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Research Article

Continuity of the Solution Maps for Generalized Parametric Set-Valued Ky Fan Inequality Problems

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Under new assumptions, we provide suffcient conditions for the (upper and lower) semicontinuity and continuity of the solution mappings to a class of generalized parametric set-valued Ky Fan inequality problems in linear metric space. These results extend and improve some known results in the literature (e.g., Gong, 2008; Gong and Yoa, 2008; Chen and Gong, 2010; Li and Fang, 2010). Some examples are given to illustrate our results.

1. Introduction

The Ky Fan inequality is a very general mathematical format, which embraces the formats of several disciplines, as those for equilibrium problems of mathematical physics, those from game theory, those from (vector) optimization and (vector) variational inequalities, and so on (see [1, 2]). Since Ky Fan inequality was introduced in [1, 2], it has been extended and generalized to vector or set-valued mappings. The Ky Fan Inequality for a set-/vector-valued mapping is known as the (weak) generalized Ky Fan inequality ((W)GKFI, in short). In the literature, existing results for various types of (generalized) Ky Fan inequalities have been investigated intensively, see [3–5] and the references therein.

It is well known that the stability analysis of solution maps for parametric Ky Fan inequality (PKFI, in short) is an important topic in optimization theory and applications. There are some papers to discuss the upper and/or lower semicontinuity of solution maps. Cheng and Zhu [6] discussed the upper semicontinuity and the lower semicontinuity of the solution map for a PKFI in finite-dimensional spaces. Anh and Khanh [7, 8] studied the stability of solution sets for two classes of parametric quasi-KFIs. Huang et al. [9]

discussed the upper semicontinuity and lower semicontinuity of the solution map for a parametric implicit KFI. By virtue of a density result and scalarization technique, Gong [10] first discussed the lower semicontinuity of the set of efficient solutions for a parametric KFI with vector-valued maps. By using the ideas of Cheng and Zhu [6], Gong and Yao [11] studied the continuity of the solution map for a class of weak parametric KFI in topological vector spaces. Then, Kimura and Yao [12] discussed the semicontinuity of solution maps for parametric quasi-KFIs. Based on the work of [6, 10], the continuity of solution sets for PKFIs was discussed in [13] without the uniform compactness assumption. Recently, Li and Fang [14] obtained a new sufficient condition for the lower semicontinuity of the solution maps to a generalized PKFI with vector-valued mappings, where their key assumption is different from the ones in [11, 13].

Motivated by the work reported in [10, 11, 14, 15], this paper aims at studying the stability of the solution maps for a class of generalized PKFI with set-valued mappings. We obtain some new sufficient conditions for the semicontinuity of the solution sets to the generalized PKFI. Our results are new and different from the corresponding ones in [6, 10, 11, 13–17].

The rest of the paper is organized as follows. In Section 2, we introduce a class of generalized set-valued PKFI and recall some concepts and their properties which are needed in the sequel. In Section 3, we discuss the upper semicontinuity and lower semicontinuity of the solution mappings for the class of generalized PKFI and compare our main results with the corresponding ones in the recent literature ([10, 11, 13–15]). We also give two examples to illustrate that our main results are applicable.

2. Preliminaries

Throughout this paper, if not, otherwise, specified, $d(\cdot,\cdot)$ denotes the metric in any metric space. Let $B(0,\delta)$ denote the closed ball with radius $\delta \geq 0$ and center 0 in any metric linear spaces. Let X and Y be two real linear metric spaces. Let Z be a linear metric space and let Λ be a nonempty subset of Z. Let Y^* be the topological dual space of Y, and let C be a closed, convex, and pointed cone in Y with $\operatorname{int} C \neq \emptyset$, where $\operatorname{int} C$ denotes the interior of C. Let

$$C^* := \{ f \in Y^* : f(y) \geq 0,\ \forall y \in C \} \tag{2.1}$$

be the dual cone of C.

Let A be a nonempty subset of X, and let $F : A \times A \rightrightarrows Y \setminus \{\emptyset\}$ be a set-valued mapping. We consider the following generalized KFI which consist in finding $x \in A(\lambda)$ such that

$$F(x,y) \cap (-\operatorname{int} C) = \emptyset, \quad \forall y \in A(\lambda). \tag{KFI}$$

When the set A and the function F are perturbed by a parameter λ which varies over a set Λ of Z, we consider the following weak generalized PKFI which consist in finding $x \in A(\lambda)$ such that

$$F(x,y,\lambda) \cap (-\operatorname{int} C) = \emptyset, \quad \forall y \in A(\lambda), \tag{PKFI}$$

where $A : \Lambda \rightrightarrows X \setminus \{\emptyset\}$ is a set-valued mapping and $F : B \times B \times \Lambda \subset X \times X \times Z \rightrightarrows Y \setminus \{\emptyset\}$ is a set-valued mapping with $A(\Lambda) = \bigcup_{\lambda \in \Lambda} A(\lambda) \subset B$.

For each $\lambda \in \Lambda$, the solution set of (PKFI) is defined by

$$V(F, \lambda) := \{x \in A(\lambda) \mid F(x, y, \lambda) \cap (-\operatorname{int} C) = \emptyset, \ \forall y \in A(\lambda)\}. \tag{2.2}$$

For each $f \in C^* \setminus \{0\}$ and for each $\lambda \in \Lambda$, the f-solution set of (PKFI) is defined by

$$V_f(F, \lambda) := \left\{x \in A(\lambda) \mid \inf_{z \in F(x,y,\lambda)} f(z) \geq 0, \ \forall y \in A(\lambda)\right\}. \tag{2.3}$$

Special Case

(i) If for any $\lambda \in \Lambda, x, y \in A(\lambda)$, $F(x, y, \lambda) := \varphi(x, y, \lambda) + \psi(y, \lambda) - \psi(x, \lambda)$, where $\varphi : A(\mu) \times A(\mu) \times \Lambda \to 2^Y$ is a set-valued mapping and $\psi : A(\mu) \times \Lambda \to Y$ is a single-valued mapping, the (PKFI) reduces to the weak parametric vector equilibrium problem ((W)PVEP) considered in [15].

(ii) When F is a vector-valued mapping, that is, $F : B \times B \times \Lambda \subset X \times X \times Z \to Y$, the (PKFI) reduces to the parametric Ky Fan inequality in [14].

(iii) If for any $\lambda \in \Lambda, x, y \in A(\lambda)$, $F(x, y, \lambda) := \varphi(x, y, \lambda) + \psi(y, \lambda) - \psi(x, \lambda)$, where $\varphi : A(\mu) \times A(\mu) \times \Lambda \to Y$ and $\psi : A(\mu) \times \Lambda \to Y$ are two vector-valued maps, the (PKFI) reduces to the parametric (weak) vector equilibrium problem (PVEP) considered in [10, 11, 13, 16].

Throughout this paper, we always assume $V(F, \lambda) \neq \emptyset$ for all $\lambda \in \Lambda$. This paper aims at investigating the semicontinuity and continuity of the solution mapping $V(F, \lambda)$ as set-valued map from the set Λ into X. Now, we recall some basic definitions and their properties which are needed in this paper.

Definition 2.1. Let $F : X \times X \times Z \rightrightarrows Y \setminus \{\emptyset\}$ be a trifunction.

(i) $F(x, \cdot, \lambda)$ is called C-convex function on $A(\lambda)$, if and only if for every $x_1, x_2 \in A(\lambda)$ and $t \in [0, 1]$, $tF(x, x_1, \lambda) + (1 - t)F(x, x_2, \lambda) \subset F(x, tx_1 + (1 - t)x_2, \lambda) + C$.

(ii) $F(x, \cdot, \lambda)$ is called C-like-convex function on $A(\lambda)$, if and only if for any $x_1, x_2 \in A(\lambda)$ and any $t \in [0, 1]$, there exists $x_3 \in A(\lambda)$ such that $tF(x, x_1, \lambda) + (1 - t)F(x, x_2, \lambda) \subset F(x, x_3, \lambda) + C$.

(iii) $F(\cdot, \cdot, \cdot)$ is called C-monotone on $A(\Lambda) \times A(\Lambda) \times \Lambda$, if and only if for any $\lambda \in \Lambda$ and $x, y \in A(\lambda)$, $F(x, y, \lambda) + F(y, x, \lambda) \subset -C$. The mapping F is called C-strictly monotone (or called C-strongly monotone in [10]) on $A(\Lambda) \times A(\Lambda) \times \Lambda$ if F is C-monotone and if for any given $\lambda \in \Lambda$, for all $x, y \in A(\lambda)$ and $x \neq y$, s.t. $F(x, y, \lambda) + F(y, x, \lambda) \subset -\operatorname{int} C$.

Definition 2.2 (see [18]). Let X and Y be topological spaces, $T : X \rightrightarrows Y \setminus \{\emptyset\}$ be a set-valued mapping.

(i) T is said to be upper semicontinuous (u.s.c., for short) at $x_0 \in X$ if and only if for any open set V containing $T(x_0)$, there exists an open set U containing x_0 such that $T(x) \subseteq V$ for all $x \in U$.

(ii) T is said to be lower semicontinuous (l.s.c., for short) at $x_0 \in X$ if and only if for any open set V with $T(x_0) \cap V \neq \emptyset$, there exists an open set U containing x_0 such that $T(x) \cap V \neq \emptyset$ for all $x \in U$.

(iii) T is said to be continuous at $x_0 \in X$, if it is both l.s.c. and u.s.c. at $x_0 \in X$. T is said to be l.s.c. (resp. u.s.c.) on X, if and only if it is l.s.c. (resp., u.s.c.) at each $x \in X$.

From [19, 20], we have the following properties for Definition 2.2.

Proposition 2.3. *Let X and Y be topological spaces, let $T : X \rightrightarrows Y \setminus \{\emptyset\}$ be a set-valued mapping.*

(i) *T is l.s.c. at $x_0 \in X$ if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and any $y_0 \in T(x_0)$, there exists $y_\alpha \in T(x_\alpha)$ such that $y_\alpha \to y_0$.*

(ii) *If T has compact values (i.e., $T(x)$ is a compact set for each $x \in X$), then T is u.s.c. at x_0 if and only if for any net $\{x_\alpha\} \subset X$ with $x_\alpha \to x_0$ and for any $y_\alpha \in T(x_\alpha)$, there exist $y_0 \in T(x_0)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$, such that $y_\beta \to y_0$.*

3. Semicontinuity and Continuity of the Solution Map for (PKFI)

In this section, we obtain some new sufficient conditions for the semicontinuity and continuity of the solution maps to the (PKFI).

Firstly, we provide a new result of sufficient condition for the upper semicontinuity and closeness of the solution mapping to the (PKFI).

Theorem 3.1. *For the problem* (PKFI)*, suppose that the following conditions are satisfied:*

(i) *$A(\cdot)$ is continuous with nonempty compact value on Λ;*

(ii) *$F(\cdot,\cdot,\cdot)$ is l.s.c. on $B \times B \times \Lambda$.*

Then, $V(F,\cdot)$ is u.s.c. and closed on Λ.

Proof. (i) Firstly, we prove $V(F,\cdot)$ is u.s.c. on Λ. Suppose to the contrary, there exists some $\mu_0 \in \Lambda$ such that $V(F,\cdot)$ is not u.s.c. at μ_0. Then, there exist an open set V satisfying $V(F,\mu_0) \subset V$ and sequences $\mu_n \to \mu_0$ and $x_n \in V(F,\mu_n)$, such that

$$x_n \notin V, \quad \forall n. \tag{3.1}$$

Since $x_n \in A(\mu_n)$ and $A(\cdot)$ are u.s.c. at μ_0 with compact values by Proposition 2.3, there is an $x_0 \in A(\mu_0)$ such that $x_n \to x_0$ (here, we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ if necessary).

Now, we need to show that $x_0 \in V(F,\mu_0)$. By contradiction, assume that $x_0 \notin V(F,\mu_0)$. Then, there exists $y_0 \in A(\mu_0)$ such that

$$F(x_0, y_0, \mu_0) \cap (-\operatorname{int} C) \neq \emptyset, \tag{3.2}$$

that is,

$$\exists z_0 \in F(x_0, y_0, \mu_0), \quad \text{s.t. } z_0 \in -\operatorname{int} C. \tag{3.3}$$

By the lower semicontinuity of $A(\cdot)$ at μ_0, for $y_0 \in A(\mu_0)$, there exists $y_n \in A(\mu_n)$ such that $y_n \to y_0$.

It follows from $x_n \in V(F, \mu_n)$ and $y_n \in A(\mu_n)$ that

$$F(x_n, y_n, \mu_n) \cap (-\operatorname{int} C) = \emptyset. \tag{3.4}$$

Since $F(\cdot, \cdot, \cdot)$ is l.s.c. at (x_0, y_0, λ_0), for $z_0 \in F(x_0, y_0, \mu_0)$, there exists $z_n \in F(x_n, y_n, \mu_n)$ such that

$$z_n \longrightarrow z_0. \tag{3.5}$$

From (3.3), (3.5), and the openness of $\operatorname{int} C$, there exists a positive integer N sufficiently large such that for all $n \geq N$,

$$z_n \in -\operatorname{int} C, \quad \text{for } z_n \in F(x_n, y_n, \mu_n), \tag{3.6}$$

which contradicts (3.4). So, we have

$$x_0 \in V(F, \mu_0) \subset V. \tag{3.7}$$

Since $x_n \to x_0$ (here we can take a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ if necessary), we can find (3.7) contradicts (3.1). Consequently, $V(F, \cdot)$ is u.s.c. on Λ.

(ii) In a similar way to the proof of (i), we can easily obtain the closeness of $V(F, \cdot)$ on Λ. This completes the proof. □

Remark 3.2. Theorem 3.1 generalizes and improves the corresponding results of Gong [10, Theorem 3.1] in the following four aspects:

(i) the condition that $A(\cdot)$ is convex values is removed;

(ii) the vector-valued mapping $F(\cdot, \cdot, \cdot)$ is extended to set-valued mapping, and the condition that C-monotone of mapping is removed;

(iii) the assumption (iii) of Theorem 3.1 in [10] is removed;

(iv) the condition that $A(\cdot)$ is uniformly compact near $\mu \in \Lambda$ is not required.

Moreover, we also can see that the obtained result extends Theorem 2.1 of [15].

Now, we give an example to illustrate that Theorem 3.1 is applicable.

Example 3.3. Let $X = Z = Y = \mathbb{R}$, $C = \mathbb{R}_+$, $\Lambda = [0, 2^{1/2}]$ be a subset of Z. Let $F : X \times X \times \Lambda \rightrightarrows Y$ be a set-valued mapping defined by $F(x, y, \lambda) = [(y + 1)(\lambda^2 + 1)(x - \lambda), 10 + \lambda^2]$ and let $A : \Lambda \rightrightarrows X$ defined by $A(\lambda) = [\lambda^2, 2]$.

It follows from direct computation that

$$A(\Lambda) = [0, 2], \quad V(F, \lambda) = [\lambda, 2], \quad \forall \lambda \in \Lambda = \left[0, 2^{1/2}\right]. \tag{3.8}$$

Then, we can verify that all assumptions of Theorem 3.1 are satisfied. By Theorem 3.1, $V(F, \cdot)$ is u.s.c. and closed on Λ. Therefore, Theorem 3.1 is applicable.

When $F : X \times X \times Z \to Y$ is a vector-valued mapping, one can get the following corollary.

Corollary 3.4. *For the problem* (PKFI), *suppose that* $F : X \times X \times Z \to Y$ *is a vector-valued mapping and the following conditions are satisfied:*

(i) $A(\cdot)$ *is continuous with nonempty compact value on* Λ*;*

(ii) $F(\cdot,\cdot,\cdot)$ *is continuous on* $B \times B \times \Lambda$.

Then, $V(F,\cdot)$ *is u.s.c. and closed on* Λ.

Now, we give a sufficient condition for the lower semicontinuity of the solution maps to the (PKFI).

Theorem 3.5. *Let* $f \in C^* \setminus \{0\}$. *Suppose that the following conditions are satisfied:*

(i) $A(\cdot)$ *is continuous with nonempty compact value on* Λ*;*

(ii) $F(\cdot,\cdot,\cdot)$ *is u.s.c. with nonempty compact values on* $B \times B \times \Lambda$*;*

(iii) *for each* $\lambda \in \Lambda$, $x \in A(\lambda) \setminus V_f(F,\lambda)$, *there exists* $y \in V_f(F,\lambda)$, *such that*

$$F(x,y,\lambda) + F(y,x,\lambda) + B\left(0, d^{\gamma}(x,y)\right) \subset -C, \tag{3.9}$$

where $\gamma > 0$ *is a positive constant.*

Then, $V_f(F,\cdot)$ is l.s.c. on Λ.

Proof. By the contrary, assume that there exists $\lambda_0 \in \Lambda$, such that $V_f(F,\cdot)$ is not l.s.c. at λ_0. Then, there exist λ_α with $\lambda_\alpha \to \lambda_0$ and $x_0 \in V_f(F,\lambda_0)$, such that for any $x_\alpha \in V_f(F,\lambda_\alpha)$ with $x_\alpha \nrightarrow x_0$.

Since $x_0 \in A(\lambda_0)$ and $A(\cdot)$ are l.s.c. at λ_0, there exists $\widehat{x}_\alpha \in A(\lambda_\alpha)$ such that $\widehat{x}_\alpha \to x_0$. We claim that $\widehat{x}_\alpha \in A(\lambda) \setminus V_f(F,\lambda_\alpha)$. If not, for $\widehat{x}_\alpha \in V_f(F,\lambda_\alpha)$, it follows from above-mentioned assumption that $\widehat{x}_\alpha \nrightarrow x_0$, which is a contradiction.

By (iii), there exists $y_\alpha \in V_f(F,\lambda_\alpha)$, such that

$$F\left(\widehat{x}_\alpha, y_\alpha, \lambda_\alpha\right) + F\left(y_\alpha, \widehat{x}_\alpha, \lambda_\alpha\right) + B\left(0, d^{\gamma}\left(\widehat{x}_\alpha, y_\alpha\right)\right) \subset -C. \tag{3.10}$$

For $y_\alpha \in V_f(F,\lambda_\alpha) \subset A(\lambda_\alpha)$, because $A(\cdot)$ is u.s.c. at λ_0 with compact values by Proposition 2.3, there exist $y_0 \in A(\lambda_0)$ and a subsequence $\{y_{\alpha_k}\}$ of $\{y_\alpha\}$ such that $y_{\alpha_k} \to y_0$. In particular, for (3.10), we have

$$F\left(\widehat{x}_{\alpha_k}, y_{\alpha_k}, \lambda_{\alpha_k}\right) + F\left(y_{\alpha_k}, \widehat{x}_{\alpha_k}, \lambda_{\alpha_k}\right) + B\left(0, d^{\gamma}\left(\widehat{x}_{\alpha_k}, y_{\alpha_k}\right)\right) \subset -C. \tag{3.11}$$

Then, there exist $\widehat{z}_{\alpha_k} \in F(\widehat{x}_{\alpha_k}, y_{\alpha_k}, \lambda_{\alpha_k})$ and $\widetilde{z}_{\alpha_k} \in F(y_{\alpha_k}, \widehat{x}_{\alpha_k}, \lambda_{\alpha_k})$ such that

$$\widehat{z}_{\alpha_k} + \widetilde{z}_{\alpha_k} + B\left(0, d^{\gamma}\left(\widehat{x}_{\alpha_k}, y_{\alpha_k}\right)\right) \subset -C. \tag{3.12}$$

Since $F(\cdot,\cdot,\cdot)$ is u.s.c. with compact values on $B\times B\times\Lambda$ by Proposition 2.3, there exist $\widehat{z}_0\in F(x_0,y_0,\lambda_0)$ and $\widetilde{z}_0\in F(y_0,x_0,\lambda_0)$ such that $\widehat{z}_{\alpha_k}\to\widehat{z}_0$, $\widetilde{z}_{\alpha_k}\to\widetilde{z}_0$. From $\widehat{x}_{\alpha_k}\to x_0$, $y_{\alpha_k}\to y_0$, the continuity of $d(\cdot,\cdot)$, and the closedness of C, we have

$$\widehat{z}_0+\widetilde{z}_0+B\left(0,d^{\gamma}(x_0,y_0)\right)\subset -C. \tag{3.13}$$

It follows from $x_0\in V_f(F,\lambda_0)$ and $y_0\in A(\lambda_0)$ that $\inf_{z\in F(x_0,y_0,\lambda_0)}f(z)\geq 0$. Particularly, we have

$$f(\widehat{z}_0)\geq 0. \tag{3.14}$$

On the other hand, since $y_{\alpha_k}\in V_f(F,\lambda_{\alpha_k})$ and $\widehat{x}_{\alpha_k}\in A(\lambda_{\alpha_k})$, we have $\inf_{z\in F(y_{\alpha_k},\widehat{x}_{\alpha_k},\lambda_{\alpha_k})}f(z)\geq 0$. Also, we have $f(\widetilde{z}_{\alpha_k})\geq 0$. It follows from the continuity of f that we have

$$f(\widetilde{z}_0)\geq 0. \tag{3.15}$$

By (3.14), (3.15), and the linearity of f, we get

$$f(\widehat{z}_0+\widetilde{z}_0)=f(\widehat{z}_0)+f(\widetilde{z}_0)\geq 0. \tag{3.16}$$

For the above x_0 and y_0, we consider two cases:

Case i. If $x_0\neq y_0$, by (3.13), we can obtain that

$$\widehat{z}_0+\widetilde{z}_0\in -\operatorname{int}C. \tag{3.17}$$

Then, it follows from $f\in C^*\setminus\{0\}$ that

$$f(\widehat{z}_0+\widetilde{z}_0)<0, \tag{3.18}$$

which is a contradiction to (3.16).

Case ii. If $x_0=y_0$, since $y_\alpha\in V_f(F,\lambda_\alpha)$, $y_\alpha\to y_0=x_0$, this contradicts that for any $x_\alpha\in V_f(F,\lambda_\alpha)$, x_α do not converge to x_0. Thus, $V_f(F,\cdot)$ is l.s.c. on Λ. The proof is completed.

□

Remark 3.6. Theorem 3.5 generalizes and improves the corresponding results of [14, Lemma 3.1] in the following three aspects:

(i) the condition that $A(\cdot)$ is convex values is removed;

(ii) the vector-valued mapping $F(\cdot,\cdot,\cdot)$ is extended to set-valued map;

(iii) the constant γ can be any positive constant ($\gamma>0$) in Theorem 3.5, while it should be strictly restricted to $\gamma=1$ in Lemma 3.1 of [14].

Moreover, we also can see that the obtained result extends the ones of Gong and Yao [11, Theorem 2.1], where a strong assumption that C-strict/strong monotonicity of the mappings is required.

The following example illustrates that the assumption (iii) of Theorem 3.5 is essential.

Example 3.7. Let $X = Y = \mathbb{R}, C = \mathbb{R}_+$. Let $\Lambda = [3,5]$ be a subset of Z. For each $\lambda \in \Lambda, x, y \in X$, let $A(\lambda) = [\lambda - 3, 2]$ and $F : X \times X \times \Lambda \rightrightarrows Y \setminus \{\emptyset\}$ be a set-valued mapping defined by

$$F(x,y,\lambda) = \left[\left(39 - \lambda^2 - \lambda\right)\frac{\sqrt{\lambda+6}}{3}x(x-y), 68 + (2\lambda-1)^2 + \frac{\lambda}{3}\right]. \tag{3.19}$$

Obviously, assumptions (i) and (ii) of Theorem 3.5 are satisfied, and $A(\lambda) = [0,2]$, for all $\lambda \in \Lambda$. For any given $\lambda \in \Lambda$, let $f(F(x,y,\lambda)) = z/3$, for all $z \in F(x,y,\lambda)$. Then, it follows from a direct computation that

$$V_f(F,3) = \{0,2\}, \quad V_f(F,\lambda) = 2, \quad \forall \lambda \in (3,5]. \tag{3.20}$$

However, $V_f(F,\lambda)$ is even not l.c.s. at $\lambda = 3$. The reason is that the assumption (iii) is violated. Indeed, if $x = 0 \in V_f(F,\lambda)$, for $\lambda = 3$ and for all $\gamma > 0$, there exist $y = 1/2 \in A(\lambda) \setminus V_f(F,\lambda) = (0,2)$, such that

$$\begin{aligned}
&F(x,y,\lambda) + F(y,x,\lambda) + B(0,d^\gamma(x,y)) \\
&\quad = \left[\left(39 - \lambda^2 - \lambda\right)\frac{\sqrt{\lambda+6}}{3}x(x-y), 68 + (2\lambda-1)^2 + \frac{\lambda}{3}\right] \\
&\qquad + \left[\left(39 - \lambda^2 - \lambda\right)\frac{\sqrt{\lambda+6}}{3}y(y-x), 68 + (2\lambda-1)^2 + \frac{\lambda}{3}\right] + B(0,d^\gamma(x,y)) \\
&\quad = \left[\left(39 - \lambda^2 - \lambda\right)\frac{\sqrt{\lambda+6}}{3}(x-y)^2, 136 + 2(2\lambda-1)^2 + \frac{2\lambda}{3}\right] + B(0,d^\gamma(x,y)) \\
&\quad = \left[\frac{27}{4} - \left|0 - \frac{1}{2}\right|^\gamma, 188 + \left|0 - \frac{1}{2}\right|^\gamma\right] \not\subseteq -C;
\end{aligned} \tag{3.21}$$

if $x = 2 \in V_f(F,\lambda)$, for $\lambda = 3$ and for all $\gamma > 0$, there exist $y = 1/2 \in A(\lambda) \setminus V_f(F,\lambda)$, using a similar method, we have $F(x,y,\lambda) + F(y,x,\lambda) + B(0,d^\gamma(x,y)) \not\subseteq -C$. Therefore, (iii) is violated.

Now, we show that $V_f(F,\cdot)$ is not l.s.c. at $\lambda = 3$. Indeed, there exists $0 \in V_f(F,3)$ and there exists a neighborhood $(-2/9, 2/9)$ of 0, for any neighborhood $N(3)$ of 3, there exists $3 < \tilde{\lambda} < 5$ such that $\tilde{\lambda} \in N(3)$ and

$$V_f\left(F,\tilde{\lambda}\right) = 2 \notin \left(-\frac{2}{9}, \frac{2}{9}\right). \tag{3.22}$$

Thus,

$$V_f\left(F,\tilde{\lambda}\right) \cap \left(-\frac{2}{9}, \frac{2}{9}\right) = \emptyset. \tag{3.23}$$

By Definition 2.2 (or page 108 in [18]), we know that $V_f(F,\cdot)$ is not l.c.s. at $\lambda = 3$. So, the assumption (iii) of Theorem 3.5 is essential.

By virtue of Theorem 1.1 in [15] (or Lemma 2.1 in [16]), we can get the following proposition.

Proposition 3.8. *Suppose that for each* $\lambda \in \Lambda$ *and* $x \in A(\lambda)$, $F(x, A(\lambda), \lambda) + C$ *is a convex set, then*

$$V(F,\lambda) = \bigcup_{f \in C^* \setminus \{0\}} V_f(F,\lambda). \tag{3.24}$$

Theorem 3.9. *For the problem* (PKFI), *suppose that the following conditions are satisfied:*

(i) $A(\cdot)$ *is continuous with nonempty compact convex value on* Λ*;*

(ii) $F(\cdot,\cdot,\cdot)$ *is continuous with nonempty compact values on* $B \times B \times \Lambda$*;*

(iii) *for each* $\lambda \in \Lambda$, $x \in A(\lambda) \setminus V_f(F,\lambda)$, *there exists* $y \in V_f(F,\lambda)$, *such that*

$$F(x,y,\lambda) + F(y,x,\lambda) + B\left(0, d^{\gamma}(x,y)\right) \subset -C, \tag{3.25}$$

where $\gamma > 0$ *is a positive constant.*

(iv) *for each* $\lambda \in \Lambda$ *and for each* $x \in A(\lambda)$, $F(x,\cdot,\lambda)$ *is* C*-like-convex on* $A(\lambda)$.

Then, $V(F,\cdot)$ is closed and continuous (i.e., both l.s.c. and u.s.c.) on Λ.

Proof. From Theorem 3.1, it is easy to see that $V(F,\cdot)$ is u.s.c. and closed on Λ. Now, we will only prove that $V(F,\cdot)$ is l.s.c. on Λ. For each $\lambda \in \Lambda$ and for each $x \in A(\lambda)$, since $F(x,\cdot,\lambda)$ is C-like-convex on $A(\lambda)$, $F(x, A(\lambda), \lambda) + C$ is convex. Thus, by virtue of Proposition 3.8, for each $\lambda \in \Lambda$, it holds

$$V(F,\lambda) = \bigcup_{f \in C^* \setminus \{0\}} V_f(F,\lambda). \tag{3.26}$$

By Theorem 3.5, for each $f \in C^* \setminus \{0\}$, $V_f(F,\cdot)$ is l.s.c. on Λ. Therefore, in view of Theorem 2 in [20, page 114], we have $V(F,\cdot)$ is l.s.c. on Λ. This completes the proof. □

Remark 3.10. Theorem 3.9 generalizes and improves the work in [15, Theorems 3.4-3.5]. Our approach on the (semi)continuity of the solution mapping $V(F,\cdot)$ is totally different from the ones by Chen and Gong [15]. In [15], the $V_f(F,\lambda)$ is strictly to be a singleton, while it may be a set-valued one in our paper. In addition, the assumption that C-strictly monotonicity of the mapping F is not required and the C-convexity of F is generalized to the C-like-convexity.

When the mapping F is vector-valued, we obtain the following corollary.

Corollary 3.11. *For the problem* (PKFI), *suppose that* $F : X \times X \times Z \to Y$ *is a vector-valued mapping and the following conditions are satisfied:*

(i) $A(\cdot)$ *is continuous with nonempty compact convex value on* Λ*;*

(ii) $F(\cdot,\cdot,\cdot)$ *is continuous on* $B \times B \times \Lambda$*;*

(iii) *for each* $\lambda \in \Lambda$, $x \in A(\lambda) \setminus V_f(F,\lambda)$, *there exists* $y \in V_f(F,\lambda)$, *such that*

$$F(x,y,\lambda) + F(y,x,\lambda) + B(0, d^{\gamma}(x,y)) \subset -C, \tag{3.27}$$

where $\gamma > 0$ *is a positive constant.*

(iv) *for each* $\lambda \in \Lambda$ *and for each* $x \in A(\lambda)$, $F(x,\cdot,\lambda)$ *is C-like-convex on* $A(\lambda)$.

Then, $V(F,\cdot)$ *is closed and continuous (i.e., both l.s.c. and u.s.c.) on* Λ.

Remark 3.12. Corollary 3.11 generalizes and improves [10, Theorem 4.2] and [13, Theorem 4.2], respectively, because the assumption that C-strict monotonicity of the mapping F is not required.

Next,we give the following example to illustrate the case.

Example 3.13. Let $X = Z = \mathbb{R}, Y = \mathbb{R}^2, C = \mathbb{R}^2_+, \Lambda = [-1,1]$ be a subset of Z. Let $F : X \times X \times \Lambda \to Y$ be a mapping defined by

$$F(x,y,\lambda) = \left(-\frac{3}{2} - \lambda^2, \left(\lambda^2 + 1\right)x\right) \tag{3.28}$$

and define $A : \Lambda \to 2^Y$ by $A(\lambda) = [-1,1]$.

Obviously, $A(\cdot)$ is a continuous set-valued mapping from Λ to R with nonempty compact convex values, and conditions (ii) and (iv) of Corollary 3.11 are satisfied.

Let $f = (0,2) \in C^* \setminus \{0\}$, it follows from a direct computation that $V_f(F,\lambda) = [0,1]$ for any $\lambda \in \Lambda$. Hence, for any $x \in A(\lambda) \setminus V_f(F,\lambda)$, there exists $y = 0 \in V_f(F,\lambda)$, such that,

$$\begin{aligned} &F(x,y,\lambda) + F(y,x,\lambda) + B(0, d^{\gamma}(x,y)) \\ &\quad = \left(-\frac{3}{2} - \lambda^2, \left(\lambda^2 + 1\right)x\right) + \left(-\frac{3}{2} - \lambda^2, \left(\lambda^2 + 1\right)y\right) + B(0, d^{\gamma}(x,y)) \\ &\quad = \left(-3 - 2\lambda^2, \left(\lambda^2 + 1\right)x\right) + B(0, |x-0|^{\gamma}) \\ &\quad \in -C. \end{aligned} \tag{3.29}$$

Thus, the condition (iii) of Corollary 3.11 is also satisfied. By Corollary 3.11, $V(F,\cdot)$ is closed and continuous (i.e., both l.s.c. and u.s.c.) on Λ.

However, the condition that F is a C-strictly monotone mapping is violated. Indeed, for any $\lambda \in \Lambda = [-1,1]$ and $x \in A(\lambda) \setminus V_f(F,\cdot)$, there exist $y = -x \in V_f(F,\cdot)$ with $y = -x$, such that

$$F(x,y,\lambda) + F(y,x,\lambda) = \left(-3 - 2\lambda^2, 0\right) \notin -\operatorname{int} C, \tag{3.30}$$

which implies that $F(\cdot,\cdot,\cdot)$ is not $\mathbb{R}^2_+$-strictly monotone on $A(\Lambda) \times A(\Lambda) \times \Lambda$. Then, Theorem 4.2 in [10] and Theorem 4.2 in [13] are not applicable, and the corresponding results in references (e.g., [11, Lemma 2.2, Theorem 2.1]) are also not applicable.

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