{\phi}(s) = \frac{g(s)(s-\rho_1)(s-\rho_2)T_sT_{\rho_2}T_{\rho_1}f_2(0)}{g(s)\left(\widehat{h}_1(s)-\widehat{h}_2(s)\right)}. \quad (52)$$

The common denominator of (52), denoted by $D_5(s)$, is a polynomial of degree 5 with the leading coefficient μ_X^2, given by

$$\begin{aligned} D_5(s) = {} & (\omega+s)(\nu+s)(\nu+\mu+s) \times \left(1-s\mu_X-\frac{\lambda_1}{\lambda_1+\lambda_2+\delta}\right)^2 \\ & -(\nu+\mu+s)(1-s\mu_X) \times \frac{\lambda_2\nu((1-q)(\omega+s)+q\omega)}{\lambda_1+\lambda_2+\delta} \times \left(1-s\mu_X-\frac{\lambda_1}{\lambda_1+\lambda_2+\delta}\right). \end{aligned} \quad (53)$$

Obviously, $D_5(s)$ has five roots on the complex plane and all the complex roots are in conjugate pairs. Noting that $s=\rho_1$, $s=\rho_2$, and $s=-(\nu+\mu)$ are three roots, we have

$$D_5(s) = \mu_X^2(s-\rho_1)(s-\rho_2)(s+\nu+\mu)(s+R_1)(s+R_2). \quad (54)$$

Note also that all R_is have positive real parts, since, otherwise, they also are roots of (22) which is a contradiction to the conclusion of Lemma 1.

Denote $R_0 = \nu+\mu$. Furthermore, if R_0, R_1, and R_2 are distinct, we obtain, by partial fractions, that

$$\begin{aligned} \frac{as^2+bs+c}{(s+R_0)(s+R_1)(s+R_2)} &= \frac{r_1(a,b,c)}{s+R_0}+\frac{r_2(a,b,c)}{s+R_1}+\frac{r_3(a,b,c)}{s+R_2}, \\ \frac{(s+\omega)(s+\nu)}{(s+R_1)(s+R_2)} &= 1+\frac{h_1}{s+R_1}+\frac{h_2}{s+R_2}, \end{aligned} \quad (55)$$

where

$$\begin{aligned} r_1(a,b,c) &= \frac{aR_0^2-bR_0+c}{(R_1-R_0)(R_2-R_0)}, \\ r_2(a,b,c) &= \frac{aR_1^2-bR_1+c}{(R_0-R_1)(R_2-R_1)}, \\ r_3(a,b,c) &= \frac{aR_2^2-bR_2+c}{(R_0-R_2)(R_1-R_2)}, \\ h_1 &= \frac{(\nu-R_1)(\omega-R_1)}{R_2-R_1}, \\ h_2 &= \frac{(\nu-R_2)(\omega-R_2)}{R_1-R_2}. \end{aligned} \quad (56)$$

Then (52) can be simplified to

$$\begin{aligned} \widehat{\phi}(s) &= \frac{1}{\mu_X^2}\left(1+\sum_{i=1}^{2}\frac{h_i}{s+R_i}\right) \\ &\times \left\{ \frac{\mu_X}{\lambda_1+\lambda_2+\delta} \times \left[(1-\mu_X\rho_1)T_sT_{\rho_1}\omega(0)+\mu_XT_s\omega(0)\right] \right. \\ &\quad - \frac{\lambda_1^2\lambda_2 q}{(\lambda_1+\lambda_2+\delta)^4} \times \left[T_sT_{\rho_2}T_{\rho_1}B_1(0)-T_sT_{\rho_2}T_{\rho_1}B_2(0)\right] \end{aligned}$$

$$
\begin{aligned}
&\quad - \frac{\lambda_2 q \mu_X [2-\mu_X(\rho_1+\rho_2)]}{(\lambda_1+\lambda_2+\delta)^2} \\
&\quad \times \left[T_s T_{\rho_1} B_1(0) - T_s T_{\rho_1} B_2(0)\right] \\
&\quad \left. - \frac{\lambda_2 q \mu_X^2}{(\lambda_1+\lambda_2+\delta)^2} [T_s B_1(0) - T_s B_2(0)] \right\} \\
&+ \frac{\lambda_1 \lambda_2 q}{\mu_X^2 (\lambda_1+\lambda_2+\delta)^2} \widehat{\phi}\left(\frac{1}{\mu_X}\right) \\
&\cdot \left[\frac{\lambda_1 \omega \nu \mu}{(\lambda_1+\lambda_2+\delta) g(\rho_1) g(\rho_2)} \right. \\
&\quad \times \sum_{j=0}^{2} \frac{r_j (c_0, c_0 a_2 + c_1, c_0 a_1 + c_1 a_2 + c_2)}{s+R_j} \\
&\quad \left. + \mu_X \frac{\omega \nu \mu}{g(\rho_1)} \sum_{j=0}^{2} \frac{r_j \left(1, a_2+\rho_1, \rho_1^2 + a_2 \rho_1 + a_1\right)}{s+R_j} \right] \\
&- \frac{\lambda_1 \lambda_2 q}{\mu_X^2 (\lambda_1+\lambda_2+\delta)^2} \widehat{\phi}_1\left(\frac{1}{\mu_X}\right) \\
&\cdot \Big[(\lambda_1 \nu \mu) \times ((\lambda_1+\lambda_2+\delta)(\nu+\rho_1) \\
&\qquad \times (\nu+\mu+\rho_1)(\nu+\rho_2) \\
&\qquad \times (\nu+\mu+\rho_2))^{-1} \\
&\quad \times \sum_{j=0}^{2} \frac{r_j (d_1, d_2, d_3)}{s+R_j} + \mu_X \frac{\nu \mu}{(\nu+\rho_1)(\nu+\mu+\rho_1)} \\
&\quad \times \sum_{j=0}^{2} \frac{r_j (1, 2\nu+\mu+\omega+\rho_1, \omega(2\nu+\mu+\rho_1))}{s+R_j} \Big],
\end{aligned} \tag{57}
$$

where

$$
\begin{aligned}
d_1 &= 2\nu + \mu + \rho_1 + \rho_2, \\
d_2 &= (2\nu+\mu+\omega) d_1 - \nu(\nu+\mu) + \rho_1 \rho_2, \\
d_3 &= \omega (d_2 - \omega d_1).
\end{aligned} \tag{58}
$$

Taking the inverse Laplace transforms, we can derive explicit expressions for $\phi(u)$:

$$
\begin{aligned}
&\phi(u) \\
&= \frac{1}{\mu_X^2} \Lambda(u) + \frac{1}{\mu_X^2} \sum_{i=1}^{2} h_i e^{-R_i u} \underline{*} \Lambda(u) \\
&\quad + \frac{\lambda_1 \lambda_2 q}{\mu_X^2 (\lambda_1+\lambda_2+\delta)^2} \widehat{\phi}\left(\frac{1}{\mu_X}\right) \\
&\quad \cdot \left[\frac{\lambda_1 \omega \nu \mu}{(\lambda_1+\lambda_2+\delta) g(\rho_1) g(\rho_2)} \right. \\
&\qquad \times \sum_{j=0}^{2} r_j (c_0, c_0 a_2 + c_1, c_0 a_1 + c_1 a_2 + c_2) e^{-R_j u} \\
&\qquad \left. + \mu_X \frac{\omega \nu \mu}{g(\rho_1)} \sum_{j=0}^{2} r_j \left(1, a_2+\rho_1, \rho_1^2 + a_2 \rho_1 + a_1\right) e^{-R_j u} \right] \\
&\quad - \frac{\lambda_1 \lambda_2 q}{\mu_X^2 (\lambda_1+\lambda_2+\delta)^2} \widehat{\phi}_1\left(\frac{1}{\mu_X}\right) \\
&\quad \cdot \Big[(\lambda_1 \nu \mu) \times ((\lambda_1+\lambda_2+\delta)(\nu+\rho_1) \\
&\qquad\qquad \times (\nu+\mu+\rho_1)(\nu+\rho_2)(\nu+\mu+\rho_2)) \\
&\qquad \times \sum_{j=0}^{2} r_j (d_1, d_2, d_3) e^{-R_j u} \\
&\qquad + \mu_X \frac{\nu \mu}{(\nu+\rho_1)(\nu+\mu+\rho_1)} \\
&\qquad \times \sum_{j=0}^{2} r_j (1, 2\nu+\mu+\omega+\rho_1, \omega(2\nu+\mu+\rho_1)) e^{-R_j u} \Big],
\end{aligned} \tag{59}
$$

where $\underline{*}$ denotes the operation of convolution which is different from the distribution functions convolution:

$$
\begin{aligned}
&\Lambda(u) \\
&\quad = \frac{\mu_X}{\lambda_1+\lambda_2+\delta} \\
&\qquad \times \left[(1-\mu_X \rho_1) T_{\rho_1} \omega(u) + \mu_X \omega(u)\right] \\
&\qquad - \frac{\lambda_1^2 \lambda_2 q}{(\lambda_1+\lambda_2+\delta)^4} \\
&\qquad \times \left[T_{\rho_2} T_{\rho_1} B_1(u) - T_{\rho_2} T_{\rho_1} B_2(u)\right] \\
&\qquad - \frac{\lambda_2 q \mu_X [2-\mu_X(\rho_1+\rho_2)]}{(\lambda_1+\lambda_2+\delta)^2} \\
&\qquad \times \left[T_{\rho_1} B_1(u) - T_{\rho_1} B_2(u)\right] \\
&\qquad - \frac{\lambda_2 q \mu_X^2}{(\lambda_1+\lambda_2+\delta)^2} \\
&\qquad \times [B_1(u) - B_2(u)].
\end{aligned} \tag{60}
$$

Equation (59) is the explicit expression for $\phi(u)$ with the case where both the claim sizes are exponentially distributed.

6. Conclusions

We have generalized the results in [11, 16]. It is assumed that the premium income process is a compound Poisson process; moreover, every main claim will produce a byclaim with a certain probability and the occurrence of the byclaim may be delayed depending on associated main claim amount. We not only derive the integral equation satisfied by the expected discounted penalty function, but also obtain the explicit expression for the Laplace transform of the expected discounted penalty function when the premium size is exponentially distributed. Finally, for the exponential claim sizes, we present the explicit formula for the expected discounted penalty function.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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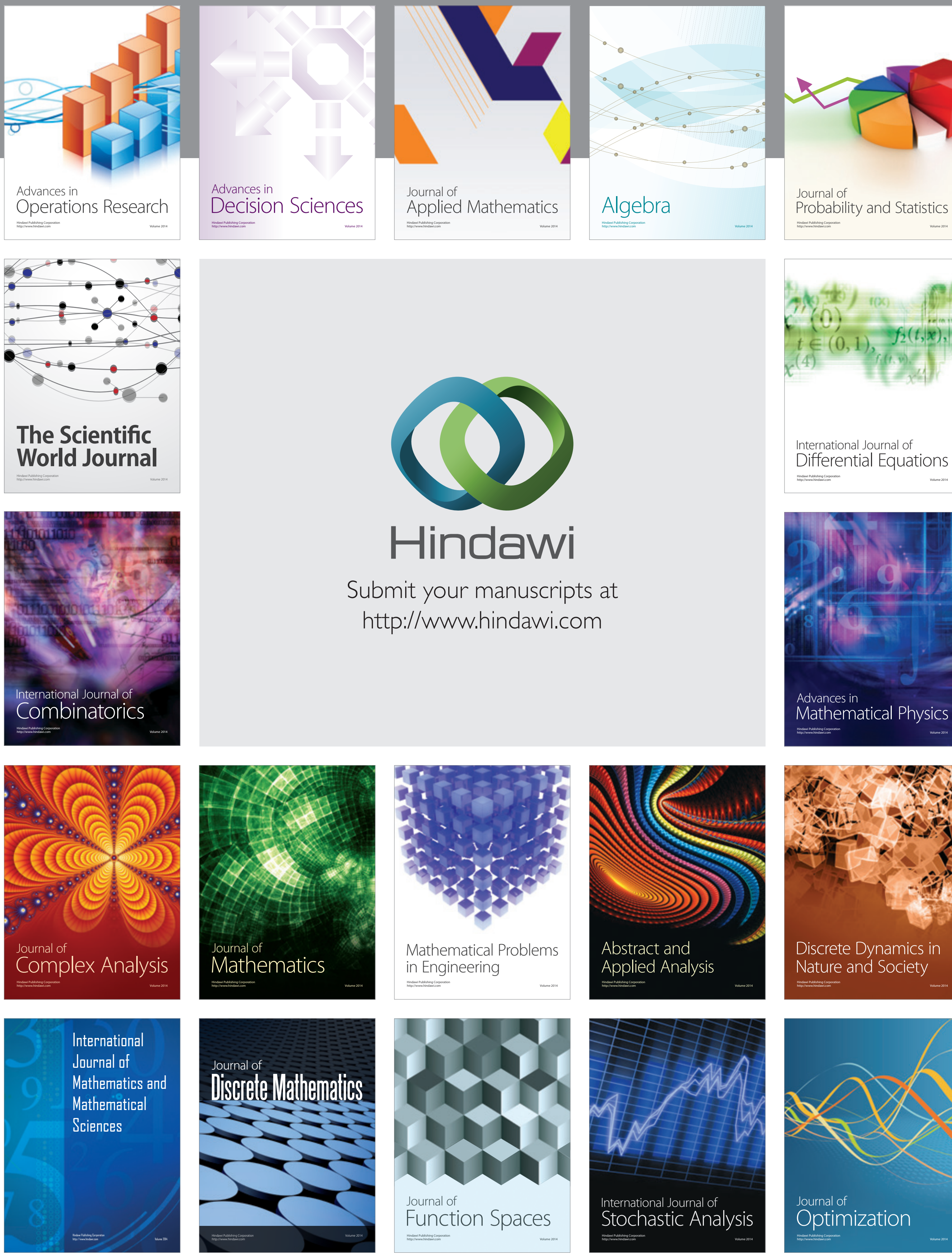
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