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*Research Article*

# Bounds for Combinations of Toader Mean and Arithmetic Mean in Terms of Centroidal Mean

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The authors find the greatest value $\lambda$ and the least value $\mu$, such that the double inequality $\overline{C}(\lambda a+(1-\lambda b), \lambda b+(1-\lambda)a) < \alpha A(a,b)+(1-\alpha)T(a,b) < \overline{C}(\mu a+(1-\mu)b, \mu b+(1-\mu)a)$ holds for all $\alpha \in (0,1)$ and $a, b > 0$ with $a \neq b$, where $\overline{C}(a,b) = 2(a^2+ab+b^2)/3(a+b)$, $A(a,b) = (a+b)/2$, and $T(a,b) = (2/\pi)\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}d\theta$ denote, respectively, the centroidal, arithmetic, and Toader means of the two positive numbers $a$ and $b$.

## 1. Introduction

In [1], Toader introduced a mean

$$T(a,b) = \frac{2}{\pi}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}d\theta = \begin{cases} 2a\mathscr{E}\dfrac{\sqrt{1-(b/a)^2}}{\pi}, & a > b, \\ 2b\mathscr{E}\dfrac{\sqrt{1-(a/b)^2}}{\pi}, & a < b, \\ a, & a = b, \end{cases} \quad (1)$$

where

$$\mathscr{E} = \mathscr{E}(r) = \int_0^{\pi/2} \left(1 - r^2\sin^2\theta\right)^{1/2} d\theta, \quad (2)$$

for $r \in [0,1]$ is the complete elliptic integral of the second kind.

In recent years, there have been plenty of literature, such as [2–6], dedicated to the Toader mean.

For $p \in \mathbb{R}$ and $a, b > 0$, the centroidal mean $\overline{C}(a,b)$ and $p$th power mean $M_p(a,b)$ are, respectively, defined by

$$\overline{C}(a,b) = \frac{2\left(a^2+ab+b^2\right)}{3(a+b)}, \qquad M_p(a,b) = \begin{cases} \left(\dfrac{a^p + a^p}{2}\right)^{1/p}, & p \neq 0, \\ \sqrt{ab}, & p = 0. \end{cases} \quad (3)$$

In [7], Vuorinen conjectured that

$$M_{3/2}(a,b) < T(a,b), \quad (4)$$

for all $a, b > 0$ with $a \neq b$. This conjecture was verified by Qiu and Shen [8] and by Barnard et al. [9], respectively.

In [10], Alzer and Qiu presented a best possible upper power mean bound for the Toader mean as follows:

$$T(a,b) < M_{\log 2/\log(\pi/2)}(a,b), \quad (5)$$

for all $a, b > 0$ with $a \neq b$.

Chu et al. [5] proved that the double inequality

$$C(\alpha a + (1-\alpha)b, \alpha b + (1-\alpha)a) < T(a,b) < C(\beta a + (1-\beta)b, \beta b + (1-\beta)a) \quad (6)$$

holds for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 3/4$ and $\beta \geq 1/2 + \sqrt{4\pi - \pi^2}/(2\pi)$.

Very recently, Hua and Qi [11] proved that the double inequality

$$\begin{aligned} &\alpha\overline{C}(a,b) + (1-\alpha)A(a,b) \\ &\quad < T(a,b) \\ &\quad < \beta\overline{C}(a,b) + (1-\beta)A(a,b) \end{aligned} \tag{7}$$

is valid for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 3/4$ and $\beta \geq (12/\pi) - 3$. Where $A(a,b) = (a+b)/2$ denote the arithmetic mean.

For positive numbers $a, b > 0$ with $a \neq b$, let

$$J(x) = \overline{C}(xa + (1-x)b, xb + (1-x)a) \tag{8}$$

be on $[1/2, 1]$. It is not difficult to directly verify that $J(x)$ is continuous and strictly increasing on $[1/2, 1]$.

The main purpose of the paper is to find the greatest value $\lambda$ and the least value $\mu$, such that the double inequality $\overline{C}(\lambda a + (1-\lambda b), \lambda b + (1-\lambda)a) < \alpha A(a,b) + (1-\alpha)T(a,b) < \overline{C}(\mu a + (1-\mu)b, \mu b + (1-\mu)a)$ holds for all $\alpha \in (0,1)$ and $a, b > 0$ with $a \neq b$. As applications, we also present new bounds for the complete elliptic integral of the second kind.

## 2. Preliminaries and Lemmas

In order to establish our main result, we need several formulas and Lemmas below.

For $0 < r < 1$ and $r' = \sqrt{1-r^2}$, Legendre's complete elliptic integrals of the first and second kinds are defined in [12, 13] by

$$\begin{aligned} &\mathscr{K} = \mathscr{K}(r) = \int_0^{\pi/2} \left(1 - r^2\sin^2\theta\right)^{-1/2} d\theta, \\ &\mathscr{K}' = \mathscr{K}'(r) = \mathscr{K}(r'), \\ &\mathscr{K}(0) = \frac{\pi}{2}, \qquad \mathscr{K}(1) = \infty, \\ &\mathscr{E} = \mathscr{E}(r) = \int_0^{\pi/2} \left(1 - r^2\sin^2\theta\right)^{1/2} d\theta, \\ &\mathscr{E}' = \mathscr{E}'(r) = \mathscr{E}(r'), \\ &\mathscr{E}(0) = \frac{\pi}{2}, \qquad \mathscr{E}(1) = 1, \end{aligned} \tag{9}$$

respectively.

For $0 < r < 1$, the formulas

$$\begin{aligned} &\frac{d\mathscr{K}}{dr} = \frac{\mathscr{E} - r'^2\mathscr{K}}{rr'^2}, \qquad \frac{d\mathscr{E}}{dr} = \frac{\mathscr{E} - \mathscr{K}}{r}, \\ &\frac{d\left(\mathscr{E} - r'^2\mathscr{K}\right)}{dr} = r\mathscr{K}, \qquad \frac{d(\mathscr{K} - \mathscr{E})}{dr} = \frac{r\mathscr{E}}{r'^2}, \\ &\mathscr{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathscr{E} - r'^2\mathscr{K}}{1+r} \end{aligned} \tag{10}$$

were presented in [14, Appendix E, pages 474-475].

**Lemma 1** (see [14, Theorem 3.21(1), 3.43 exercises 13(a)]). *The function $(\mathscr{E} - r'^2\mathscr{K})/r^2$ is strictly increasing from $(0,1)$ to $(\pi/4, 1)$, and the function $2\mathscr{E} - r'^2\mathscr{K}$ is increasing from $(0,1)$ to $(\pi/2, 2)$.*

**Lemma 2.** *Let $u, \alpha \in (0,1)$ and*

$$\begin{aligned} f_{u,\alpha}(r) = &\frac{1}{3}ur^2 \\ &- (1-\alpha)\left(\frac{2}{\pi}\left(2\mathscr{E}(r) - \left(1 - r^2\right)\mathscr{K}(r)\right) - 1\right). \end{aligned} \tag{11}$$

*Then, $f_{u,\alpha} > 0$, for all $r \in (0,1)$ if and only if $u \geq 3(1-\alpha)(4/\pi - 1)$, and $f_{u,\alpha} < 0$, for all $r \in (0,1)$ if and only if $u \leq 3(1-\alpha)/4$.*

*Proof.* From (11), one has

$$f_{u,\alpha}(0^+) = 0, \tag{12}$$

$$f_{u,\alpha}(1^-) = \frac{1}{3}u - (1-\alpha)\left(\frac{4}{\pi} - 1\right), \tag{13}$$

$$f'_{u,\alpha}(r) = \frac{2}{3}r\left[u - 3(1-\alpha)g(r)\right], \tag{14}$$

where $g(r) = (1/\pi)((\mathscr{E} - r'^2\mathscr{K})/r^2)$.

We divide the proof into four cases.

*Case 1 ($u \geq 3(1-\alpha)/\pi$).* From (14) and Lemma 1 together with the monotonicity of $g(r)$, we clearly see that $f_{u,\alpha}(r)$ is strictly increasing on $(0,1)$. Therefore, $f_{u,\alpha}(r) > 0$, for all $r \in (0,1)$.

*Case 2 ($u \leq 3(1-\alpha)/4$).* From (14) and Lemma 1 together with the monotonicity of $g(r)$, we obtain that $f_{u,\alpha}(r)$ is strictly decreasing on $(0,1)$. Therefore, $f_{u,\alpha}(r) < 0$, for all $r \in (0,1)$.

*Case 3 ($3(1-\alpha)/4 < u \leq 3(1-\alpha)(4/\pi - 1)$).* From (13) and (14) together with the monotonicity of $g(r)$, we see that there exists $\lambda \in (0,1)$, such that $f_{u,\alpha}(r)$ is strictly increasing in $(0, \lambda]$ and strictly decreasing in $[\lambda, 1)$ and

$$f_{u,\alpha}(1^-) \leq 0. \tag{15}$$

Therefore, making use of (12) and inequality (15) together with the piecewise monotonicity of $f_{u,\alpha}(r)$ leads to the conclusion that there exists $0 < \lambda < \eta < 1$, such that $f_{u,\alpha}(r) > 0$ for $r \in (0, \eta)$ and $f_{u,\alpha}(r) < 0$ for $r \in (\eta, 1)$.

*Case 4 ($3(1-\alpha)(4/\pi - 1) \leq u < 3(1-\alpha)/\pi$).* Equation (13) leads to

$$f_{u,\alpha}(1^-) \geq 0. \tag{16}$$

From (13) and (14) together with the monotonicity of $g(r)$, we clearly see that there exists $\lambda \in (0,1)$, such that $f_{u,\alpha}(r)$ is strictly increasing in $(0, \lambda]$ and strictly decreasing in $[\lambda, 1)$. Therefore, $f_{u,\alpha}(r) > 0$ for $r \in (0,1)$ follows from (12) and (16) together with the piecewise monotonicity of $f_{u,\alpha}(r)$. □

## 3. Main Results

Now, we are in a position to state and prove our main results.

**Theorem 3.** *If* $\alpha \in (0,1)$ *and* $\lambda, \mu \in (1/2, 1)$, *then the double inequality*

$$\begin{aligned}&\overline{C}(\lambda a+(1-\lambda)b, \lambda b+(1-\lambda)a)\\&\quad<\alpha A(a,b)+(1-\alpha)T(a,b)\\&\quad<\overline{C}(\mu a+(1-\mu)b, \mu b+(1-\mu)a)\end{aligned} \tag{17}$$

*holds for all* $a, b > 0$ *with* $a \neq b$ *if and only if*

$$\begin{aligned}\lambda &\leq \frac{1}{2}+\frac{\sqrt{3(1-\alpha)}}{4},\\ \mu &\geq \frac{1}{2}\left(1+\sqrt{3(1-\alpha)\left(\frac{4}{\pi}-1\right)}\right).\end{aligned} \tag{18}$$

*Proof.* Since $A(a,b)$, $T(a,b)$, and $\overline{C}(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a > b$. Let $p \in (1/2, 1)$, $t = b/a \in (0,1)$, and $r = (1-t)/(1+t)$. Then,

$$\begin{aligned}&\overline{C}(pa+(1-p)b, pb+(1-p)a)\\&\quad-\alpha A(a,b)-(1-\alpha)T(a,b)\\&=a\frac{2}{3}\left(\left(p+(1-p)\frac{b}{a}\right)^2\right.\\&\qquad+\left(p+(1-p)\frac{b}{a}\right)\left(p\frac{b}{a}+1-p\right)\\&\qquad\left.+\left(p\frac{b}{a}+1-p\right)^2\right)\left(1+\frac{b}{a}\right)^{-1}\\&\quad-\alpha a\frac{1+(b/a)}{2}\\&\quad-(1-\alpha)\frac{2a}{\pi}\mathscr{E}\left(\sqrt{1-\left(\frac{b}{a}\right)^2}\right)\\&=a\left\{\frac{2}{3}\left((p+(1-p)t)^2\right.\right.\\&\qquad+(p+(1-p)t)(pt+1-p)\\&\qquad\left.+(pt+1-p)^2\right)(1+t)^{-1}\\&\qquad\left.-\alpha\frac{1+t}{2}-(1-\alpha)\frac{2}{\pi}\mathscr{E}\left(\sqrt{1-t^2}\right)\right\}\\&=a\left\{\frac{(1-2p)^2r^2+3}{3(1+r)}-\alpha\frac{1}{1+r}\right.\\&\qquad\left.-(1-\alpha)\frac{2}{\pi}\frac{2\mathscr{E}-r'^2\mathscr{K}}{1+r}\right\}\\&=\frac{a}{1+r}\left[\frac{1}{3}(1-2p)^2r^2+1-\alpha\right.\\&\qquad\left.-(1-\alpha)\frac{2}{\pi}\left(2\mathscr{E}-r'^2\mathscr{K}\right)\right].\end{aligned} \tag{19}$$

Therefore, Theorem 3 follows easily from Lemma 2 and (19). □

Let $\alpha = 1/4$, $\lambda = 7/8$, $\mu = (1/2)(1+(3\sqrt{4/\pi-1}/2))$. Then, from Theorem 3, we get new bounds for the complete elliptic integral $\mathscr{E}(r)$ of the second kind in terms of elementary functions as follows.

**Corollary 4.** *For* $r \in (0,1)$ *and* $r' = \sqrt{1-r^2}$, *one has*

$$\frac{\pi}{2}\left[\frac{5+6r'+5r'^2}{8(1+r')}\right]<\mathscr{E}(r)<\pi\left[\frac{r'+(2/\pi)(1-r')^2}{1+r'}\right]. \tag{20}$$

## 4. Remarks

*Remark 5.* In the recent past, the complete elliptic integrals have attracted the attention of numerous mathematicians. In [4], it was established that

$$\begin{aligned}&\frac{\pi}{2}\left[\frac{1}{2}\sqrt{\frac{1+r'^2}{2}}+\frac{1+r'}{4}\right]\\&\quad<\mathscr{E}(r)\\&\quad<\frac{\pi}{2}\left[\frac{4-\pi}{(\sqrt{2}-1)\pi}\sqrt{\frac{1+r'^2}{2}}+\frac{(\sqrt{2}\pi-4)(1+r')}{2(\sqrt{2}-1)\pi}\right],\end{aligned} \tag{21}$$

for all $r \in (0,1)$.

Guo and Qi [15] proved that

$$\frac{\pi}{2}-\frac{1}{2}\log\frac{(1+r)^{1-r}}{(1-r)^{1+r}}<\mathscr{E}(r)<\frac{\pi-1}{2}+\frac{1-r^2}{4r}\log\frac{1+r}{1-r}, \tag{22}$$

for all $r \in (0,1)$.

Yin and Qi [16] presented that

$$\frac{\pi}{2}\frac{\sqrt{6+2\sqrt{1-r^2}-3r^2}}{2\sqrt{2}}\leq\mathscr{E}(r)\leq\frac{\pi}{2}\frac{\sqrt{10-2\sqrt{1-r^2}-5r^2}}{2\sqrt{2}}, \tag{23}$$

for all $r \in (0,1)$.

It was pointed out in [4] that the bounds in (21) for $\mathscr{E}(r)$ are better than the bounds in (22) for some $r \in (0,1)$.

*Remark 6.* The lower bound in (20) for $\mathscr{E}(r)$ is better than the lower bound in (21). Indeed,

$$\frac{5+6x+5x^2}{8(1+x)} - \left[\frac{1}{2}\sqrt{\frac{1+x^2}{2}} + \frac{1+x}{4}\right] = \frac{3x^2+2x+3-2\sqrt{2(1+x^2)}(1+x)}{8(1+x)}, \tag{24}$$

$$\left(3x^2+2x+3\right)^2 - \left(2\sqrt{2(1+x^2)}(1+x)\right)^2 = (1-x)^4 > 0,$$

for all $x \in (0,1)$.

*Remark 7.* The following equivalence relations for $x \in (0,1)$ show that the lower bound in (20) for $\mathscr{E}(r)$ is better than the lower bound in (23):

$$\frac{5+6x+5x^2}{8(1+x)} > \frac{\sqrt{6+2x-3(1-x^2)}}{2\sqrt{2}} \Longleftrightarrow \left(5x^2+6x+5\right)^2 > 8(x+1)^2\left(3x^2+2x+3\right) \Longleftrightarrow (x-1)^4 > 0. \tag{25}$$

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