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Research Article

Interpolation by Hankel Translates of a Basis Function: Inversion Formulas and Polynomial Bounds

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For $\mu \geq -1/2$, the authors have developed elsewhere a scheme for interpolation by Hankel translates of a basis function Φ in certain spaces of continuous functions Y_n $(n \in \mathbb{N})$ depending on a weight w. The functions Φ and w are connected through the distributional identity $t^{4n}(h'_\mu\Phi)(t) = 1/w(t)$, where h'_μ denotes the generalized Hankel transform of order μ. In this paper, we use the projection operators associated with an appropriate direct sum decomposition of the Zemanian space $\mathscr{H}_\mu$ in order to derive explicit representations of the derivatives $S_\mu^m\Phi$ and their Hankel transforms, the former ones being valid when $m \in \mathbb{Z}_+$ is restricted to a suitable interval for which $S_\mu^m\Phi$ is continuous. Here, S_μ^m denotes the mth iterate of the Bessel differential operator S_μ if $m \in \mathbb{N}$, while S_μ^0 is the identity operator. These formulas, which can be regarded as inverses of generalizations of the equation $(h'_\mu\Phi)(t) = 1/t^{4n}w(t)$, will allow us to get some polynomial bounds for such derivatives. Corresponding results are obtained for the members of the interpolation space Y_n.

1. Introduction

The method of radial basis function interpolation has seen substantial developments, both theoretical and computational, and in applications; compare [1–3] and references therein. A radially symmetric function in Euclidean space $\mathbb{R}^d$ can be identified with a function on the positive real axis. The d-dimensional Fourier transform of a radial function is also radial and reduces to a 1-dimensional Hankel transform of order $d/2 - 1$ [4, Theorem 3.3]. The natural convolution structure in the positive real axis is not that of a group but is given by the family of the so-called Bessel-Kingman hypergroups, depending on a parameter $\mu \geq -1/2$. The Kingman convolution is defined upon a generalized translation operator, known as Delsarte translation, and for $\mu = -1/2$ coincides with the standard one. Recently [5, 6], the authors have benefited from the hypergroup structure in order to provide a new approach to the problem of radial basis function interpolation, which extends the usual scheme. Such an approach yields a greater variety of manageable kernels, which could be useful in handling mathematical models built upon classes of radial basis functions depending on the order μ and whose performance is expected to improve by suitably adjusting μ, as it happens, for instance, with the family of Matérn kernels in [7, Supplement, page 6]. The examples and numerical experiments included in [5] seem to support this view.

Our scheme actually considers a variant of the Delsarte translation, the so-called Hankel translation, in order to accommodate the usual definition of the Hankel integral transformation, namely,

$$\left(h_\mu\varphi\right)(x) = \int_0^\infty \varphi(t)\,\mathscr{J}_\mu(xt)\,dt \quad (x \in I), \tag{1}$$

where $I =]0, \infty[$, $\mathscr{J}_\mu(z) = z^{1/2}J_\mu(z)$ $(z \in I)$ and J_μ denotes the Bessel function of the first kind and order $\mu \in \mathbb{R}$.

Aiming to define the Hankel transformation in spaces of distributions, Zemanian [8] introduced the space $\mathscr{H}_\mu$ of all those smooth, complex-valued functions $\varphi = \varphi(x)$ $(x \in I)$ such that

$$\nu_{\mu,r}(\varphi) = \max_{0\le k\le r} \sup_{x\in I} \left|\left(1+x^2\right)^r \left(x^{-1}D\right)^k x^{-\mu-1/2}\varphi(x)\right| < \infty \quad (r \in \mathbb{Z}_+). \tag{2}$$

Here, and in the sequel, $D = D_x = d/dx$ and $(x^{-1}D)^k$ is the operator $x^{-1}D$ iterated k times ($k \in \mathbb{N}$) or the identity operator ($k = 0$). When topologized by the family of norms $\{\nu_{\mu,r}\}_{r\in\mathbb{Z}_+}$, $\mathscr{H}_\mu$ becomes a Fréchet space where h_μ is an automorphism provided that $\mu \ge -1/2$. Then the generalized Hankel transformation h'_μ, defined by transposition on the dual $\mathscr{H}'_\mu$ of $\mathscr{H}_\mu$, is an automorphism of $\mathscr{H}'_\mu$ when this latter space is endowed with either its weak* or its strong topology.

The space $\mathscr{O}$ consists of all those smooth, complex-valued functions θ on I such that $\theta\psi \in \mathscr{H}_\mu$ whenever $\psi \in \mathscr{H}_\mu$ and the linear operator $\psi \mapsto \theta\psi$ is a continuous mapping of $\mathscr{H}_\mu$ into itself. This $\mathscr{O}$ is also the space of multipliers of $\mathscr{H}'_\mu$, the corresponding multiplication operators being defined by transposition [9].

Denote by $L^1_{\mu,l}$ the class of all those Lebesgue measurable functions $u = u(t)$ $(t \in I)$ such that

$$\int_0^a |u(t)|\, t^{\mu+1/2} dt < \infty \quad (a > 0). \tag{3}$$

The following spaces were introduced in [5].

Definition 1. Let $w = w(t) > 0$ $(t \in I)$ be a continuous function, let

$$S_\mu = S_{\mu,t} = t^{-\mu-1/2} D_t t^{2\mu+1} D_t t^{-\mu-1/2} \tag{4}$$

be the Bessel differential operator, and let

$$Y_n = \left\{ f \in \mathscr{H}'_\mu : h'_\mu S^n_\mu f \in L^1_{\mu,l} \cap L^2_{\mu,w} \right\} \quad (n \in \mathbb{Z}_+), \tag{5}$$

where S^0_μ is the identity operator, S^n_μ $(n \in \mathbb{N})$ is the operator S_μ iterated n times, and $L^2_{\mu,w}$ stands for the class of all measurable functions $u = u(t)$ $(t \in I)$ satisfying

$$\|u\|_{\mu,w} = \left(\int_0^\infty |u(t)|^2 w(t)\, t^{\mu+1/2} dt \right)^{1/2} < \infty. \tag{6}$$

A seminorm (norm if $n = 0$) is defined on Y_n by setting

$$\begin{aligned} |f|_n &= \left\| h'_\mu S^n_\mu f \right\|_{\mu,w} \\ &= \left(\int_0^\infty \left| \left(h'_\mu S^n_\mu f\right)(t) \right|^2 w(t) t^{\mu+1/2} dt \right)^{1/2} \quad (f \in Y_n). \end{aligned} \tag{7}$$

The function w will be called a weight function for Y_n.

In [5], for $n \in \mathbb{N}$ and suitable conditions on the weight w related to the values of n, the spaces Y_n were shown to consist of continuous functions on I. Also, interpolants to $f \in Y_n$ of the form

$$(Uf)(x) = \sum_{i=1}^m \alpha_i \left(\tau_{a_i}\Phi\right)(x) + \sum_{j=0}^{n-1} \beta_j p_{\mu,j}(x) \quad (x \in I) \tag{8}$$

were obtained, where $\{a_1, \ldots, a_m\} \subset I$ is the set of interpolation points; $\Phi \in \mathscr{H}'_\mu$ is a complex function defined on I (the so-called basis function), connected with w through the distributional identity

$$t^{4n} \left(h'_\mu \Phi\right)(t) = \frac{1}{w(t)}; \tag{9}$$

$p_{\mu,j}(x) = x^{2j+\mu+1/2}$ $(j \in \mathbb{Z}_+,\ 0 \le j \le n-1)$ are Müntz monomials; τ_z $(z \in I)$ denotes the Hankel translation operator of order μ; and α_i, β_j $(i, j \in \mathbb{Z}_+,\ 1 \le i \le m,\ 0 \le j \le n-1)$ are complex coefficients.

In [5] the regularity results for the basis distribution Φ [5, Theorem 4.4] and the members of Y_n [5, Theorems 3.2 and 3.6] were achieved with the aid of the Lagrange interpolation projector onto the space of Müntz polynomials

$$\begin{aligned} \pi_{\mu,n-1} = \operatorname{span} \big\{ p_{\mu,j}(t) = t^{2j+\mu+1/2} \ (t \in I) : j \in \mathbb{Z}_+, \\ 0 \le j \le n-1 \big\}. \end{aligned} \tag{10}$$

In this paper we use, instead, the projectors associated with a suitable direct sum decomposition $\mathscr{H}_\mu = \mathscr{H}_{\mu,n} \oplus \Pi_{\mu,n}(\rho)$ (which will be described in Section 2) to obtain conditions guaranteeing the regularity of Φ, the distributions in Y_n, and their S^m_μ-derivatives. In spite of the conditions obtained being stronger than those in [5], this new approach has the advantage of providing an explicit representation of these functions, their S^m_μ-derivatives, and their Hankel transforms, along with some polynomial bounds. The formulas for $S^m_\mu\Phi$ hold when $m \in \mathbb{Z}_+$ ranges over a suitable interval and may be considered as inverses of generalizations of the equation

$$\left(h'_\mu \Phi\right)(t) = \frac{1}{t^{4n} w(t)}, \tag{11}$$

valid on $\mathscr{H}'_{\mu,2n}$.

The paper is organized as follows. In Section 2 we introduce the direct sum decomposition $\mathscr{H}_\mu = \mathscr{H}_{\mu,n} \oplus \Pi_{\mu,n}(\rho)$ along with the projection operators P_n onto $\Pi_{\mu,n}(\rho)$ and Q_n onto $\mathscr{H}_{\mu,n}$. Section 3 is devoted to studying the properties of their adjoints P'_n and Q'_n. Our main results are in Section 5, where a Hankel inversion formula is presented in a general setting and then specialized to basis distributions and members of the interpolation space Y_n. Section 4 contains some auxiliary results of a rather technical nature.

Throughout the rest of this paper $n \in \mathbb{N}$ will be fixed. The positive real axis will be always denoted by I, while μ will stand for a real number not less than $-1/2$ and C will represent a suitable positive constant, depending only on the opportune subscripts (if any), whose value may vary from line to line. Furthermore, we will adhere to the notations

$\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$ for the set of nonnegative integers and $\mathcal{J}_\mu(z) = z^{1/2} J_\mu(z)$ $(z \in I)$ for the function giving the kernel of the Hankel transformation h_μ. The symbol $\mathscr{C}$ (resp., $\mathscr{C}^{n-1}$) will denote the space of all continuous (resp., of class $n-1$) functions on I (note that $\mathscr{C} = \mathscr{C}^0$). For the operational rules of the Hankel transformation that eventually might be required, both in the classical and the generalized senses, the reader is mainly referred to [10].

2. The Operators P_n and Q_n

In this section, we introduce the direct sum decomposition $\mathscr{H}_\mu = \mathscr{H}_{\mu,n} \oplus \Pi_{\mu,n}(\rho)$ along with the projectors P_n onto $\Pi_{\mu,n}(\rho)$ and Q_n onto $\mathscr{H}_{\mu,n}$. The main properties of these projectors are established.

We begin by recalling some definitions and results from [5] which will be needed in the sequel.

Proposition 2. *Assume $\varphi \in \mathscr{H}_\mu$. Then, for every $m \in \mathbb{Z}_+$ one has*

$$x^{-\mu-1/2}\varphi(x) = \sum_{j=0}^{m} a_{2j} x^{2j} + R_{2m}(x) \quad (x \in I), \tag{12}$$

where

$$a_{2j} = \frac{1}{2^j j!} \lim_{z \to 0+} \left(z^{-1}D\right)^j z^{-\mu-1/2}\varphi(z) \qquad \left(j \in \mathbb{Z}_+,\ 0 \le j \le m\right), \tag{13}$$

and the remainder term satisfies

$$\left|R_{2m}(x)\right| \le C_m x^{2m+1} \times \max_{0\le k\le m} \sup_{z\in I} \left| z^{2(m-k)+1} \left(z^{-1}D\right)^{2m-k+1} z^{-\mu-1/2}\varphi(z)\right| \quad (x \in I) \tag{14}$$

for some $C_m > 0$.

Definition 3. Let

$$\mathscr{H}_{\mu,n} = \left\{\varphi \in \mathscr{H}_\mu : \lim_{x\to 0+} \left(x^{-1}D\right)^j x^{-\mu-1/2}\varphi(x) = 0 \quad \left(j \in \mathbb{Z}_+,\ 0 \le j \le n-1\right)\right\}. \tag{15}$$

The space $\mathscr{H}_{\mu,n}$ is endowed with the topology inherited from that of $\mathscr{H}_\mu$.

In view of Proposition 2, loosely speaking, one can say that $\mathscr{H}_{\mu,n}$ consists of all those $\varphi \in \mathscr{H}_\mu$ such that $x^{-\mu-1/2}\varphi(x)$ has a Maclaurin series expansion starting at x^{2n}.

Definition 4. For $j \in \mathbb{Z}_+$, the distribution $\Lambda_j \in \mathscr{H}'_\mu$ is defined by

$$\left\langle \Lambda_j, \varphi \right\rangle = (-1)^j c_{\mu,j} \lim_{x\to 0+} \left(x^{-1}D\right)^j x^{-\mu-1/2}\varphi(x) \qquad \left(\varphi \in \mathscr{H}_\mu\right), \tag{16}$$

where $c_{\mu,j} = 2^{\mu+j}\Gamma(\mu + j + 1)$.

Theorem 5. *The following hold:*

(i) *Given $j \in \mathbb{Z}_+$, one has $h'_\mu \Lambda_j = p_{\mu,j}$, where $p_{\mu,j}(t) = t^{2j+\mu+1/2}$ $(t \in I)$.*

(ii) *The kernel of the operator S^n_μ in $\mathscr{H}'_\mu$ is $\pi_{\mu,n-1}$.*

Next we introduce new spaces and mappings.

Definition 6. By $\mathscr{V}_{\mu,n}$ we denote the space of all complex-valued functions $\phi \in \mathscr{C}^{n-1}$ such that the limit

$$\lim_{x\to 0+} \left(x^{-1}D\right)^j x^{-\mu-1/2}\phi(x) \tag{17}$$

exists for all $j \in \mathbb{Z}_+$, $0 \le j \le n-1$.

Note that the functionals Λ_j $(j \in \mathbb{Z}_+,\ 0 \le j \le n-1)$ are well defined on $\mathscr{V}_{\mu,n}$.

Definition 7. The space $\mathscr{H}_{\mu,n,*}$ consists of all those $\rho \in \mathscr{H}_\mu$ such that

$$\lim_{x\to 0+} x^{-\mu-1/2}\rho(x) = 1, \qquad \lim_{x\to 0+} \left(x^{-1}D\right)^j x^{-\mu-1/2}\rho(x) = 0 \quad \left(j \in \mathbb{N},\ 1 \le j \le n-1\right). \tag{18}$$

Given $\rho \in \mathscr{H}_{\mu,n,*}$, we set

$$\Pi_{\mu,n}(\rho) = \left\{x^{-\mu-1/2}\rho(x)\, p(x) : p \in \pi_{\mu,n-1}\right\}. \tag{19}$$

Definition 8. Let $\rho \in \mathscr{H}_{\mu,n,*}$ be fixed. The mappings $P_n : \mathscr{V}_{\mu,n} \to \mathscr{V}_{\mu,n}$ and $Q_n = \mathbf{I} - P_n : \mathscr{V}_{\mu,n} \to \mathscr{V}_{\mu,n}$ ($\mathbf{I}$ the identity operator) are, respectively, defined by

$$\left(P_n\varphi\right)(x) = \rho(x) \sum_{j=0}^{n-1} \left\langle L_j, \varphi\right\rangle x^{2j} \quad \left(\varphi \in \mathscr{V}_{\mu,n},\ x \in I\right),$$
$$\left(Q_n\varphi\right)(x) = \varphi(x) - \rho(x) \sum_{j=0}^{n-1} \left\langle L_j, \varphi\right\rangle x^{2j} \quad \left(\varphi \in \mathscr{V}_{\mu,n},\ x \in I\right), \tag{20}$$

where

$$L_j = \frac{(-1)^j}{2^j j! c_{\mu,j}} \Lambda_j \quad \left(j \in \mathbb{Z}_+,\ 0 \le j \le n-1\right). \tag{21}$$

The next theorem proves some useful properties of the operators P_n and Q_n and also clarifies the relationship between $\mathscr{H}_\mu$ and $\mathscr{H}_{\mu,n}$. In what follows we will adopt the usual notation $\mathscr{N}(P)$ and $\mathscr{R}(P)$ for the kernel and range of a linear operator P. Recall that a projector or projection P of a vector space X is a linear transformation $P : X \to X$ which is idempotent, meaning that $P^2 = P$.

Theorem 9. *Let P_n, Q_n be as in Definition 8.*

(i) *P_n and Q_n are continuous linear mappings from $\mathscr{H}_\mu$ into $\mathscr{H}_\mu$.*

(ii) *P_n and Q_n are projections into $\mathscr{H}_\mu$ which satisfy*

$$\mathscr{N}(P_n) = \mathscr{R}(Q_n) = \mathscr{H}_{\mu,n}, \qquad \mathscr{R}(P_n) = \mathscr{N}(Q_n) = \Pi_{\mu,n}(\rho). \tag{22}$$

(iii) $\mathscr{H}_\mu = \mathscr{H}_{\mu,n} \oplus \Pi_{\mu,n}(\rho)$.

(iv) *Finally,*

$$\begin{aligned}\mathscr{H}_\mu &= h_\mu(\mathscr{H}_{\mu,n}) \oplus h_\mu[\Pi_{\mu,n}(\rho)] \\ &= h_\mu(\mathscr{H}_{\mu,n}) \\ &\quad \oplus \left\{ \sum_{j=0}^{n-1} a_j(-S_\mu)^j(h_\mu\rho) \right. \\ &\qquad \left. (a_j \in \mathbb{C},\ j \in \mathbb{Z}_+,\ 0 \le j \le n-1) \right\}.\end{aligned} \tag{23}$$

Proof. To prove (i), let $\varphi \in \mathscr{H}_\mu$ and let $r, m \in \mathbb{Z}_+$, with $0 \le m \le r$. Then

$$\begin{aligned}&(1+x^2)^r(x^{-1}D)^m x^{-\mu-1/2}(P_n\varphi)(x) \\ &= (1+x^2)^r(x^{-1}D)^m \left[x^{-\mu-1/2}\rho(x) \sum_{j=0}^{n-1} \langle L_j, \varphi\rangle x^{2j} \right] \\ &= \sum_{j=0}^{n-1} \langle L_j, \varphi\rangle (1+x^2)^r(x^{-1}D)^m x^{-\mu-1/2}x^{2j}\rho(x) \\ &\qquad (x \in I).\end{aligned} \tag{24}$$

Since $\rho \in \mathscr{H}_\mu$ and all even polynomials lie in $\mathscr{O}$ [10, Lemma 5.3-1], we may write

$$\begin{aligned}&\sup_{x\in I} \left| (1+x^2)^r(x^{-1}D)^m x^{-\mu-1/2}(P_n\varphi)(x) \right| \\ &\le \sum_{j=0}^{n-1} \sup_{z\in I} \left| (1+z^2)^r(z^{-1}D)^m z^{-\mu-1/2}z^{2j}\rho(z) \right| \\ &\quad \times \left| \langle L_j, \varphi\rangle \right| \quad (\varphi \in \mathscr{H}_\mu).\end{aligned} \tag{25}$$

Now it suffices to take into account that $L_j \in \mathscr{H}'_\mu$ ($j \in \mathbb{Z}_+$, $0 \le j \le n-1$).

Next, let $\varphi \in \mathscr{H}_\mu$. A simple manipulation yields

$$\begin{aligned}(P_n^2\varphi)(x) &= \rho(x) \sum_{j=0}^{n-1} \langle L_j, P_n\varphi\rangle x^{2j} \\ &= \rho(x) \sum_{j=0}^{n-1}\sum_{k=0}^{n-1} \langle L_{j,z}, z^{2k}\rho(z)\rangle \langle L_k, \varphi\rangle x^{2j} \\ &\qquad (x \in I).\end{aligned} \tag{26}$$

Since $\rho \in \mathscr{H}_{\mu,n,*}$, for $j, k \in \mathbb{Z}_+$, $0 \le j,\ k \le n-1$, we have

$$\begin{aligned}\langle L_{j,z}, z^{2k}\rho(z)\rangle &= \frac{1}{2^j j!} \lim_{z\to 0+} (z^{-1}D)^j z^{-\mu-1/2}z^{2k}\rho(z) \\ &= \frac{1}{2^j j!} \sum_{i=0}^{j} \binom{j}{i} \lim_{z\to 0+} \left[(z^{-1}D)^i z^{2k} \right] \\ &\quad \times \lim_{z\to 0+} \left[(z^{-1}D)^{j-i} z^{-\mu-1/2}\rho(z) \right] \\ &= \frac{1}{2^j j!} \lim_{z\to 0+} (z^{-1}D)^j z^{2k} \\ &= \frac{2^k k!}{2^j j! \cdot 2^{k-j}(k-j)!} \lim_{z\to 0+} z^{2(k-j)} \\ &= \begin{cases} 1, & k = j \\ 0, & k \ne j. \end{cases}\end{aligned} \tag{27}$$

Plugging (27) into (26),

$$(P_n^2\varphi)(x) = \rho(x) \sum_{j=0}^{n-1} \langle L_j, \varphi\rangle x^{2j} = (P_n\varphi)(x) \quad (x \in I). \tag{28}$$

Therefore, P_n is a projection, and hence so is Q_n.

Let us show that $P_n : \mathscr{H}_\mu \to \Pi_{\mu,n}(\rho)$ is onto. If $\varphi(x) = x^{-\mu-1/2}\rho(x)p(x) \in \Pi_{\mu,n}(\rho)$, with $p \in \pi_{\mu,n-1}$, then $\varphi \in \mathscr{H}_\mu$ and, using (27), it is easily seen that $P_n\varphi = \varphi$.

In view of Definition 8, it is apparent that $\varphi \in \mathscr{H}_{\mu,n}$ implies $P_n\varphi = 0$, so that $\mathscr{H}_{\mu,n} \subset \mathscr{N}(P_n)$. To prove the reverse inclusion, take $\varphi \in \mathscr{N}(P_n)$. Then

$$\rho(x) \sum_{j=0}^{n-1} \langle L_j, \varphi\rangle x^{2j} = 0 \quad (x \in I). \tag{29}$$

As $\rho \in \mathscr{H}_{\mu,n,*}$ we have $\lim_{x\to 0+} x^{-\mu-1/2}\rho(x) = 1$, and hence there exists $\delta > 0$ such that $x^{-\mu-1/2}\rho(x) > 0$ ($0 < x < \delta$). Consequently,

$$\rho(x) \sum_{j=0}^{n-1} \langle L_j, \varphi\rangle x^{2j} = 0 \quad (0 < x < \delta) \tag{30}$$

forces

$$\left\langle L_j, \varphi \right\rangle = 0 \quad \left(j \in \mathbb{Z}_+,\ 0 \leq j \leq n-1\right). \qquad (31)$$

From (21) we conclude that $\varphi \in \mathscr{H}_{\mu,n}$.

Since $Q_n = \mathbf{I} - P_n$, we also have

$$\mathscr{N}\left(Q_n\right) = \mathscr{R}\left(P_n\right) = \Pi_{\mu,n}\left(\rho\right), \qquad \mathscr{R}\left(Q_n\right) = \mathscr{N}\left(P_n\right) = \mathscr{H}_{\mu,n}. \qquad (32)$$

With the map P_n being a projection of $\mathscr{H}_\mu$, we may write

$$\mathscr{H}_\mu = \mathscr{N}\left(P_n\right) \oplus \mathscr{R}\left(P_n\right) = \mathscr{H}_{\mu,n} \oplus \Pi_{\mu,n}\left(\rho\right). \qquad (33)$$

From the fact that h_μ is an automorphism of $\mathscr{H}_\mu$ we infer

$$\mathscr{H}_\mu = h_\mu\left(\mathscr{H}_\mu\right) = h_\mu\left(\mathscr{H}_{\mu,n}\right) \oplus h_\mu\left[\Pi_{\mu,n}\left(\rho\right)\right]. \qquad (34)$$

Let $p \in \pi_{\mu,n-1}$, $p(x) = \sum_{j=0}^{n-1} a_j x^{2j+\mu+1/2}$ $(x \in I,\ a_j \in \mathbb{C},\ j \in \mathbb{Z}_+,\ 0 \leq j \leq n-1)$, and consider $x^{-\mu-1/2}\rho(x)p(x) \in \Pi_{\mu,n}(\rho)$. Then

$$x^{-\mu-1/2}\rho\left(x\right)p\left(x\right) = \sum_{j=0}^{n-1} a_j x^{2j}\rho\left(x\right), \qquad (35)$$

so that

$$h_\mu\left[x^{-\mu-1/2}\rho\left(x\right)p\left(x\right)\right] = \sum_{j=0}^{n-1} a_j\left(-S_\mu\right)^j\left(h_\mu\rho\right). \qquad (36)$$

We thus conclude

$$h_\mu\left[\Pi_{\mu,n}\left(\rho\right)\right] = \left\{\sum_{j=0}^{n-1} a_j\left(-S_\mu\right)^j\left(h_\mu\rho\right) \quad \left(a_j \in \mathbb{C},\ j \in \mathbb{Z}_+,\ 0 \leq j \leq n-1\right)\right\}. \qquad (37)$$

This completes the proof. □

3. The Distribution Adjoints of P_n and Q_n

This section is devoted to studying the definition and properties of the distribution adjoints of P_n and Q_n. A new space of functions must be introduced first.

Definition 10. Set

$$\mathscr{K}_{\mu,n} = \left\{\varphi \in \mathscr{C}^{n-1} : \lim_{x \to 0+}\left(x^{-1}D\right)^j\varphi\left(x\right) = 0 \quad \left(j \in \mathbb{Z}_+,\ 0 \leq j \leq n-1\right)\right\}. \qquad (38)$$

Definition 11. The adjoints $P_n', Q_n' : \mathscr{H}_\mu' \to \mathscr{H}_\mu'$ of $P_n, Q_n : \mathscr{H}_\mu \to \mathscr{H}_\mu$ are defined by transposition:

$$\begin{aligned}\left\langle P_n'u, \varphi\right\rangle &= \left\langle u, P_n\varphi\right\rangle \quad \left(u \in \mathscr{H}_\mu',\ \varphi \in \mathscr{H}_\mu\right),\\ \left\langle Q_n'u, \varphi\right\rangle &= \left\langle u, Q_n\varphi\right\rangle \quad \left(u \in \mathscr{H}_\mu',\ \varphi \in \mathscr{H}_\mu\right).\end{aligned} \qquad (39)$$

Theorem 12. *The operators P_n' and Q_n' have the following properties:*

(i) *P_n' and Q_n' are projections and $P_n' + Q_n' = \mathbf{I}'$, the identity on $\mathscr{H}_\mu'$.*

(ii) *If $u \in \mathscr{H}_\mu'$, then*

$$P_n'u = \sum_{j=0}^{n-1} b_j\left(u\right)L_j, \qquad (40)$$

where $b_j(u) = \langle u_x, x^{2j}\rho(x)\rangle$ $(j \in \mathbb{Z}_+,\ 0 \leq j \leq n-1)$.

(iii) *$\mathscr{R}(P_n') = \mathscr{N}(Q_n') = h_\mu'(\pi_{\mu,n-1}) = \operatorname{span}\{\Lambda_j : j \in \mathbb{Z}_+,\ 0 \leq j \leq n-1\}$.*

(iv) *$\mathscr{N}(P_n') = \mathscr{R}(Q_n') = \{u \in \mathscr{H}_\mu' : \langle u_x, x^{2j}\rho(x)\rangle = 0\ (j \in \mathbb{Z}_+,\ 0 \leq j \leq n-1)\} = \Pi_{\mu,n}^{\perp}(\rho)$.*

(v) *$\mathscr{H}_\mu' = \Pi_{\mu,n}^{\perp}(\rho) \oplus \operatorname{span}\{\Lambda_j : j \in \mathbb{Z}_+,\ 0 \leq j \leq n-1\}$.*

(vi) *$\psi u = \psi(Q_n'u)$ whenever $u \in \mathscr{H}_\mu'$ and $\psi \in \mathscr{O} \cap \mathscr{K}_{\mu,n}$.*

Proof. Part (i) is a direct consequence of P_n and Q_n being projections of $\mathscr{H}_\mu$, with $P_n + Q_n = \mathbf{I}$ on $\mathscr{H}_\mu$:

$$\begin{aligned}\left\langle\left(P_n' + Q_n'\right)u, \varphi\right\rangle &= \left\langle u, \left(P_n + Q_n\right)\varphi\right\rangle\\ &= \left\langle u, \mathbf{I}\varphi\right\rangle = \left\langle \mathbf{I}'u, \varphi\right\rangle \quad \left(\varphi \in \mathscr{H}_\mu\right).\end{aligned} \qquad (41)$$

To establish (ii), let $u \in \mathscr{H}_\mu'$. Then

$$\begin{aligned}\left\langle P_n'u, \varphi\right\rangle &= \left\langle u, P_n\varphi\right\rangle = \left\langle u_x, \rho\left(x\right)\sum_{j=0}^{n-1}\left\langle L_j, \varphi\right\rangle x^{2j}\right\rangle\\ &= \sum_{j=0}^{n-1}\left\langle L_j, \varphi\right\rangle\left\langle u_x, x^{2j}\rho\left(x\right)\right\rangle = \sum_{j=0}^{n-1} b_j\left(u\right)\left\langle L_j, \varphi\right\rangle\\ &= \left\langle \sum_{j=0}^{n-1} b_j\left(u\right)L_j, \varphi\right\rangle \quad \left(\varphi \in \mathscr{H}_\mu\right),\end{aligned} \qquad (42)$$

where $b_j(u) = \langle u_x, x^{2j}\rho(x)\rangle$ $(j \in \mathbb{Z}_+,\ 0 \leq j \leq n-1)$.

Since, distributionally, $(h_\mu'\Lambda_j)(x) = x^{2j+\mu+1/2}$ for all $j \in \mathbb{Z}_+$, $0 \leq j \leq n-1$ (Theorem 5), from (ii) and (21) it follows that

$$\mathscr{R}\left(P_n'\right) \subset h_\mu'\left(\pi_{\mu,n-1}\right). \qquad (43)$$

Next we prove

$$h_\mu'\left(\pi_{\mu,n-1}\right) \subset \mathscr{N}\left(Q_n'\right). \qquad (44)$$

Take $u \in h_\mu'(\pi_{\mu,n-1})$, so that

$$u = \sum_{j=0}^{n-1} a_j\Lambda_j \quad \left(a_j \in \mathbb{C},\ j \in \mathbb{Z}_+,\ 0 \leq j \leq n-1\right), \qquad (45)$$

and let $\varphi \in \mathscr{H}_\mu$. Then $Q_n\varphi \in \mathscr{H}_{\mu,n}$ (Theorem 9) implies

$$\left\langle Q_n' u, \varphi \right\rangle = \left\langle u, Q_n\varphi \right\rangle = \sum_{j=0}^{n-1} a_j \left\langle \Lambda_j, Q_n\varphi \right\rangle = 0. \quad (46)$$

As $\mathscr{R}(P_n') = \mathscr{N}(Q_n')$, by virtue of (43) and (44) we may write

$$\mathscr{R}\left(P_n'\right) = \mathscr{N}\left(Q_n'\right) = h_\mu'\left(\pi_{\mu,n-1}\right). \quad (47)$$

On the other hand, given $u \in \mathscr{H}_\mu'$ we have $P_n'u = 0$ if, and only if, $h_\mu'(P_n'u) = 0$, or, from (ii) and (21),

$$\begin{aligned}\sum_{j=0}^{n-1} b_j(u)\, h_\mu' L_j &= \sum_{j=0}^{n-1} \frac{(-1)^j b_j(u)}{2^j j! c_{\mu,j}} h_\mu' \Lambda_j \\ &= \sum_{j=0}^{n-1} \frac{(-1)^j b_j(u)}{2^j j! c_{\mu,j}} x^{2j+\mu+1/2} = 0.\end{aligned} \quad (48)$$

This happens if, and only if, $\langle u_x, x^{2j}\rho(x)\rangle = b_j(u) = 0$ ($j \in \mathbb{Z}_+$, $0 \le j \le n-1$). Therefore,

$$\begin{aligned}R\left(Q_n'\right) = \mathscr{N}\left(P_n'\right) = &\left\{u \in \mathscr{H}_\mu' : \left\langle u_x, x^{2j}\rho(x)\right\rangle = 0\right. \\ &\left.\left(j \in \mathbb{Z}_+,\ 0 \le j \le n-1\right)\right\}.\end{aligned} \quad (49)$$

From the identity $P_n' = \mathbf{I}' - Q_n'$ we arrive at

$$\begin{aligned}\mathscr{H}_\mu' &= \mathscr{N}\left(P_n'\right) \oplus \mathscr{R}\left(P_n'\right) \\ &= \left\{u \in \mathscr{H}_\mu' : \left\langle u_x, x^{2j}\rho(x)\right\rangle = 0\right. \\ &\quad \left.\left(j \in \mathbb{Z}_+,\ 0 \le j \le n-1\right)\right\} \oplus h_\mu'\left(\pi_{\mu,n-1}\right).\end{aligned} \quad (50)$$

To complete the proof, let $\varphi \in \mathscr{H}_\mu$ and $\psi \in \mathscr{O} \cap \mathscr{K}_{\mu,n}$. The Leibniz rule ensures that $\psi\varphi \in \mathscr{H}_{\mu,n} = \mathscr{R}(Q_n)$, so that $\psi\varphi = Q_n(\psi\varphi)$ (Theorem 9). Thus, for any $u \in \mathscr{H}_\mu'$ we may write

$$\begin{aligned}\langle \psi u, \varphi\rangle &= \langle u, \psi\varphi\rangle = \langle u, Q_n(\psi\varphi)\rangle \\ &= \left\langle Q_n' u, \psi\varphi\right\rangle = \left\langle \psi\left(Q_n' u\right), \varphi\right\rangle.\end{aligned} \quad (51)$$

The arbitrariness of $\varphi \in \mathscr{H}_\mu$ leads us to conclude that $\psi u = \psi(Q_n' u)$ as distributions over $\mathscr{H}_\mu$. □

4. Auxiliary Results

Here we prove two auxiliary lemmas.

Lemma 13. *For $\psi \in \mathscr{H}_\mu$, there holds*

$$\begin{aligned}P_n\left(h_\mu\psi\right)(t) &= \int_0^\infty P_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\,\psi(x)\,dx \quad (t \in I), \\ Q_n\left(h_\mu\psi\right)(t) &= \int_0^\infty Q_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\,\psi(x)\,dx \quad (t \in I).\end{aligned} \quad (52)$$

Proof. Let $\psi \in \mathscr{H}_\mu$ and fix $x \in I$. According to Definition 8, we have

$$P_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t) = \rho(t)\sum_{j=0}^{n-1}\left\langle L_{j,\xi}, \mathscr{J}_\mu(x\xi)\right\rangle t^{2j} \quad (t \in I), \quad (53)$$

where

$$\begin{aligned}\left\langle L_{j,\xi}, \mathscr{J}_\mu(x\xi)\right\rangle &= \frac{(-1)^j}{2^j j! c_{\mu,j}}\left\langle \Lambda_{j,\xi}, \mathscr{J}_\mu(x\xi)\right\rangle \\ &= \frac{1}{2^j j!}\lim_{\xi\to 0+}\left(\xi^{-1}D_\xi\right)^j \xi^{-\mu-1/2}(x\xi)^{1/2}J_\mu(x\xi) \\ &= \frac{x^{\mu+1/2}}{2^j j!}\lim_{\xi\to 0+}\left(\xi^{-1}D_\xi\right)^j (x\xi)^{-\mu}J_\mu(x\xi) \\ &= \frac{(-1)^j x^{2j+\mu+1/2}}{2^j j!}\lim_{\xi\to 0+}(x\xi)^{-\mu-j}J_{\mu+j}(x\xi) \\ &= \frac{(-1)^j x^{2j+\mu+1/2}}{2^j j! c_{\mu,j}}.\end{aligned} \quad (54)$$

Hence,

$$P_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t) = \rho(t)\sum_{j=0}^{n-1}\frac{(-1)^j x^{2j+\mu+1/2}}{2^j j! c_{\mu,j}} t^{2j} \quad (t \in I). \quad (55)$$

Now,

$$\begin{aligned}P_n\left(h_\mu\psi\right)(t) &= \rho(t)\sum_{j=0}^{n-1}\left\langle L_j, h_\mu\psi\right\rangle t^{2j} \\ &= \rho(t)\sum_{j=0}^{n-1}\frac{(-1)^j}{2^j j! c_{\mu,j}}\left\langle \Lambda_j, h_\mu\psi\right\rangle t^{2j} \\ &= \rho(t)\sum_{j=0}^{n-1}\frac{(-1)^j}{2^j j! c_{\mu,j}}\left\langle h_\mu'\Lambda_j, \psi\right\rangle t^{2j} \\ &= \rho(t)\sum_{j=0}^{n-1}\frac{(-1)^j t^{2j}}{2^j j! c_{\mu,j}}\int_0^\infty \psi(x)\, x^{2j+\mu+1/2}dx \\ &= \int_0^\infty\left[\rho(t)\sum_{j=0}^{n-1}\frac{(-1)^j x^{2j+\mu+1/2}}{2^j j! c_{\mu,j}} t^{2j}\right]\psi(x)\,dx \\ &= \int_0^\infty P_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\,\psi(x)\,dx \quad (t \in I).\end{aligned} \quad (56)$$

Consequently

$$\begin{aligned}Q_n\left(h_\mu\psi\right)(t) &= \left(h_\mu\psi\right)(t) - P_n\left(h_\mu\psi\right)(t) \\ &= \int_0^\infty \mathscr{J}_\mu(xt)\,\psi(x)\,dx\end{aligned}$$

$$\begin{aligned} &-\int_0^\infty P_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\,\psi(x)\,dx \\ &=\int_0^\infty \left[\mathcal{J}_\mu(xt)-P_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\psi(x)\,dx \\ &=\int_0^\infty Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\,\psi(x)\,dx \quad (t\in I), \end{aligned} \tag{57}$$

as asserted. □

Remark 14. Since $\rho \in \mathcal{H}_\mu$ and all even polynomials lie in $\mathcal{O}$, from (55) it follows that

$$P_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right) \in \mathcal{H}_\mu \subset \mathcal{H}'_\mu \tag{58}$$

for each $x \in I$. On the other hand, it is apparent that $\mathcal{J}_\mu(x\cdot) \in \mathcal{H}'_\mu$ for each $x \in I$. Therefore, $Q_{n,\xi}(\mathcal{J}_\mu(x\xi)) \in \mathcal{H}'_\mu$ for each $x \in I$.

We close this section with some useful estimates.

Lemma 15. *Suppose that the function ρ used to define the projection operator Q_n (cf. Definition 8) also satisfies $0 \le t^{-\mu-1/2}\rho(t) \le 1$ $(t \in I)$ and* supp $\rho \subset [0,1]$. *Then:*

(i) *for all $x, t \in I$, one has*

$$\begin{aligned} &\left|\left(x^{-1}D_x\right)^m x^{-\mu-1/2}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right| \\ &\quad\le C\left(1+x^2\right)^n t^{2m+\mu+1/2}, \\ &\left|S^m_{\mu,x}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right| \\ &\quad\le Cx^{\mu+1/2}\left(1+x^2\right)^{n+m}\sum_{i=0}^{m} t^{2(m+i)+\mu+1/2}, \end{aligned} \tag{59}$$

(ii) *for $x \in I$ and $0 < t \le a$, one has*

$$\begin{aligned} &\left|\left(x^{-1}D_x\right)^m x^{-\mu-1/2}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right| \\ &\quad\le C\left(1+x^2\right)^n t^{2n+\mu+1/2}, \end{aligned} \tag{60}$$

$$\left|S^m_{\mu,x}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right| \le Cx^{\mu+1/2}\left(1+x^2\right)^{n+m} t^{2n+\mu+1/2}. \tag{61}$$

Proof. Fix $x, t \in I$. Equation (55) gives

$$\begin{aligned} Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t) &= \mathcal{J}_\mu(xt) - P_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t) \\ &= \mathcal{J}_\mu(xt) - \rho(t)\sum_{j=0}^{n-1}\frac{(-1)^j x^{2j+\mu+1/2}}{2^j j! c_{\mu,j}} t^{2j}, \end{aligned} \tag{62}$$

so that

$$\begin{aligned} &\left(x^{-1}D_x\right)^m x^{-\mu-1/2}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right] \\ &= \begin{cases} \left(x^{-1}D_x\right)^m x^{-\mu-1/2}\mathcal{J}_\mu(xt) \\ \quad -\rho(t)\displaystyle\sum_{k=0}^{n-m-1}\frac{(-1)^{m+k}x^{2k}}{2^k k! c_{\mu,m+k}} t^{2(m+k)}, & m<n \\ \left(x^{-1}D_x\right)^m x^{-\mu-1/2}\mathcal{J}_\mu(xt), & m \ge n. \end{cases} \end{aligned} \tag{63}$$

If $m \ge n$,

$$\begin{aligned} &\left(x^{-1}D_x\right)^m x^{-\mu-1/2}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right] \\ &\quad = (-1)^m t^{2m+\mu+1/2}(xt)^{-\mu-m} J_{\mu+m}(xt), \end{aligned} \tag{64}$$

whence

$$\left|\left(x^{-1}D_x\right)^m x^{-\mu-1/2}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right| \le c^{-1}_{\mu,m} t^{2m+\mu+1/2}. \tag{65}$$

If $m < n$,

$$\begin{aligned} &\left(x^{-1}D_x\right)^m x^{-\mu-1/2}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right] \\ &\quad = (-1)^m t^{2m+\mu+1/2}(xt)^{-\mu-m} J_{\mu+m}(xt) \\ &\qquad - (-1)^m t^{2m}\rho(t)\sum_{k=0}^{n-m-1}\frac{(-1)^k x^{2k}}{2^k k! c_{\mu,m+k}} t^{2k}, \end{aligned} \tag{66}$$

whence

$$\begin{aligned} &\left|\left(x^{-1}D_x\right)^m x^{-\mu-1/2}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right| \\ &\quad \le \left(c^{-1}_{\mu,m} + \sum_{k=0}^{n-m-1}\frac{x^{2k}}{2^k k! c_{\mu,m+k}}\right) t^{2m+\mu+1/2} \\ &\quad \le \left[c^{-1}_{\mu,m} + \sum_{k=0}^{n-m-1}\frac{\left(1+x^2\right)^k}{2^k k! c_{\mu,m+k}}\right] t^{2m+\mu+1/2} \\ &\quad \le C\left(1+x^2\right)^{n-m-1} t^{2m+\mu+1/2}. \end{aligned} \tag{67}$$

Here we have used our hypotheses on ρ. To summarize,

$$\begin{aligned} &\left|\left(x^{-1}D_x\right)^m x^{-\mu-1/2}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right| \\ &\quad \le \begin{cases} C\left(1+x^2\right)^{n-m-1} t^{2m+\mu+1/2}, & m<n \\ Ct^{2m+\mu+1/2}, & m \ge n. \end{cases} \end{aligned} \tag{68}$$

Thus we find

$$\begin{aligned} &\left|\left(x^{-1}D_x\right)^m x^{-\mu-1/2}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right| \\ &\quad \le C\left(1+x^2\right)^n t^{2m+\mu+1/2}. \end{aligned} \tag{69}$$

Assume now $0 < t \le a$. When $m \ge n$, (65) yields

$$\begin{aligned} &\left|\left(x^{-1}D_x\right)^m x^{-\mu-1/2}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right| \\ &\quad \le c^{-1}_{\mu,m} a^{2m+\mu+1/2}\frac{t^{2m+\mu+1/2}}{a^{2m+\mu+1/2}} \\ &\quad \le c^{-1}_{\mu,m} a^{2(m-n)} t^{2n+\mu+1/2} = Ct^{2n+\mu+1/2}. \end{aligned} \tag{70}$$

If $m < n$, (66) can be written as

$$\begin{aligned}
&\left(x^{-1}D_x\right)^m x^{-\mu-1/2}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\\
&=\Bigg(\left[1-t^{-\mu-1/2}\rho(t)\right](xt)^{-\mu-m}J_{\mu+m}(xt)+t^{-\mu-1/2}\rho(t)\\
&\quad\times\left[(xt)^{-\mu-m}J_{\mu+m}(xt)-\sum_{k=0}^{n-m-1}\frac{(-1)^k(xt)^{2k}}{2^k k! c_{\mu+m,k}}\right]\Bigg)\\
&\quad\times(-1)^m t^{2m+\mu+1/2}\\
&=\Bigg(\left[1-t^{-\mu-1/2}\rho(t)\right](xt)^{-\mu-m}J_{\mu+m}(xt)\\
&\quad+t^{-\mu-1/2}\rho(t)\sum_{k=n-m}^{\infty}\frac{(-1)^k(xt)^{2k}}{2^k k! c_{\mu+m,k}}\Bigg)(-1)^m t^{2m+\mu+1/2}.
\end{aligned}\tag{71}$$

Since $\rho\in\mathcal{H}_{\mu,n,*}$, from Proposition 2 it follows that

$$\begin{aligned}
&\left|\left(x^{-1}D_x\right)^m x^{-\mu-1/2}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right|\\
&\le C\left(\left[1-t^{-\mu-1/2}\rho(t)\right]+t^{-\mu-1/2}\rho(t)(xt)^{2(n-m)}\right)t^{2m+\mu+1/2}\\
&=C\Bigg(\frac{t^{-\mu-1/2}\left[1-t^{-\mu-1/2}\rho(t)\right]}{t^{2n}}t^{2m+\mu+1/2}\\
&\quad+t^{-\mu-1/2}\rho(t)\,x^{2(n-m)}\Bigg)t^{2n+\mu+1/2}\\
&\le C\left[\frac{1-t^{-\mu-1/2}\rho(t)}{t^{2n}}a^{2m}+t^{-\mu-1/2}\rho(t)\right]\\
&\quad\times\left(1+x^2\right)^{n-m}t^{2n+\mu+1/2}\\
&\le C\left(1+x^2\right)^{n-m}t^{2n+\mu+1/2}.
\end{aligned}\tag{72}$$

To summarize,

$$\begin{aligned}
&\left|\left(x^{-1}D_x\right)^m x^{-\mu-1/2}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right|\\
&\quad\le\begin{cases}C\left(1+x^2\right)^{n-m}t^{2n+\mu+1/2}, & m<n\\ Ct^{2n+\mu+1/2}, & m\ge n.\end{cases}
\end{aligned}\tag{73}$$

In any case, we get

$$\begin{aligned}
&\left|\left(x^{-1}D_x\right)^m x^{-\mu-1/2}\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right|\\
&\quad\le C\left(1+x^2\right)^{n}t^{2n+\mu+1/2}.
\end{aligned}\tag{74}$$

To complete the proof we note that if $\phi\in\mathcal{C}^{2m}$, then

$$x^{-\mu-1/2}\left(S_\mu^m\phi\right)(x)=\sum_{i=0}^{m}a_{m,i}x^{2i}\left(x^{-1}D\right)^{m+i}x^{-\mu-1/2}\phi(x)\tag{75}$$

for suitable coefficients $a_{m,i}$ $(i\in\mathbb{Z}_+,\ 0\le i\le m)$. Hence

$$\begin{aligned}
&\left|x^{-\mu-1/2}S_{\mu,x}^m\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right|\\
&\quad\le C\sum_{i=0}^{m}x^{2i}\left(1+x^2\right)^n t^{2(m+i)+\mu+1/2}\\
&\quad\le C\left(1+x^2\right)^{n+m}\sum_{i=0}^{m}t^{2(m+i)+\mu+1/2}
\end{aligned}\tag{76}$$

for any $t\in I$, while

$$\begin{aligned}
&\left|x^{-\mu-1/2}S_{\mu,x}^m\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right|\\
&\quad\le C\sum_{i=0}^{m}x^{2i}\left(1+x^2\right)^n t^{2n+\mu+1/2}\\
&\quad\le C\left(1+x^2\right)^{n+m}t^{2n+\mu+1/2}
\end{aligned}\tag{77}$$

if $0<t\le a$ for some $a\in I$. □

5. Main Results

Throughout this section we will assume that the function ρ used in Definition 8 satisfies $0\le x^{-\mu-1/2}\rho(x)\le 1$ $(x\in I)$ and $\operatorname{supp}\rho\subset[0,1]$, so that Lemma 15 applies.

First we prove a regularity result, along with a Hankel inversion formula and a polynomial estimate, in a general setting.

Theorem 16. *Let $f\in\mathcal{H}'_\mu$ be such that the distribution $h'_\mu f$ is regular on $\mathcal{H}_{\mu,n}$. Then, for all $m\in\mathbb{Z}_+$,*

$$\begin{aligned}
\left\langle h'_\mu\left(S_\mu^m f\right),\psi\right\rangle&=\int_0^\infty\left(h'_\mu f\right)(\xi)\left(-\xi^2\right)^m\left(Q_n\psi\right)(\xi)\,d\xi\\
&\quad+\left\langle h'_\mu\left(S_\mu^m p\right),\psi\right\rangle\quad\left(\psi\in\mathcal{H}_\mu\right),
\end{aligned}\tag{78}$$

where

$$p(\xi)=\sum_{j=0}^{n-1}\frac{(-1)^j\left\langle\left(h'_\mu f\right)(x),x^{2j}\rho(x)\right\rangle}{2^j j! c_{\mu,j}}\xi^{2j+\mu+1/2}\quad(\xi\in I).\tag{79}$$

Further, if there exists $r\in\mathbb{Z}_+$ and a function G integrable on I for which

$$\begin{aligned}
&\left|\left(h'_\mu f\right)(t)\,S_{\mu,x}^m\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]\right|\\
&\quad\le C\left(1+x^2\right)^r G(t)\quad(x,t\in I),
\end{aligned}\tag{80}$$

where C is independent of t, then $S_\mu^m f\in\mathcal{C}$, with

$$\begin{aligned}
\left(S_\mu^m f\right)(x)&=\int_0^\infty\left(h'_\mu f\right)(t)\,S_{\mu,x}^m\left[Q_{n,\xi}\left(\mathcal{J}_\mu(x\xi)\right)(t)\right]dt\\
&\quad+\left(S_\mu^m p\right)(x)\quad(x\in I),
\end{aligned}\tag{81}$$

and the estimate

$$\left|\left(S_\mu^m f\right)(x)\right| \leq C\left(1+x^2\right)^r + \left|\left(S_\mu^m p\right)(x)\right| \quad (x \in I) \tag{82}$$

holds.

Proof. Fix $m \in \mathbb{Z}_+$ and let $\psi \in \mathscr{H}_\mu$. We have

$$\begin{aligned}\left\langle h'_\mu\left(S_\mu^m f\right), \psi\right\rangle &= \left\langle\left(-\xi^2\right)^m\left(h'_\mu f\right)(\xi), \psi(\xi)\right\rangle \\ &= \left\langle\left(h'_\mu f\right)(\xi), \left(-\xi^2\right)^m \psi(\xi)\right\rangle \\ &= \left\langle\left(h'_\mu f\right)(\xi), \left(-\xi^2\right)^m \left(Q_n\psi\right)(\xi)\right\rangle \\ &\quad + \left\langle\left(h'_\mu f\right)(\xi), \left(-\xi^2\right)^m \left(P_n\psi\right)(\xi)\right\rangle.\end{aligned} \tag{83}$$

On the one hand, from our hypotheses on f and since $Q_n\psi \in \mathscr{H}_{\mu,n}$, we may write

$$\begin{aligned}&\left\langle\left(h'_\mu f\right)(\xi), \left(-\xi^2\right)^m \left(Q_n\psi\right)(\xi)\right\rangle \\ &\quad = \int_0^\infty \left(h'_\mu f\right)(\xi)\left(-\xi^2\right)^m \left(Q_n\psi\right)(\xi)\, d\xi.\end{aligned} \tag{84}$$

On the other hand,

$$\begin{aligned}&h'_\mu\left[P'_n\left(h'_\mu f\right)\right](\xi) \\ &\quad = \sum_{j=0}^{n-1} \frac{(-1)^j\left\langle\left(h'_\mu f\right)(x), x^{2j}\rho(x)\right\rangle}{2^j j! c_{\mu,j}} \xi^{2j+\mu+1/2} = p(\xi) \\ &\qquad (\xi \in I),\end{aligned} \tag{85}$$

or

$$P'_n\left(h'_\mu f\right) = h'_\mu p. \tag{86}$$

Thus,

$$\begin{aligned}&\left\langle\left(h'_\mu f\right)(\xi), \left(-\xi^2\right)^m \left(P_n\psi\right)(\xi)\right\rangle \\ &\quad = \left\langle\left(h'_\mu f\right)(\xi), P_n\left[\left(-t^2\right)^m \psi(t)\right](\xi)\right\rangle \\ &\quad = \left\langle\left(-\xi^2\right)^m P'_n\left(h'_\mu f\right)(\xi), \psi(\xi)\right\rangle \\ &\quad = \left\langle\left(-\xi^2\right)^m\left(h'_\mu p\right)(\xi), \psi(\xi)\right\rangle = \left\langle h'_\mu\left(S_\mu^m p\right), \psi\right\rangle.\end{aligned} \tag{87}$$

A combination of (83), (84), and (87) gives (78).

Note that (80) and the fact that $(h'_\mu f)S_{\mu,x}^m[Q_{n,\xi}(\mathscr{J}_\mu(x\xi))] \in \mathscr{C}$ as a function of $x \in I$ ensure that

$$\int_0^\infty \left(h'_\mu f\right)(t)\, S_{\mu,x}^m\left[Q_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right] dt \tag{88}$$

also belongs to $\mathscr{C}$ as a function of $x \in I$ (cf. [11, Proposition 7.8.3]). Furthermore, (80) entails

$$\begin{aligned}&\int_0^\infty \left|\left(h'_\mu f\right)(t)\, S_{\mu,x}^m\left[Q_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right]\right| dt \\ &\quad \leq C\left(1+x^2\right)^r \quad (x \in I).\end{aligned} \tag{89}$$

Now we have

$$\begin{aligned}\left\langle S_\mu^m f, \psi\right\rangle &= \left\langle h'_\mu f, h_\mu\left(S_\mu^m \psi\right)\right\rangle \\ &= \left\langle h'_\mu f, Q_n\left[h_\mu\left(S_\mu^m\psi\right)\right]\right\rangle + \left\langle h'_\mu f, P_n\left[h_\mu\left(S_\mu^m\psi\right)\right]\right\rangle.\end{aligned} \tag{90}$$

Lemma 13 allows us to write

$$\begin{aligned}&\left\langle h'_\mu f, Q_n\left[h_\mu\left(S_\mu^m\psi\right)\right]\right\rangle \\ &\quad = \int_0^\infty \left(h'_\mu f\right)(t)\, Q_n\left[h_\mu\left(S_\mu^m\psi\right)\right](t)\, dt \\ &\quad = \int_0^\infty \left(h'_\mu f\right)(t)\, dt \int_0^\infty Q_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\left(S_\mu^m\psi\right)(x)\, dx.\end{aligned} \tag{91}$$

In view of Remark 14,

$$\begin{aligned}&\int_0^\infty Q_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\left(S_\mu^m\psi\right)(x)\, dx \\ &\quad = \left\langle Q_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t), \left(S_\mu^m\psi\right)(x)\right\rangle \\ &\quad = \left\langle S_{\mu,x}^m\left[Q_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right], \psi(x)\right\rangle \\ &\quad = \int_0^\infty S_{\mu,x}^m\left[Q_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right]\psi(x)\, dx.\end{aligned} \tag{92}$$

Hence

$$\begin{aligned}&\left\langle h'_\mu f, Q_n\left[h_\mu\left(S_\mu^m\psi\right)\right]\right\rangle \\ &\quad = \int_0^\infty \left(h'_\mu f\right)(t)\, dt \int_0^\infty S_{\mu,x}^m\left[Q_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right]\psi(x)\, dx \\ &\quad = \int_0^\infty \psi(x)\, dx \int_0^\infty \left(h'_\mu f\right)(t)\, S_{\mu,x}^m\left[Q_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right] dt \\ &\quad = \left\langle \int_0^\infty \left(h'_\mu f\right)(t)\, S_{\mu,x}^m\left[Q_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right] dt, \psi(x)\right\rangle,\end{aligned} \tag{93}$$

the change in the order of integration being justified by (89). Lastly, from (86),

$$\begin{aligned}\left\langle h'_\mu f, P_n\left[h_\mu\left(S_\mu^m\psi\right)\right]\right\rangle &= \left\langle h'_\mu\left[P'_n\left(h'_\mu f\right)\right], S_\mu^m\psi\right\rangle \\ &= \left\langle p, S_\mu^m\psi\right\rangle = \left\langle S_\mu^m p, \psi\right\rangle.\end{aligned} \tag{94}$$

Equations (90), (93), and (94) lead us to (81), while the estimate (82) follows immediately from (89) and (81). The proof is thus complete. □

At this point, let us formalize the definition of a basis distribution.

Definition 17. We call $\Phi \in \mathscr{H}'_\mu$ a basis distribution if $t^{4n}(h'_\mu\Phi)(t) = 1/w(t)$ for some weight w (cf. Definition 1) such that $1/w \in L^1_{\mu,l}$ and there exists $\gamma \in \mathbb{R}$ with $1/w(t) = O(t^{-\gamma})$ as $t \to \infty$.

The existence of basis distributions is guaranteed by the next Lemma 18, which also proves that a basis distribution is unique modulo a polynomial in $\pi_{\mu,2n-1}$. Although Lemma 18 is substantially Lemma 3.3 in [5], its proof illustrates the ideas behind our main results below, and we include it for the sake of completeness.

Lemma 18. *Assume $F \in L^1_{\mu,l}$ and there exists $\gamma \in \mathbb{R}$ such that $F(x) = O(x^{-\gamma})$ as $x \to \infty$. Let $r \in \mathbb{Z}_+$. On $\mathscr{H}_{\mu,r}$ we define the linear functional F_r by*

$$\langle F_r, \psi\rangle = \int_0^\infty \frac{F(t)}{(-t^2)^r}\psi(t)\,dt \quad (\psi \in \mathscr{H}_{\mu,r}). \tag{95}$$

Then

(i) $F_r \in \mathscr{H}'_{\mu,r}$.

(ii) *Any extension $F_r^e \in \mathscr{H}'_\mu$ of F_r to $\mathscr{H}_\mu$ satisfies*

$$\left\langle \left(-t^2\right)^r F_r^e(t), \varphi(t)\right\rangle = \langle F, \varphi\rangle \quad (\varphi \in \mathscr{H}_\mu). \tag{96}$$

(iii) *If $F_{2r}^e \in \mathscr{H}'_\mu$ is an extension of $F_{2r} \in \mathscr{H}'_{\mu,2r}$ to $\mathscr{H}_\mu$, then*

$$\left\langle \left(-t^2\right)^r F_{2r}^e(t), \psi(t)\right\rangle = \langle F_r, \psi\rangle \quad (\psi \in \mathscr{H}_{\mu,r}). \tag{97}$$

(iv) *If F_r^1 and F_r^2 are two extensions of F_r to $\mathscr{H}_\mu$, then $h'_\mu(F_r^1) - h'_\mu(F_r^2) \in \pi_{\mu,r-1}$.*

Proof. Note that (ii) gives $h'_\mu S_\mu^r(h'_\mu F_r^e)(t) = (-t^2)^r F_r^e(t) = F(t)$. Therefore, part (iv) is a consequence of Theorem 5.

Let $a, C > 0$ be such that $F(x) \le Cx^{-\gamma}$ $(x > a)$. To prove (i), take $\psi \in \mathscr{H}_{\mu,r}$ and write

$$\langle F_r, \psi\rangle = \int_0^a \frac{F(t)}{(-t^2)^r}\psi(t)\,dt + \int_a^\infty \frac{F(t)}{(-t^2)^r}\psi(t)\,dt. \tag{98}$$

Using Proposition 2 and the hypothesis that $F \in L^1_{\mu,l}$, for the first integral we obtain

$$\begin{aligned}
&\left|\int_0^a \frac{F(t)}{(-t^2)^r}\psi(t)\,dt\right| \\
&\quad\le \int_0^a \frac{\left|t^{-\mu-1/2}\psi(t)\right|}{t^{2r}}\left|F(t)\right| t^{\mu+1/2}\,dt \\
&\quad\le C\left[\sup_{z\in I}\left|\left(z^{-1}D\right)^r z^{-\mu-1/2}\psi(z)\right|\right. \\
&\qquad\left. + \max_{0\le k\le r}\sup_{z\in I}\left|z^{2(r-k)+1}\left(z^{-1}D\right)^{2r-k+1} z^{-\mu-1/2}\psi(z)\right|\right].
\end{aligned} \tag{99}$$

As to the second integral, we get

$$\begin{aligned}
&\left|\int_a^\infty \frac{F(t)}{(-t^2)^r}\psi(t)\,dt\right| \\
&\quad\le C\int_a^\infty \frac{\left|t^{k-\mu-1/2}\psi(t)\right|}{t^{k+2r+\gamma-\mu-1/2}}\,dt \\
&\quad\le C\sup_{z\in I}\left|z^{k-\mu-1/2}\psi(z)\right|\int_a^\infty \frac{dt}{t^{k+2r+\gamma-\mu-1/2}} \\
&\quad= C\sup_{z\in I}\left|z^{k-\mu-1/2}\psi(z)\right|,
\end{aligned} \tag{100}$$

provided $k \in \mathbb{Z}_+$ is chosen so that $k > -2r - \gamma + \mu + 3/2$. A combination of (98), (99), and (100) along with the arbitrariness of $\psi \in \mathscr{H}_{\mu,r}$ completes the proof of (i).

Now we establish (ii). First of all, we note that specializing $r = 0$ in (i) we obtain $F \in \mathscr{H}'_\mu$. Next, let $F_r^e \in \mathscr{H}'_\mu$ be an extension of F_r to $\mathscr{H}_\mu$, and let $\varphi \in \mathscr{H}_\mu$. Since $(-t^2)^r\varphi(t) \in \mathscr{H}_{\mu,r}$, we may write

$$\begin{aligned}
&\left\langle \left(-t^2\right)^r F_r^e(t), \varphi(t)\right\rangle \\
&\quad= \left\langle F_r^e(t), \left(-t^2\right)^r\varphi(t)\right\rangle = \int_0^\infty \frac{F(t)}{(-t^2)^r}\left(-t^2\right)^r\varphi(t)\,dt \\
&\quad= \int_0^\infty F(t)\varphi(t)\,dt = \langle F, \varphi\rangle.
\end{aligned} \tag{101}$$

The arbitrariness of $\varphi \in \mathscr{H}_\mu$ gives (ii).

Finally, we prove (iii). Define $F_{2r} \in \mathscr{H}'_{\mu,2r}$ by (95), and let $F_{2r}^e \in \mathscr{H}'_\mu$ be an extension of F_{2r} to $\mathscr{H}_\mu$. Then

$$\begin{aligned}
&\left\langle \left(-t^2\right)^r F_{2r}^e(t), \psi(t)\right\rangle \\
&\quad= \left\langle F_{2r}^e(t), \left(-t^2\right)^r\psi(t)\right\rangle = \int_0^\infty \frac{F(t)}{t^{4r}}\left(-t^2\right)^r\psi(t)\,dt \\
&\quad= \int_0^\infty \frac{F(t)}{(-t^2)^r}\psi(t)\,dt = \langle F_r, \psi\rangle
\end{aligned} \tag{102}$$

whenever $\psi \in \mathscr{H}_{\mu,r}$. Thus we are done. □

When applied to a basis distribution Φ, Theorem 16 yields our second main result.

Theorem 19. *Pick a weight function w with the properties that $1/w \in L^1_{\mu,l}$ and $1/w(t) = O(t^{-\gamma})$ as $t \to \infty$ for some $\gamma \in \mathbb{R}$. Let $\Phi \in \mathscr{H}'_\mu$ satisfy $t^{4n}(h'_\mu\Phi)(t) = 1/w(t)$, so that $h'_\mu\Phi \in \mathscr{H}'_{\mu,2n}$ and*

$$\left\langle h'_\mu\Phi, \varphi\right\rangle = \int_0^\infty \frac{\varphi(t)}{t^{4n}w(t)}\,dt \quad (\varphi \in \mathscr{H}_{\mu,2n}) \tag{103}$$

(Lemma 18). Then, for all $m \in \mathbb{Z}_+$ one has

$$\begin{aligned}
\left\langle h'_\mu\left(S_\mu^m\Phi\right), \psi\right\rangle &= (-1)^m\int_0^\infty \frac{\xi^{2(m-2n)}\left(Q_{2n}\psi\right)(\xi)}{w(\xi)}\,d\xi \\
&\quad+ \left\langle h'_\mu\left(S_\mu^m p\right), \psi\right\rangle \quad (\psi \in \mathscr{H}_\mu),
\end{aligned} \tag{104}$$

where

$$p(\xi)=\sum_{j=0}^{2n-1}\frac{(-1)^j\left\langle\left(h'_\mu\Phi\right)(x),x^{2j}\rho(x)\right\rangle}{2^j j!c_{\mu,j}}\xi^{2j+\mu+1/2}\quad(\xi\in I). \tag{105}$$

Further, if

$$4m<4n+\gamma-\mu-\frac{3}{2}, \tag{106}$$

then $S_\mu^m\Phi\in\mathscr{C}$,

$$\left(S_\mu^m\Phi\right)(x)=\int_0^\infty\frac{S_{\mu,x}^m\left[Q_{2n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right]}{t^{4n}w(t)}dt+\left(S_\mu^m p\right)(x)\quad(x\in I), \tag{107}$$

and the inequality

$$\left|\left(S_\mu^m\Phi\right)(x)\right|\leq C\left(1+x^2\right)^r+\left|\left(S_\mu^m p\right)(x)\right|\quad(x\in I) \tag{108}$$

holds for some $r\in\mathbb{Z}_+$.

Proof. In order to derive this result from Theorem 16 it suffices to establish, under (106), an estimate like (80), with Φ instead of f.

Let $C>0$, $a>1$ be such that $1/w(t)\leq Ct^{-\gamma}$ $(t>a)$, with $\gamma\in\mathbb{R}$. Use Lemma 15 to choose $r\in\mathbb{Z}_+$ satisfying

$$\left|S_{\mu,x}^m\left[Q_{2n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right]\right|\leq C\left(1+x^2\right)^r t^{4n+\mu+1/2}\quad(x\in I,\ 0<t\leq a), \tag{109}$$

$$\left|S_{\mu,x}^m\left[Q_{2n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right]\right|\leq C\left(1+x^2\right)^r\sum_{i=0}^m t^{2(m+i)+\mu+1/2}\quad(x\in I,\ t>a). \tag{110}$$

Note that

$$\begin{aligned}\left(h'_\mu\Phi\right)(t)\,S_{\mu,x}^m\left[Q_{2n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right]&=t^{4n}\left(h'_\mu\Phi\right)(t)\frac{S_{\mu,x}^m\left[Q_{2n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right]}{t^{4n}}\\&=\frac{S_{\mu,x}^m\left[Q_{2n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right]}{t^{4n}w(t)}\quad(x,t\in I).\end{aligned} \tag{111}$$

Now

$$\begin{aligned}\left|\frac{S_{\mu,x}^m\left[Q_{2n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right]}{t^{4n}w(t)}\right|&\leq C\left(1+x^2\right)^r\frac{t^{4n+\mu+1/2}}{t^{4n}w(t)}\\&=C\left(1+x^2\right)^r\frac{t^{\mu+1/2}}{w(t)}\quad(x\in I)\end{aligned} \tag{112}$$

if $0<t\leq a$, while

$$\begin{aligned}\left|\frac{S_{\mu,x}^m\left[Q_{2n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right]}{t^{4n}w(t)}\right|&\leq C\left(1+x^2\right)^r\sum_{i=0}^m\frac{t^{2(m+i)+\mu+1/2}}{t^{4n}w(t)}\\&\leq C\left(1+x^2\right)^r t^{4m-4n-\gamma+\mu+1/2}\quad(x\in I)\end{aligned} \tag{113}$$

if $t>a$. Set

$$G(t)=\begin{cases}\dfrac{t^{\mu+1/2}}{w(t)}, & 0<t\leq a\\ t^{4m-4n-\gamma+\mu+1/2}, & t>a.\end{cases} \tag{114}$$

This function is integrable on I as long as $1/w\in L_{\mu,l}^1$ and (106) holds. The required estimate is established by combining (111), (112), and (113). □

Remark 20. In [5, Theorem 4.4], under the same hypotheses on the weight w as in Theorem 19, we proved that any basis distribution Φ has the property that $S_\mu^m\Phi\in\mathscr{C}$ whenever $m\in\mathbb{Z}_+$ satisfies $2m<4n+\gamma-\mu-3/2$, a condition in fact weaker than (106). However, interestingly enough, Theorem 19 provides us with explicit expressions and polynomial bounds for the Bessel derivatives $S_\mu^m\Phi$ whenever $m\in\mathbb{Z}_+$ satisfies (106).

Next we apply Theorem 16 to the distributions in Y_n in order to obtain Theorem 21, our last main result.

Theorem 21. *Assume* $1/w\in L_{\mu,l}^1$ *and there exists* $\gamma\in\mathbb{R}$ *such that* $1/w(t)=O(t^{-\gamma})$ *as* $t\to\infty$. *Let* $f\in Y_n$, *so that* $(-t^2)^n(h'_\mu f)(t)\in L_{\mu,l}^1$. *Then* $h'_\mu f\in\mathscr{H}'_{\mu,n}$, *and for all* $m\in\mathbb{Z}_+$ *one has*

$$\left\langle h'_\mu\left(S_\mu^m f\right),\psi\right\rangle=\int_0^\infty\left(h'_\mu f\right)(\xi)\left(-\xi^2\right)^m\left(Q_n\psi\right)(\xi)\,d\xi+\left\langle h'_\mu\left(S_\mu^m p\right),\psi\right\rangle\quad\left(\psi\in\mathscr{H}_\mu\right), \tag{115}$$

where

$$p(\xi)=\sum_{j=0}^{n-1}\frac{(-1)^j\left\langle\left(h'_\mu f\right)(x),x^{2j}\rho(x)\right\rangle}{2^j j!c_{\mu,j}}\xi^{2j+\mu+1/2}\quad(\xi\in I). \tag{116}$$

Further, if

$$8m<4n+\gamma-\mu-\frac{3}{2}, \tag{117}$$

then $S_\mu^m f\in\mathscr{C}$, *with*

$$\left(S_\mu^m f\right)(x)=\int_0^\infty\left(h'_\mu f\right)(t)\,S_{\mu,x}^m\left[Q_{n,\xi}\left(\mathscr{J}_\mu(x\xi)\right)(t)\right]dt+\left(S_\mu^m p\right)(x)\quad(x\in I), \tag{118}$$

and the estimate

$$\left|\left(S_\mu^m f\right)(x)\right| \le C\left(1+x^2\right)^r + \left|\left(S_\mu^m p\right)(x)\right| \quad (x \in I) \qquad (119)$$

holds for some $r \in \mathbb{Z}_+$.

Proof. Since $h'_\mu(S_\mu^n f) \in L^2_{\mu,w}$, we have

$$|f|_n = \left(\int_0^\infty \left|\left(-\xi^2\right)^n \left(h'_\mu f\right)(\xi)\right|^2 w(\xi)\,\xi^{\mu+1/2} d\xi\right)^{1/2} < \infty. \qquad (120)$$

Choose $C > 0$, $a > 1$ such that $1/w(t) \le Ct^{-\gamma}$ $(t > a)$, with $\gamma \in \mathbb{R}$. For any $\psi \in \mathscr{H}_{\mu,n}$, the Cauchy-Schwarz inequality gives

$$\begin{aligned}
&\int_0^\infty \left|\left(h'_\mu f\right)(\xi)\,\psi(\xi)\right| d\xi \\
&\le |f|_n \left(\int_0^\infty \frac{\left|\xi^{-\mu-1/2}\psi(\xi)\right|^2}{\xi^{4n} w(\xi)} \xi^{\mu+1/2} d\xi\right)^{1/2} \\
&\le |f|_n \left[\left(\int_0^a \frac{\left|\xi^{-\mu-1/2}\psi(\xi)\right|^2}{\xi^{4n} w(\xi)} \xi^{\mu+1/2} d\xi\right)^{1/2} + \left(\int_a^\infty \frac{\left|\xi^{-\mu-1/2}\psi(\xi)\right|^2}{\xi^{4n} w(\xi)} \xi^{\mu+1/2} d\xi\right)^{1/2}\right].
\end{aligned} \qquad (121)$$

Using Proposition 2, for the first bounding integral we obtain

$$\begin{aligned}
&\int_0^a \frac{\left|\xi^{-\mu-1/2}\psi(\xi)\right|^2}{\xi^{4n} w(\xi)} \xi^{\mu+1/2} d\xi \\
&\le C\left[\sup_{z\in I}\left|\left(z^{-1}D\right)^n z^{-\mu-1/2}\psi(z)\right| + \max_{0\le k\le n}\sup_{z\in I}\left|z^{2(n-k)+1}\left(z^{-1}D\right)^{2n-k+1} z^{-\mu-1/2}\psi(z)\right|\right]^2.
\end{aligned} \qquad (122)$$

As to the second one,

$$\begin{aligned}
&\int_a^\infty \frac{\left|\xi^{-\mu-1/2}\psi(\xi)\right|^2}{\xi^{4n} w(\xi)} \xi^{\mu+1/2} d\xi \\
&\le C \sup_{z\in I}\left|z^{k-\mu-1/2}\psi(z)\right|^2 \int_a^\infty \frac{d\xi}{\xi^{2k+4n+\gamma-\mu-1/2}} \\
&= C \sup_{z\in I}\left|z^{k-\mu-1/2}\psi(z)\right|^2,
\end{aligned} \qquad (123)$$

provided $k \in \mathbb{Z}_+$ is chosen so that $2k > -4n - \gamma + \mu + 3/2$. This proves that $h'_\mu f \in \mathscr{H}'_{\mu,n}$. Now (115) follows from (78).

To complete the proof it suffices to establish, under (117), an estimate like (80) and apply Theorem 16.

With this purpose, fix $x \in I$. As above, factorize

$$\left(h'_\mu f\right)(t)\, S^m_{\mu,x}\left[Q_{n,t}\left(\mathscr{J}_\mu(xt)\right)(t)\right] = F(t)\,R(t) \quad (t \in I), \qquad (124)$$

where

$$\begin{aligned}
F(t) &= \left(-t^2\right)^n \left(h'_\mu f\right)(t)\, w^{1/2}(t)\, t^{(\mu+1/2)/2} \quad (t \in I), \\
R(t) &= \frac{S^m_{\mu,x}\left[Q_{n,t}\left(\mathscr{J}_\mu(xt)\right)(t)\right]}{\left(-t^2\right)^n w^{1/2}(t)} t^{-(\mu+1/2)/2} \quad (t \in I).
\end{aligned} \qquad (125)$$

Because of (120), F is square-integrable on I. On the other hand, Lemma 15 and the argument in the proof of Theorem 19 show that

$$\begin{aligned}
|R(t)| &= \left|\frac{S^m_{\mu,x}\left[Q_{n,t}\left(\mathscr{J}_\mu(xt)\right)(t)\right]}{t^{2n} w^{1/2}(t)}\right| t^{-(\mu+1/2)/2} \\
&\le C\left(1+x^2\right)^r H(t) \quad (t \in I)
\end{aligned} \qquad (126)$$

for some $r \in \mathbb{Z}_+$, where

$$H^2(t) = \begin{cases} \dfrac{t^{\mu+1/2}}{w(t)}, & 0 < t \le a \\ t^{8m-4n-\gamma+\mu+1/2}, & t > a \end{cases} \qquad (127)$$

is integrable on I, as far as $1/w \in L^1_{\mu,l}$ and (117) holds. Letting $G(t) = |F(t)|H(t)$ $(t \in I)$, from (124) we find that G is integrable and

$$\begin{aligned}
&\left|\left(h'_\mu f\right)(t)\, S^m_{\mu,x}\left[Q_{n,t}\left(\mathscr{J}_\mu(xt)\right)(t)\right]\right| \\
&\le C\left(1+x^2\right)^r G(t) \quad (t \in I).
\end{aligned} \qquad (128)$$

This ends the proof. □

Remark 22. In [5, Theorem 3.2], under the same hypotheses on the weight w as in Theorem 21, we proved that every $f \in Y_n$ has the property that $S_\mu^m f \in \mathscr{C}$ whenever $m \in \mathbb{Z}_+$ satisfies $4m < 4n + \gamma - \mu - 3/2$. Although this condition is actually weaker than (117), Theorem 21 provides us with explicit expressions and polynomial bounds for those S_μ-derivatives.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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