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Research Article

Note on the Regularity of Nonadditive Measures

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We consider the regularity for nonadditive measures. We prove that the non-additive measures which satisfy Egoroff's theorem and have pseudometric generating property possess Radon property (strong regularity) on a complete or a locally compact, separable metric space.

1. Introduction

The relations of continuity and regularity of nonadditive measures are considered in several papers [1–4]. In [5], Li et al. investigated the regularity in nonadditive measures. They proved that the null-additive fuzzy measures possess a Radon property (strong regularity) on a complete metric space. In [6], Kawabe also investigated the regularity in fuzzy measures taking value in Riesz spaces. He proved that every weakly null-additive Riesz space valued fuzzy measure on a complete or a locally compact, separable metric space is Radon, provided that the Riesz space has the multiple Egoroff property.

On the other hand Li and Mesiar [7] proved the regularity of nonadditive monotone measures. They proved that the equivalence condition of Egoroff's theorem implies regularity for the nonadditive measures by using pseudometric generating property of a set function. For information on real valued nonadditive measures, see [8–10].

In this paper, as notes, we prove that Egoroff's theorem implies Radon property (strong regularity) for nonadditive measures which have pseudometric generating property on a complete or a locally compact, separable metric space.

2. Preliminaries

Let R be the set of real numbers and N the set of natural numbers. In what follows, let $(X, \mathscr{F})$ be a measurable space.

Definition 1. A set function $\mu : \mathscr{F} \to R$ is called a nonadditive measure if it satisfies the following two conditions:

(1) $\mu(\emptyset) = 0$,

(2) if $A, B \in \mathscr{F}$ and $A \subset B$, then $\mu(A) \leq \mu(B)$.

Definition 2. Let $\mu : \mathscr{F} \to R$ be a nonadditive measure.

(1) μ is said to be continuous from above if for any $\{A_n\} \subset \mathscr{F}$ and $A \in \mathscr{F}$ satisfying $A_n \searrow A$ and there exists n_0 with $\mu(A_{n_0}) < \infty$ it holds that $\lim_{n\to\infty}\mu(A_n) = \mu(A)$.

(2) μ is said to be continuous from below if for any $\{A_n\} \subset \mathscr{F}$ and $A \in \mathscr{F}$ satisfying $A_n \nearrow A$ it holds that $\lim_{n\to\infty}\mu(A_n) = \mu(A)$.

(3) μ is said to be fuzzy measure if it is continuous from above and below.

(4) μ is said to be strongly order continuous if it is continuous from above at measurable sets of measure 0; that is, for any $\{A_n\} \subset \mathscr{F}$ and $A \in \mathscr{F}$ satisfying $A_n \searrow A$ and $\mu(A) = 0$ it holds that $\lim_{n\to\infty}\mu(A_n) = 0$.

(5) μ is said to be weakly null-additive if $\mu(A \cup B) = 0$ whenever $A, B \in \mathscr{F}$ and $\mu(A) = \mu(B) = 0$.

(6) μ has property (S) if for any sequence $\{A_n\} \subset \mathscr{F}$ with $\lim_{n\to\infty}\mu(A_n) = 0$ there exists a subsequence $\{A_{n_k}\}$ such that $\mu(\bigcap_{i=1}^{\infty}\bigcup_{k=i}^{\infty}A_{n_k}) = 0$; see [11].

(7) μ is said to be autocontinuous from above if $\lim_{n\to\infty}\mu(A \cup B_n) = \mu(A)$ for each $A \in \mathcal{F}$ and $\{B_n\} \subset \mathcal{F}$ with $\lim_{n\to\infty}\mu(B_n) = 0$.

(8) μ is said to be autocontinuous from below if $\lim_{n\to\infty}\mu(A \setminus B_n) = \mu(A)$ for each $A \in \mathcal{F}$ and $\{B_n\} \subset \mathcal{F}$ with $\lim_{n\to\infty}\mu(B_n) = 0$.

(9) μ is said to be autocontinuous if it is autocontinuous from above and below.

Definition 3. Let $\mu : \mathcal{F} \to R$ be a nonadditive measure.

(1) A double sequence $\{A_{m,n}\} \subset \mathcal{F}$ is said to be a μ-regulator if it satisfies the following two conditions:

(D1) $A_{m,n} \supset A_{m,n'}$ whenever $n \le n'$,

(D2) $\mu(\cup_{m=1}^{\infty}\cap_{n=1}^{\infty}A_{m,n}) = 0$.

(2) μ satisfies the Egoroff condition if for any μ-regulator $\{A_{m,n}\}$ and for every $\varepsilon > 0$ there exists a sequence $\{n_m\}$ of natural numbers such that $\mu(\cup_{m=1}^{\infty}A_{m,n_m}) < \varepsilon$.

Remark 4. A nonadditive measure μ satisfies the Egoroff condition if (and only if) for any double sequence $\{A_{m,n}\} \subset \mathcal{F}$ satisfying (D2) and the following (D1′) it holds that for every $\varepsilon > 0$ there exists a sequence $\{n_m\}$ of natural numbers such that $\mu(\cup_{m=1}^{\infty}A_{m,n_m}) < \varepsilon$:

(D1′) $A_{m,n} \supset A_{m',n'}$ whenever $m \ge m'$ and $n \le n'$.

3. Compact Measure and Regularity of Measure

In this section, we pick up several results for compact nonadditive measures and regularity of measures.

Definition 5. Let $\mu : \mathcal{F} \to R$ be a nonadditive measure.

(1) A nonempty family $\mathcal{K}$ of subsets of X is called a compact system if for any sequence $\{K_n\} \subset \mathcal{K}$ with $\cap_{n=1}^{\infty}K_n = \emptyset$ there is $n_0 \in N$ such that $\cap_{n=1}^{n_0}K_n = \emptyset$; see [12].

(2) We say that μ is compact if there exists a compact system $\mathcal{K}$ such that for each $A \in \mathcal{F}$ there are sequences $\{K_n\} \subset \mathcal{K}$ and $\{B_n\} \subset \mathcal{F}$ such that $B_n \subset K_n \subset A$ for all $n \in N$ and $\lim_{n\to\infty}\mu(A \setminus B_n) = 0$.

Remark 6. (1) The family of all compact subsets of a Hausdorff space is a compact system.

(2) The family of all finite unions of sets in a compact system is also compact [13, Lemma 1.4]. Therefore, in (2) of the above definition, the compact system $\mathcal{K}$ and the sequences $\{K_n\} \subset \mathcal{K}$ and $\{B_n\} \subset \mathcal{F}$ may be chosen so that $\mathcal{K}$ is closed for finite unions and both $\{K_n\}$ and $\{B_n\}$ are increasing.

By [6, Theorem 1], the following result follows.

Theorem 7. *Let $\mu : \mathcal{F} \to R$ be a nonadditive measure. If μ is compact and autocontinuous, then it is continuous from above and below.*

Proof. Since μ is compact and autocontinuous, by [6, Theorem 1], the assertion follows. □

In what follows, let (X, d) be a metric space. Denote by $\mathcal{B}(X)$ the σ-field of all Borel subsets of X, that is, the σ-field generated by the open subsets of X. A nonadditive measure defined on $\mathcal{B}(X)$ is called a nonadditive Borel measure on X.

Definition 8. μ is said to have pseudometric generating property if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $A, B \in \mathcal{B}(X)$, $\mu(A) \vee \mu(B) < \delta$ implies $\mu(A \cup B) < \varepsilon$.

Proposition 9. *If μ satisfies pseudometric generating property, then it is weakly null-additive.*

Proof. It is easy to see from the definition. □

Definition 10. Let $\mu : \mathcal{B}(X) \to R$ be a nonadditive Borel measure on X.

μ is called regular if for any $\varepsilon > 0$ and $A \in \mathcal{B}(X)$, there exist a closed set F_ε and an open set G_ε such that $F_\varepsilon \subset A \subset G_\varepsilon$ and $\mu(G_\varepsilon \setminus F_\varepsilon) < \varepsilon$.

Li and Mesiar [7] also investigated the regularity on monotone measures. The following follows.

Lemma 11. *Let X be a metric space and $\mu : \mathcal{B}(X) \to R$ a nonadditive Borel measure on X. If μ has the Egoroff condition and pseudometric generating property, then μ is regular.*

Corollary 12. *Let X be a metric space and $\mu : \mathcal{B}(X) \to R$ a nonadditive Borel measure on X. If μ has property (S), is strong order continuous, and is weakly null-additive, then μ is regular.*

By Li and Yasuda [14, Theorem 1], we also have the following.

Corollary 13. *Let X be a metric space. If $\mu : \mathcal{B}(X) \to R$ is weakly null-additive fuzzy Borel measure on X, then it is regular. Moreover if μ is null-additive, we have*

$$\begin{aligned}\mu(A) &= \sup\{\mu(F) \mid F \subset A, F \textit{ is closed set}\} \\ &= \inf\{\mu(G) \mid G \supset A, G \textit{ is open set}\}.\end{aligned} \tag{1}$$

Corollary 13 above is a special case of [6, Theorem 5] and [15, Theorem 3].

For more information on regularity of nonadditive measures, see [5, 6].

4. Radon Measure

In this section, as main results, if we assume that a nonadditive Borel measure satisfies the equivalence condition of Egoroff's theorem and pseudometric generating property on a complete or a locally compact, separable metric space, then it is Radon.

Definition 14. Let μ be a nonadditive Borel measure on X.

(1) μ is said to be Radon (strongly regular) if for each $A \in \mathcal{B}(X)$ there are sequences $\{K_n\}_{n\in N}$ of compact

sets and $\{G_n\}_{n\in N}$ of open sets such that $K_n \subset A \subset G_n$ for all $n \in N$ and $\lim_{n\to\infty}\mu(G_n \setminus K_n) = 0$.

(2) μ is said to be tight if there is a sequence $\{K_n\}_{n\in N}$ of compact sets such that $\lim_{n\to\infty}\mu(X \setminus K_n) = 0$.

Remark 15. Sequences of sets in the above definition may be chosen so that $\{G_n\}_{n\in N}$ is decreasing, while $\{F_n\}_{n\in N}$ and $\{K_n\}_{n\in N}$ are increasing.

Proposition 16. *Let X be a Hausdorff space. Let μ be a nonadditive Borel measure on X which is weakly null-additive and strongly order continuous. Then, the following two conditions are equivalent:*

(i) *μ is Radon (strongly regular),*

(ii) *μ is regular and tight.*

Proof. See [6, Proposition 2]. □

It is known that every finite Borel measure on a complete or a locally compact, separable metric space is Radon; see [16, Theorem 3.2] and [17, Theorems 6 and 9, Chapter II, Part I]. Its counterpart in nonadditive measure theory can be found in [5, 9, Theorem 1, Lemma 2], which states that every Borel fuzzy measure on a complete separable metric space is tight, so that it is Radon if it is null-additive; see also [3, Theorem 2.3]. The following two theorems contain those previous results; see also [18, Theorem 12].

Theorem 17. *Let X be a complete separable metric space and $\mu : \mathscr{B}(X) \to R$ a nonadditive Borel measure on X. If μ is weakly null-additive and satisfies the Egoroff condition, then it is tight. Moreover, if μ has pseudometric generating property and satisfies the Egoroff condition, then it is Radon.*

To prove the theorem, we need the following; see [7, Proposition 3.7].

Proposition 18. *Let $\mu : \mathscr{F} \to R$ be a nonadditive measure. Then (i) implies (ii).*

(i) *μ is weakly null-additive and satisfies the Egoroff condition.*

(ii) *For each $\varepsilon > 0$ and double sequence $\{A_{m,n}\} \subset \mathscr{F}$ satisfying $A_{m,n} \searrow \emptyset$ as $n \to \infty$ for each $m \in N$, there exists a sequence $\{n_m\}$ of natural numbers such that $\mu(\cup_{m=1}^{\infty} A_{m,n_m}) < \varepsilon$.*

Proof of Theorem 17. Since μ satisfies the Egoroff condition, by [19, Proposition 3], it is strongly order continuous. By Proposition 16 and Lemma 11, we have only to prove that μ is tight. Let $\{s_i\}_{i\in N}$ be a countable dense subset of X. For each $m, i \in N$, denote by $\overline{B_m(s_i)}$ the closed ball with center s_i and radius $1/m$. For each $m, n \in N$, put $A_{m,n} := X \setminus \cup_{i=1}^{n}\overline{B_m(s_i)}$. Then, for any $\varepsilon > 0$ and $m \in N$, we have $A_{m,n} \searrow \emptyset$, so that by Proposition 18, there exists a sequence $\{n_m\}$ of natural numbers such that

$$\mu\left(\cup_{m=1}^{\infty} A_{m,n_m}\right) < \varepsilon. \tag{2}$$

Put $P_\varepsilon := \cap_{m=1}^{\infty}\cup_{i=1}^{n_m}\overline{B_m(s_i)}$. Then, each P_ε is closed and totally bounded, so that it is compact. Since $X \setminus P_\varepsilon = \cup_{m=1}^{\infty} A_{m,n_m}$, it follows from (2) that $\mu(X \setminus P_\varepsilon) < \varepsilon$. Thus μ is tight. □

Corollary 19. *Let X be a complete separable metric space and $\mu : \mathscr{B}(X) \to R$ a nonadditive Borel measure on X. If μ is weakly null-additive, strongly order continuous, and has property (S), then it is Radon.*

Proof. It follows from Theorem 17 since μ has pseudometric generating property [7, Proposition 5.1] and satisfies the Egoroff condition [19, Proposition 2]. □

Corollary 20. *Let X be a complete separable metric space and $\mu : \mathscr{B}(X) \to R$ a fuzzy measure on X. If μ is weakly null-additive, then it is Radon.*

Proof. It follows from Theorem 17 since μ satisfies the Egoroff condition [7, Proposition 3.1] and it is regular [14, Theorem 1]. □

Remark 21. Corollary 20 above is a special case of [6, Theorem 5] and [15, Theorem 3].

Theorem 22. *Let X be a locally compact, separable metric space and $\mu : \mathscr{B}(X) \to R$ a nonadditive Borel measure on X. If μ is weakly null-additive and satisfies the Egoroff condition, then it is tight. Moreover, if μ has pseudometric generating property and satisfies Egoroff condition, then it is Radon.*

Proof. By Lemma 11 and Proposition 16, we have only to prove the tightness of μ. Denote by $\mathscr{H}$ the family of all open and relatively compact subsets of X. The local compactness of X implies that $\mathscr{H}$ is an open cover of X. Since X is strongly Lindelöf, that is, every open cover of any open subset of X has a countable subcover [17, Proposition 3 and Theorem 6, Chapter II, Part I], there is a sequence $\{H_m\}_{m\in N} \subset \mathscr{H}$ such that $X = \cup_{m=1}^{\infty} H_m$. Put $K_n := \cup_{m=1}^{n}\overline{H_m}$ for all $n \in N$, where $\overline{A}$ denotes the closure of a set A. Then K_n is compact and $X \setminus K_n \searrow \emptyset$. Since μ is strongly order continuous [19, Proposition 3], $\lim_{n\to\infty}\mu(X \setminus K_n) = 0$. Thus μ is tight. □

Corollary 23. *Let X be a locally compact, separable metric space and $\mu : \mathscr{B}(X) \to R$ a nonadditive Borel measure on X. If μ is weakly-null-additive, strongly order continuous, and has property (S), then μ is Radon.*

Corollary 24. *Let X be a locally compact, separable metric space and $\mu : \mathscr{B}(X) \to R$ a fuzzy Borel measure on X. If μ is weaklly null-additive, then μ is Radon.*

Remark 25. Corollary 24 above is a special case of [6, Theorem 6] and [15, Theorem 4].

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