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Research Article

Iterative Algorithm for Common Fixed Points of Infinite Family of Nonexpansive Mappings in Banach Spaces

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Let C be a nonempty closed convex subset of a real uniformly smooth Banach space X, $\{T_k\}_{k=1}^{\infty} : C \to C$ an infinite family of nonexpansive mappings with the nonempty set of common fixed points $\bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k)$, and $f : C \to C$ a contraction. We introduce an explicit iterative algorithm $x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)L_n x_n$, where $L_n = \sum_{k=1}^{n}(\omega_k/s_n)T_k, S_n = \sum_{k=1}^{n}\omega_k$, and $\omega_k > 0$ with $\sum_{k=1}^{\infty}\omega_k = 1$. Under certain appropriate conditions on $\{\alpha_n\}$, we prove that $\{x_n\}$ converges strongly to a common fixed point x^* of $\{T_k\}_{k=1}^{\infty}$, which solves the following variational inequality: $\langle x^* - f(x^*), J(x^* - p)\rangle \le 0,\ \ p \in \bigcap_{k=1}^{\infty}\mathrm{Fix}(T_k)$, where J is the (normalized) duality mapping of X. This algorithm is brief and needs less computational work, since it does not involve W-mapping.

1. Introduction

Let X be a real Banach space, C a nonempty closed convex subset of X, and X^* the dual space of X. The (normalized) duality mapping $J : X \to 2^{X^*}$ is defined by

$$J(x) = \left\{x^* \in X^* : \langle x, x^*\rangle = \|x\|^2, \|x^*\| = \|x\|\right\}, \quad \forall x \in X. \tag{1.1}$$

If X is a Hilbert space, then $J = I$, where I is the identity mapping. It is well known that if X is smooth, then J is single valued.

Recall that a mapping $f : C \to C$ is a contraction, if there exists a constant $\alpha \in [0,1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

We use Π_C to denote the collection of all contractions on C, that is,

$$\Pi_C = \{f : f \text{ is a contraction on } C\}. \tag{1.3}$$

A mapping $T : C \to C$ is said to be nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C. \tag{1.4}$$

We use $\mathrm{Fix}(T)$ to denote the set of fixed points of T, namely, $\mathrm{Fix}(T) = \{x \in C : Tx = x\}$. One classical way to study nonexpansive mappings is to use contractions to approximate a nonexpansive mapping ([1–11]). Browder [1] first considered the following approximation in a Hilbert space. Fix $u \in C$ and define a contraction F_t from C into itself by

$$F_t x = tu + (1-t)Tx, \quad x \in C, \tag{1.5}$$

where $t \in (0,1)$. Banach contraction mapping principle guarantees that F_t has a unique fixed point in C. Denote by $z_t \in C$ the unique fixed point of F_t, that is,

$$z_t = tu + (1-t)Tz_t. \tag{1.6}$$

In the case of T having fixed points, Browder [1] proved the following.

Theorem 1.1. *In a Hilbert space, as $t \to 0$, z_t defined in* (1.6) *converges strongly to a fixed point of T that is closest to u, that is, the nearest point projection of u onto* $\mathrm{Fix}(T)$.

Halpern [3] introduced an iteration process (discretization of (1.6)) in a Hilbert as follows:

$$z_{n+1} = \alpha_n u + (1-\alpha_n)Tz_n, \quad n \geq 0, \tag{1.7}$$

where $u, z_0 \in C$ are arbitrary (but fixed) and $\{\alpha_n\}$ is a sequence in $(0,1)$. Lions [4] proved the following.

Theorem 1.2. *In a Hilbert space, if $\{\alpha_n\}$ satisfies the following conditions:*

(K1) $\lim_{n\to\infty}\alpha_n = 0$;

(K2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(K3) $\lim_{n\to\infty}|\alpha_n - \alpha_{n-1}|/\alpha_{n+1}^2 = 0$.

Then $\{z_n\}$ converges strongly to the nearest point projection of u onto $\mathrm{Fix}(T)$.

The Banach space versions of Theorems 1.1 and 1.2 were obtained by Reich [5]. He proved the following.

Theorem 1.3. *In a uniformly smooth Banach space X, both z_t defined in* (1.6) *and* $\{z_n\}$ *defined in* (1.7) *converge strongly to a same fixed point of T. If one defines* $Q : C \to \mathrm{Fix}(T)$ *by*

$$Q(u) = \lim_{t\to 0} z_t, \tag{1.8}$$

then Q is the sunny nonexpansive retraction from C onto $\mathrm{Fix}(T)$. *Namely, Q satisfies the property:*

$$\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy)\rangle, \quad x, y \in C, \tag{1.9}$$

where J is the duality mapping of X.

Moudafi [6] introduced a viscosity approximation method and proved the strong convergence of both the implicit and explicit methods in Hilbert spaces. Xu [7] extended Moudafi's results in Hilbert spaces. Given a real number $t \in (0,1)$ and a contraction $f \in \Pi_C$, define a contraction $T_t^f : C \to C$ by

$$T_t^f x = tf(x) + (1-t)Tx, \quad x \in C. \tag{1.10}$$

Let $x_t \in C$ be the unique fixed point of T_t^f. Thus,

$$x_t = tf(x_t) + (1-t)Tx_t. \tag{1.11}$$

Corresponding explicit iterative process is defined by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \tag{1.12}$$

where $x_0 \in C$ is arbitrary (but fixed) and $\{\alpha_n\}$ is a sequence in $(0,1)$. It was proved by Xu [7] that under certain appropriate conditions on $\{\alpha_n\}$, both x_t defined in (1.11) and $\{x_n\}$ defined in (1.12) converged strongly to $x^* \in C$, which is the unique solution of the variational inequality:

$$\langle (I - f)x^*, x - x^*\rangle \geq 0, \quad x \in \mathrm{Fix}(T). \tag{1.13}$$

Xu [7] also extended Moudafi's results to the setting of Banach spaces and proved the strong convergence of both the implicit method (1.11) and explicit method (1.12) in uniformly smooth Banach spaces.

In order to deal with some problems involving the common fixed points of infinite family of nonexpansive mappings, W-mapping is often used, see [12–20]. Let $\{T_k\}_{k=1}^{\infty} : C \to C$ be an infinite family of nonexpansive mappings and let $\{\xi_k\}_{k=1}^{\infty}$ be a real number sequence such that $0 < \xi_k < 1$ for every $k \in \mathbb{N}$. For any $n \in \mathbb{N}$, we define a mapping W_n of C into itself as follows:

$$\begin{aligned}
U_{n,n+1} &= I,\\
U_{n,n} &= \xi_n T_n U_{n,n+1} + (1-\xi_n) I,\\
U_{n,n-1} &= \xi_{n-1} T_{n-1} U_{n,n} + (1-\xi_{n-1}) I,\\
&\vdots\\
U_{n,k} &= \xi_k T_k U_{n,k+1} + (1-\xi_k) I,\\
U_{n,k-1} &= \xi_{k-1} T_{k-1} U_{n,k} + (1-\xi_{k-1}) I,\\
&\vdots\\
U_{n,2} &= \xi_2 T_2 U_{n,3} + (1-\xi_2) I,\\
W_n &= U_{n,1} = \xi_1 T_1 U_{n,2} + (1-\xi_1) I.
\end{aligned} \tag{1.14}$$

Such W_n is called the W-mapping generated by $\{T_k\}_{k=1}^{\infty}$ and $\{\xi_k\}_{k=1}^{\infty}$, see [12, 13].

Yao et al. [10] introduced the following iterative algorithm for infinite family of nonexpansive mappings. Let X be a uniformly convex Banach space with a uniformly Gâteaux differentiable norm and C a nonempty closed convex subset of X. Sequence $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n u + (1-\alpha_n) W_n x_n, \quad n \geq 0, \tag{1.15}$$

where $u, x_0 \in C$ are arbitrary (but fixed) and $\{\alpha_n\} \subset (0,1)$. It was proved that under certain appropriate conditions on $\{\alpha_n\}$, the sequence $\{x_n\}$ generated by (1.15) converges strongly to a common fixed point of $\{T_k\}_{k=1}^{\infty}$ [13].

Since W-mapping contains many composite operations of $\{T_k\}$, it is complicated and needs large computational work. In this paper, we introduce a new iterative algorithm for solving the common fixed point problem of infinite family of nonexpansive mappings. Let X be a real uniformly smooth Banach space, C a nonempty closed convex subset of X, $\{T_k\}_{k=1}^{\infty} : C \to C$ an infinite family of nonexpansive mappings with the nonempty set of common fixed points $\bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k)$, and $f \in \Pi_C$. Given any $x_0 \in C$, define a sequence $\{x_n\}$ by

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) L_n x_n, \quad n \geq 0, \tag{1.16}$$

where $\{\alpha_n\} \subset (0,1)$, $L_n = \sum_{k=1}^{n} (\omega_k / s_n) T_k$, $S_n = \sum_{k=1}^{n} \omega_k$ and $w_k > 0$ with $\sum_{k=1}^{\infty} \omega_k = 1$. Under certain appropriate conditions on $\{\alpha_n\}$, we prove that $\{x_n\}$ converges strongly to $x^* \in \bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k)$, which solves the following variational inequality:

$$\langle x^* - f(x^*), J(x^* - p) \rangle \leq 0, \quad p \in \bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k), \tag{1.17}$$

where J is the duality mapping of X. Because L_n doesn't contain many composite operations of $\{T_k\}$, this algorithm is brief and needs less computational wok.

We will use M to denote a constant, which may be different in different places.

2. Preliminaries

Let $B = \{x \in X : \|x\| = 1\}$ denotes the unit sphere of X. A Banach space X is said to be strictly convex, if $\|(x+y)/2\| < 1$ holds for all $x, y \in B$, $x \neq y$. A Banach space X is said to be uniformly convex if for each $\varepsilon \in (0,2]$, there exists a constant $\delta > 0$ such that for any x, $y \in B$, $\|x-y\| \geq \varepsilon$ implies $\|(x+y)/2\| \leq 1-\delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex, see [21].

The norm of X is said to be Gâteaux differentiable if

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists for each $x, y \in B$ and in this case X is said to be smooth. The norm of X is said to be uniformly Gâteaux differentiable if for each $y \in B$, the limit (2.1) is attained uniformly for $x \in B$. The norm of X is said to be Frêchet differentiable, if for each $x \in B$, the limit (2.1) is attained uniformly for $y \in B$. The norm of X is said to be uniformly Frêchet differentiable, if the limit (2.1) is attained uniformly for $x, y \in B$ and in this case X is said to be uniformly smooth.

Let D be a nonempty subset of C. A mapping $Q : C \to D$ is said to be sunny [22] if

$$Q(x + t(x - Q(x))) = Q(x), \quad \forall x \in C, \ t \geq 0, \tag{2.2}$$

whenever $x + t(x - Q(x)) \in C$. A mapping $Q : C \to D$ is called a retraction if $Qx = x$ for all $x \in D$. Furthermore, Q is sunny nonexpansive retraction from C onto D if Q is a retraction from C onto D which is also sunny and nonexpansive.

A subset D of C is called a sunny nonexpansive retraction of C if there exits a sunny nonexpansive retraction from C onto D.

Lemma 2.1 (see [22]). *Let C be a closed convex subset of a smooth Banach space X. Let D be a nonempty subset of C and $Q : C \to D$ be a retraction. Then the following are equivalent.*

(a) *Q is sunny and nonexpansive.*

(b) $\|Qx - Qy\|^2 \leq \langle x - y, J(Qx - Qy) \rangle$, *for all $x, y \in C$.*

(c) $\langle x - Qx, J(y - Qx) \rangle \leq 0$, *for all $x \in C, y \in D$.*

Lemma 2.2 (see [23]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers satisfying*

$$s_{n+1} \leq (1 - \gamma_n) s_n + \gamma_n \beta_n + \delta_n, \quad n \geq 0, \tag{2.3}$$

where $\{\gamma_n\} \subset (0,1)$, $\{\beta_n\}$ *and* $\{\delta_n\}$ *satisfy the following conditions:*

(A1) $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(A2) $\limsup_{n\to\infty} \beta_n \leq 0$;

(A3) $\delta_n \geq 0$ $(n \geq 0)$, $\sum_{n=1}^{\infty} \delta_n < \infty$.

Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 (see [24]). *In a Banach space X, the following inequality holds:*

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle, \quad \forall x, y \in X, \tag{2.4}$$

where $j(x+y) \in J(x+y)$.

Lemma 2.4 (see [25]). *Let C be a closed convex subset of a strictly convex Banach space X. Let* $\{T_n : n \in \mathbb{N}\}$ *be a sequence of nonexpansive mappings on C. Suppose* $\bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ *is nonempty. Let* $\{\lambda_n\}$ *be a sequence of positive numbers with* $\sum_{n=1}^{\infty} \lambda_n = 1$. *Then a mapping S on C defined by* $Sx = \sum_{n=1}^{\infty} \lambda_n T_n x$ *for* $x \in C$ *is well defined, nonexpansive and* $\mathrm{Fix}(S) = \bigcap_{n=1}^{\infty} \mathrm{Fix}(T_n)$ *holds.*

Lemma 2.5 (see [7]). *Let X be a uniformly smooth Banach space, C a closed convex subset of X,* $T : C \to C$ *a nonexpansive mapping with* $\mathrm{Fix}(T) \neq \emptyset$, *and* $f \in \Pi_C$. *Then* $\{x_t\}$ *defined by*

$$x_t = tf(x_t) + (1-t)Tx_t \tag{2.5}$$

converges strongly to a point in $\mathrm{Fix}(T)$. *If we define a mapping* $Q : \Pi_C \to \mathrm{Fix}(T)$ *by*

$$Q(f) := \lim_{t\to 0} x_t, \quad f \in \Pi_C, \tag{2.6}$$

then $Q(f)$ *solves the variational inequality:*

$$\langle (I-f)Q(f), J(Q(f)-p)\rangle \leq 0, \quad f \in \Pi_C,\ p \in \mathrm{Fix}(T). \tag{2.7}$$

Lemma 2.6. *Let X be a Banach space,* $\{x_k\}$ *a bounded sequence of X, and* $\{\omega_k\}$ *a sequence of positive numbers with* $\sum_{k=1}^{\infty} \omega_k = 1$. *Then* $\sum_{k=1}^{\infty} \omega_k x_k$ *is convergent in X.*

Lemma 2.7. *Let X be Banach space,* $\{T_k : k \in \mathbb{N}\}$ *a sequence of nonexpansive mappings on X with* $\bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k) \neq \emptyset$, *and* $\{\omega_k\}$ *a sequence of positive numbers with* $\sum_{k=1}^{\infty} \omega_k = 1$. *Let* $T = \sum_{k=1}^{\infty} \omega_k T_k$, $L_m = \sum_{k=1}^{m} (\omega_k / S_m) T_k$, *and* $S_m = \sum_{k=1}^{m} \omega_k$. *Then* L_m *uniformly converges to T in each bounded subset S of X.*

Proof. For all $x \in S$, we observe that

$$\begin{aligned}
\|L_m x - Tx\| &= \left\| \sum_{k=1}^{m} \frac{\omega_k}{S_m} T_k x - \sum_{k=1}^{\infty} \omega_k T_k x \right\| \\
&= \left\| \sum_{k=1}^{m} \frac{\omega_k - \omega_k S_m}{S_m} T_k x - \sum_{k=m+1}^{\infty} \omega_k T_k x \right\| \\
&\le \left\| \sum_{k=1}^{m} \frac{1 - S_m}{S_m} \omega_k T_k x \right\| + \left\| \sum_{k=m+1}^{\infty} \omega_k T_k x \right\| \\
&\le \frac{1 - S_m}{S_m} \sum_{k=1}^{m} \omega_k \|T_k x\| + \sum_{k=m+1}^{\infty} \omega_k \|T_k x\| \\
&\le \frac{1 - S_m}{S_m} M + M \sum_{k=m+1}^{\infty} \omega_k,
\end{aligned} \tag{2.8}$$

where $M = \sup_{x \in S, k \ge 1} \|T_k x\| < \infty$. Taking $m \to \infty$ in above last inequality, we have that

$$\lim_{m \to \infty} \|L_m x - Tx\| = 0 \tag{2.9}$$

holds uniformly for $x \in S$ and this completes the proof. □

3. Main Results

Theorem 3.1. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space X, $\{T_k\}_{k=1}^{\infty} : C \to C$ an infinite family of nonexpansive mappings with $\bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k) \ne \emptyset$, and $\{\omega_k\}$ a sequence of positive numbers with $\sum_{k=1}^{\infty} \omega_k = 1$. Let $L_n = \sum_{k=1}^{n} (\omega_k / S_n) T_k$, $S_n = \sum_{k=1}^{n} \omega_k$, and $f \in \Pi_C$ with coefficient $\alpha \in [0,1)$. Given any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) L_n x_n, \quad n \ge 0, \tag{3.1}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(A1) $\lim_{n \to \infty} \alpha_n = 0$;

(A2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(A3) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n \to \infty} (\alpha_{n+1} / \alpha_n) = 1$.

Then $\{x_n\}$ converges strongly to $x^* \in \bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k)$, which solves the following variational inequality:

$$\langle x^* - f(x^*), J(x^* - p) \rangle \le 0, \quad p \in \bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k). \tag{3.2}$$

Proof.

Step 1. We show that $\{x_n\}$ is bounded.

Noticing nonexpansiveness of L_n, take a $p \in \bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k)$ to derive that

$$\begin{aligned}
\|x_{n+1} - \mathrm{p}\| &= \|\alpha_n f(x_n) + (1-\alpha_n)L_n x_n - p\| \\
&\le \alpha_n \|f(x_n) - p\| + (1-\alpha_n)\|L_n x_n - p\| \\
&= \alpha_n \|f(x_n) - f(p) + f(p) - p\| + (1-\alpha_n)\|L_n x_n - p\| \\
&\le \alpha_n \|f(x_n) - f(p)\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|L_n x_n - p\| \\
&\le \alpha\alpha_n \|x_n - p\| + \alpha_n \|f(p) - p\| + (1-\alpha_n)\|x_n - p\| \\
&= (1-(1-\alpha)\alpha_n)\|x_n - p\| + \alpha_n \|(f(p) - p)\| \\
&= (1-(1-\alpha)\alpha_n)\|x_n - p\| + (1-\alpha)\alpha_n \frac{\|(f(p) - p)\|}{1-\alpha} \\
&\le \max\left\{\frac{\|(f(p) - p)\|}{1-\alpha}, \|x_n - p\|\right\}.
\end{aligned} \tag{3.3}$$

By induction, we obtain

$$\|x_{n+1} - p\| \le \max\left\{\frac{\|(f(p) - p)\|}{1-\alpha}, \|x_0 - p\|\right\}, \quad n \ge 0, \tag{3.4}$$

and $\{x_n\}$ is bounded, so are $\{T_k x_n\}$, $\{L_n x_n\}$, and $\{f(x_n)\}$.

Step 2. We prove that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

By (3.1), We have

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\alpha_n f(x_n) + (1-\alpha_n)L_n x_n - \alpha_{n-1} f(x_{n-1}) - (1-\alpha_{n-1})L_{n-1}x_{n-1}\| \\
&= \|\alpha_n f(x_n) - \alpha_n f(x_{n-1}) + \alpha_n f(x_{n-1}) - \alpha_{n-1} f(x_{n-1}) \\
&\qquad + (1-\alpha_n)L_n x_n - (1-\alpha_n)L_{n-1}x_{n-1} + (1-\alpha_n)L_{n-1}x_{n-1} - (1-\alpha_{n-1})L_{n-1}x_{n-1}\| \\
&\le \alpha\alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|f(x_{n-1})\| \\
&\quad + (1-\alpha_n)\|L_n x_n - L_{n-1}x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|L_{n-1}x_{n-1}\| \\
&= \alpha\alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|L_{n-1}x_{n-1}\|) \\
&\quad + (1-\alpha_n)\|L_n x_n - L_{n-1}x_{n-1}\| \\
&\le \alpha\alpha_n\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|(\|f(x_{n-1})\| + \|L_{n-1}x_{n-1}\|) \\
&\quad + (1-\alpha_n)\|L_n x_n - L_n x_{n-1}\| + (1-\alpha_n)\|L_n x_{n-1} - L_{n-1}x_{n-1}\|
\end{aligned}$$

$$\begin{aligned}
&\leq (1-(1-\alpha)\alpha_n)\|x_n - x_{n-1}\| + (1-\alpha_n)\|L_n x_{n-1} - L_{n-1}x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|\left(\left\|f(x_{n-1})\right\| + \|L_{n-1}x_{n-1}\|\right) \\
&\leq (1-(1-\alpha)\alpha_n)\|x_n - x_{n-1}\| + (1-\alpha_n)\|L_n x_{n-1} - L_{n-1}x_{n-1}\| \\
&\quad + |\alpha_n - \alpha_{n-1}|M,
\end{aligned} \tag{3.5}$$

where $M = \sup_{n\geq 1}(\|f(x_{n-1})\| + \|L_{n-1}x_{n-1}\|)$. At the same time, we observe that

$$\begin{aligned}
\sum_{n=1}^{\infty}\|L_n x_{n-1} - L_{n-1}x_{n-1}\| &= \sum_{n=1}^{\infty}\left\|\sum_{k=1}^{n}\frac{\omega_k}{S_n}T_k x_{n-1} - \sum_{k=1}^{n-1}\frac{\omega_k}{S_{n-1}}T_k x_{n-1}\right\| \\
&= \sum_{n=1}^{\infty}\left\|\frac{\omega_n}{S_n}T_n x_{n-1} + \sum_{k=1}^{n-1}\frac{-\omega_n\omega_k}{S_n S_{n-1}}T_k x_{n-1}\right\| \\
&\leq \sum_{n=1}^{\infty}\left\{\left\|\frac{\omega_n}{S_n}T_n x_{n-1}\right\| + \left\|\sum_{k=1}^{n-1}\frac{\omega_n\omega_k}{S_n S_{n-1}}T_k x_{n-1}\right\|\right\} \\
&\leq \sum_{n=1}^{\infty}\frac{\omega_n}{S_n}\|T_n x_{n-1}\| + \sum_{n=1}^{\infty}\sum_{k=1}^{n-1}\frac{\omega_n\omega_k}{S_n S_{n-1}}\|T_k x_{n-1}\| \\
&\leq \sum_{n=1}^{\infty}\frac{\omega_n}{S_n}M + \sum_{n=1}^{\infty}\frac{\omega_n}{S_n}M \\
&= \sum_{n=1}^{\infty}\frac{2M}{S_n}\omega_n,
\end{aligned} \tag{3.6}$$

where $M = \sup_{k\geq 1, n\geq 1}\|T_k x_{n-1}\|$. Applying Lemma 2.6 and compatibility test of series, we have

$$\sum_{n=1}^{\infty}\|L_n x_{n-1} - L_{n-1}x_{n-1}\| < \infty. \tag{3.7}$$

Put

$$\begin{gathered}
\gamma_n = (1-\alpha)\alpha_n, \qquad \beta_n = \frac{|\alpha_n - \alpha_{n-1}|M}{(1-\alpha)\alpha_n}, \\
\delta_n = (1-\alpha_n)\|L_n x_{n-1} - L_{n-1}x_{n-1}\|.
\end{gathered} \tag{3.8}$$

It follows that

$$\|x_{n+1} - x_n\| \leq (1-\gamma_n)\|x_n - x_{n-1}\| + \gamma_n\beta_n + \delta_n. \tag{3.9}$$

It is easily seen from (A2), (A3), and (3.7) that

$$\sum_{n=1}^{\infty}\gamma_n = \infty, \qquad \limsup_{n\to\infty}\beta_n \le 0, \qquad \sum_{n=1}^{\infty}\delta_n < \infty. \tag{3.10}$$

Applying Lemma 2.2 to (3.9), we obtain

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.11}$$

Step 3. We show that $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$.

Indeed we observe that

$$\begin{aligned}
\|x_n - Tx_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - Tx_n\| \\
&= \|x_n - x_{n+1}\| + \left\|\alpha_n f(x_n) + (1-\alpha_n)L_n x_n - Tx_n\right\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\left\|f(x_n) - Tx_n\right\| + (1-\alpha_n)\|L_n x_n - Tx_n\|.
\end{aligned} \tag{3.12}$$

Hence, by (3.11), (A1), and Lemma 2.7, we have

$$\lim_{n\to\infty}\|x_n - Tx_n\| = 0. \tag{3.13}$$

Step 4. We prove that

$$\limsup_{n\to\infty}\left\langle f(x^*) - x^*, J(x_n - x^*)\right\rangle \le 0, \tag{3.14}$$

where $x^* = \lim_{t\to 0}x_t$ with x_t being the fixed point of the contraction

$$x \longmapsto tf(x) + (1-t)Tx. \tag{3.15}$$

From Lemma 2.5, we have $x^* \in \mathrm{Fix}(T)$ and

$$\left\langle (I-f)x^*, J\left(x^* - p\right)\right\rangle \le 0, \quad p \in \mathrm{Fix}(T). \tag{3.16}$$

By Lemma 2.4, we have $x^* \in \bigcap_{k=1}^{\infty}\mathrm{Fix}(T_k)$ and

$$\left\langle (I-f)x^*, J\left(x^* - p\right)\right\rangle \le 0, \quad p \in \bigcap_{k=1}^{\infty}\mathrm{Fix}(T_k). \tag{3.17}$$

By $x_t = tf(x_t) + (1-t)Tx_t$, we have

$$x_t - x_n = t\left(f(x_t) - x_n\right) + (1-t)(Tx_t - x_n). \tag{3.18}$$

It follows from Lemma 2.3 that

$$\begin{aligned}
\|x_t - x_n\|^2 &= \left\|t(f(x_t) - x_n) + (1-t)(Tx_t - x_n)\right\|^2 \\
&\le (1-t)^2\|Tx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, J(x_t - x_n)\rangle \\
&\le (1-t)^2(\|Tx_t - Tx_n\| + \|Tx_n - x_n\|)^2 + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle \\
&\quad + 2t\langle x_t - x_n, J(x_t - x_n)\rangle \\
&\le (1-t)^2(\|x_t - x_n\| + \|Tx_n - x_n\|)^2 + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle \\
&\quad + 2t\langle x_t - x_n, J(x_t - x_n)\rangle \\
&\le (1-t)^2\|x_t - x_n\|^2 + b_n(t) + 2t\langle f(x_t) - x_t, J(x_t - x_n)\rangle \\
&\quad + 2t\|x_t - x_n\|^2,
\end{aligned} \tag{3.19}$$

where $b_n(t) = \|Tx_n - x_n\|(2\|x_t - x_n\| + \|Tx_n - x_n\|) \to 0$ $(n \to \infty)$. It follows from above last inequality that

$$\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}b_n(t). \tag{3.20}$$

Taking $n \to \infty$ in (3.20) yields

$$\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le \frac{t}{2}M, \tag{3.21}$$

where $M \ge \|x_t - x_n\|^2$ for all $n \ge 1$ and $t \in (0,1)$. Taking $t \to 0$ in (3.21), we have

$$\limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \le 0. \tag{3.22}$$

Noticing the fact that two limits are interchangeable due to the fact the duality mapping J is norm-to-norm uniformly continuous on bounded sets, it follows from (3.22), we have

$$\begin{aligned}
\limsup_{n\to\infty}\left\langle f(x^*) - x^*, J(x_n - x^*)\right\rangle &= \limsup_{n\to\infty}\limsup_{t\to 0}\langle x_t - f(x_t), J(x_t - x_n)\rangle \\
&= \limsup_{t\to 0}\limsup_{n\to\infty}\langle x_t - f(x_t), J(x_t - x_n)\rangle \\
&\le 0.
\end{aligned} \tag{3.23}$$

Hence (3.14) holds.

Step 5. Finally, we prove that $x_n \to x^*$ $(n \to \infty)$.

Indeed we observe that

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|\alpha_n f(x_n) + (1-\alpha_n) L_n x_n - x^*\right\|^2 \\
&= \left\|\alpha_n (f(x_n) - f(x^*)) + (1-\alpha_n)(L_n x_n - x^*) + \alpha_n (f(x^*) - x^*)\right\|^2 \\
&\le \left\|\alpha_n (f(x_n) - f(x^*)) + (1-\alpha_n)(L_n x_n - x^*)\right\|^2 \\
&\quad + 2\langle \alpha_n (f(x^*) - x^*), J(x_{n+1} - x^*)\rangle \\
&\le \left\{\alpha_n \left\|f(x_n) - f(x^*)\right\| + (1-\alpha_n)\|L_n x_n - x^*\|\right\}^2 \\
&\quad + 2\alpha_n \langle f(x^*) - x^*, J(x_{n+1} - x^*)\rangle \\
&\le (\alpha\alpha_n \|x_n - x^*\| + (1-\alpha_n)\|x_n - x^*\|)^2 \\
&\quad + 2\alpha_n \langle f(x^*) - x^*, J(x_{n+1} - x^*)\rangle \\
&\le (1 - (1-\alpha)\alpha_n)^2 \|x_n - x^*\|^2 + 2\alpha_n \langle f(x^*) - x^*, J(x_{n+1} - x^*)\rangle \\
&\le (1 - (1-\alpha)\alpha_n) \|x_n - x^*\|^2 + 2\alpha_n \langle f(x^*) - x^*, J(x_{n+1} - x^*)\rangle.
\end{aligned} \tag{3.24}$$

By view of (3.14) and condition (A2), it follows from Lemma 2.2 that $x_n \to x^*(n \to \infty)$. This completes the proof. □

Corollary 3.2. *Let C be a nonempty closed convex subset of a real uniformly smooth Banach space X, $\{T_k\}_{k=1}^{\infty} : C \to C$ an infinite family of nonexpansive mappings with $\bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k) \neq \emptyset$, $\{\omega_k\}$ a sequence of positive numbers with $\sum_{k=1}^{\infty} \omega_k = 1$. Let $L_n = \sum_{k=1}^{n} (\omega_k / S_n) T_k$, $S_n = \sum_{k=1}^{n} \omega_k$, and $u \in C$. Given any $x_0 \in C$, let $\{x_n\}$ be a sequence generated by*

$$x_{n+1} = \alpha_n u + (1-\alpha_n) L_n x_n, \quad n \ge 0, \tag{3.25}$$

where $\{\alpha_n\} \subset (0,1)$ satisfies the following conditions:

(A1) $\lim_{n\to\infty} \alpha_n = 0$;

(A2) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(A3) *either* $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ *or* $\lim_{n\to\infty} (\alpha_{n+1}/\alpha_n =)1$.

Then $\{x_n\}$ converges strongly to $x^* \in \bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k)$, which solves the following variational inequality:

$$\langle x^* - u, J(x^* - p)\rangle \le 0, \quad p \in \bigcap_{k=1}^{\infty} \mathrm{Fix}(T_k). \tag{3.26}$$

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