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Research Article

On Boundaries of Parallelizable Regions of Flows of Free Mappings

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We are interested in the first prolongational limit set of the boundary of parallelizable regions of a given flow of the plane which has no fixed points. We prove that for every point from the boundary of a maximal parallelizable region, there exists exactly one orbit contained in this region which is a subset of the first prolongational limit set of the point. Using these uniquely determined orbits, we study the structure of maximal parallelizable regions.

1. Introduction

We assume that f is a *free mapping*, that is, an orientation preserving homeomorphism of the plane onto itself without fixed points. We consider a relation in $\mathbb{R}^2$ defined in the following way:

$p \sim q$ if $p = q$ or p and q are endpoints of some arc K for which $f^n(K) \to \infty$ as $n \to \pm\infty$. By an *arc* K with endpoints p and q, we mean that the image of a homeomorphism $c : [0,1] \to c([0,1])$ satisfying conditions $c(0) = p$, $c(1) = q$, where the topology on $c([0,1])$ is induced by the topology of $\mathbb{R}^2$. It turns out that the relation defined above is an equivalence relation (see [1]) and has the same equivalence classes as the relation defined by Andrea [2]. Moreover, each equivalence class is an invariant simply connected set (see [2, 1]).

From now on, we assume that f is embeddeable in a flow $\{f^t : t \in \mathbb{R}\}$. It follows from the Jordan theorem that each orbit C of $\{f^t : t \in \mathbb{R}\}$ divides the plane into two simply connected regions. Note that each of them is invariant under f^t for $t \in \mathbb{R}$. Thus two different orbits C_p and C_q of points p and q, respectively, divide the plane into three simply connected invariant regions, one of which contains both C_p and C_q in its boundary. We will call this region the *strip* between C_p and C_q and denote it by D_{pq}.

For any distinct orbits C_{p_1}, C_{p_2}, C_{p_3} of $\{f^t : t \in \mathbb{R}\}$, one of the following two possibilities must be satisfied: exactly one of the orbits C_{p_1}, C_{p_2}, C_{p_3} is contained in the strip between the other two, or each of the orbits C_{p_1}, C_{p_2}, C_{p_3} is contained in the strip between the other two. In the first case, if C_{p_j} is the orbit which lies in the strip between C_{p_i} and C_{p_k}, we will write $C_{p_i}|C_{p_j}|C_{p_k}$ (i, j, $k \in \{1,2,3\}$ and i, j, k are different). In the second case, we will write $|C_{p_i}, C_{p_j}, C_{p_k}|$ (see [3, 4]).

Put

$$\begin{aligned}
J^+(q) := \{p \in \mathbb{R}^2 : &\text{there exist a sequence } (q_n)_{n\in\mathbb{N}} \text{ and a sequence } (t_n)_{n\in\mathbb{N}} \\
&\text{such that } q_n \longrightarrow q,\ t_n \longrightarrow +\infty,\ f^{t_n}(q_n) \longrightarrow p \text{ as } n \longrightarrow +\infty\}, \\
J^-(q) := \{p \in \mathbb{R}^2 : &\text{ there exist a sequence } (q_n)_{n\in\mathbb{N}} \text{ and a sequence } (t_n)_{n\in\mathbb{N}} \\
&\text{such that } q_n \longrightarrow q,\ t_n \longrightarrow -\infty,\ f^{t_n}(q_n) \longrightarrow p \text{ as } n \longrightarrow +\infty\}.
\end{aligned} \tag{1.1}$$

The set $J(q) := J^+(q) \cup J^-(q)$ is called the *first prolongational limit set* of q. Let us observe that $p \in J(q)$ if and only if $q \in J(p)$ for any $p,q \in \mathbb{R}^2$. For a subset $H \subset \mathbb{R}^2$, we define

$$J(H) := \bigcup_{q\in H} J(q). \tag{1.2}$$

One can observe that for each $p \in \mathbb{R}^2$, the set $J(p)$ is invariant. In [5], it has been proved that each orbit contained in $J(\mathbb{R}^2)$ is a boundary orbit of an equivalence class. Therefore every equivalence class can contain at most two orbits from $J(\mathbb{R}^2)$ (see [6]).

An invariant region $M \subset \mathbb{R}^2$ is said to be *parallelizable* if there exists a homeomorphism φ mapping M onto $\mathbb{R}^2$ such that

$$f^t(x) = \varphi^{-1}(\varphi(x) + (t,0)) \quad \text{for } x \in M. \tag{1.3}$$

It is known that a region M is parallelizable if and only if there exists a homeomorphic image K of a straight line which is a closed set in M such that K has exactly one common point with every orbit of $\{f^t : t \in \mathbb{R}\}$ contained in M (see [7, page 49], and, e.g., [6]). We will call such a set K a *section* in M.

It is known that a region M is parallelizable if and only if $J(M) \cap M = \emptyset$ (see [7, pages 46 and 49]). Hence for every parallelizable region M, we have $J(M) \subset \operatorname{fr} M$. If M is a maximal parallelizable region (i.e., M is not contained properly in any parallelizable region), then $J(M) = \operatorname{fr} M$ (see [8]). In [5], it has been proved that every maximal parallelizable region M is a union of equivalence classes of the relation $\sim$.

Now we collect the results from [5, 9] which are needed in this paper.

Proposition 1.1. *(see [5]) Let M be a parallelizable region and let $p \in \operatorname{fr} M$. Then $\operatorname{cl} M \setminus C_p$ is contained in one of the components of $\mathbb{R}^2 \setminus C_p$.*

Proposition 1.2. *(see [5]) Let M be a maximal parallelizable region and $p \in \operatorname{fr} M$. Let G_0 be the equivalence class which contains p. Assume that G_0 does not consist of just one orbit. Then $p \notin J(q)$ for each point q belonging to the component of $\mathbb{R}^2 \setminus C_p$ that does not contain M.*

Proposition 1.3. *(see [9]) Let p and q belong to different equivalence classes G_1 and G_2, respectively. Then there exists a point r lying in the strip between the orbits C_p and C_q of p and q, respectively, such that $r \notin G_1 \cup G_2$.*

Proposition 1.4. *(see [9]) Let M be a parallelizable region. Let $G_1 \cup G_2 \subset M$ and $\operatorname{fr} G_1 \cap \operatorname{fr} G_2 \neq \emptyset$. Let $p \in G_1$, $q \in G_2$. Then there exists a point $z \in D_{pq}$ such that $z \in \operatorname{fr} M$. Moreover $|C_p, C_q, C_z|$ for each $z \in D_{pq} \cap \operatorname{fr} M$.*

2. Boundary orbits of a parallelizable region

In this section, we prove some properties of boundary orbits of parallelizable regions. The main result of this section says that for every point from the boundary of a maximal parallelizable region, there exists exactly one orbit contained in this region which is a subset of the first prolongational limit set of the point.

Proposition 2.1. *Let M be a parallelizable region of $\{f^t : t \in \mathbb{R}\}$. Then $\operatorname{fr} M$ is invariant.*

Proof. Let $p \in \operatorname{fr} M$ and $t \in \mathbb{R}$. Then on account of Proposition 1.1, M is contained in one of the components of $\mathbb{R}^2 \setminus C_p$. Denote this component by H_0, and the other by H_1. Fix $\varepsilon > 0$ and consider the ball $B(f^t(p), \varepsilon)$ centered at $f^t(p)$ with radius ε. By the continuity of f^t, there exists $\delta > 0$ such that $f^t(B(p,\delta)) \subset B(f^t(p), \varepsilon)$, where $B(p,\delta)$ denotes the ball centered at p with radius δ.

Since $p \in \operatorname{fr} M$, there exists $r \in M \cap B(p,\delta)$. Thus $f^t(r) \in B(f^t(p), \varepsilon)$. Moreover, $f^t(r) \in M$ since M is invariant. Consequently, $B(f^t(p), \varepsilon)$ contains a point from M. On the other hand, $B(f^t(p), \varepsilon) \cap H_1$ does not contain any point from M. Thus $f^t(p) \in \operatorname{fr} M$. □

Proposition 2.2. *Let M be a parallelizable region of $\{f^t : t \in \mathbb{R}\}$. Then for all distinct orbits C_{p_1}, C_{p_2}, C_{p_3} contained in $\operatorname{fr} M$, the relation $|C_{p_1}, C_{p_2}, C_{p_3}|$ holds.*

Proof. Let C_{p_1}, C_{p_2}, C_{p_3} be distinct orbits which are contained in $\operatorname{fr} M$. Suppose, on the contrary, that for these orbits the relation $\cdot \mid \cdot \mid \cdot$ holds. Without loss of generality, we can consider only the case $C_{p_1} \mid C_{p_2} \mid C_{p_3}$. Then the orbits C_{p_1} and C_{p_3} are contained in different components of $\mathbb{R}^2 \setminus C_{p_2}$. On the other hand, by Proposition 1.1, $\operatorname{cl} M \setminus C_{p_2}$ is contained in the same component of $\mathbb{R}^2 \setminus C_{p_2}$. Hence C_{p_1} and C_{p_3} are contained in the same component of $\mathbb{R}^2 \setminus C_{p_2}$ since $C_{p_1} \cup C_{p_3} \subset \operatorname{cl} M \setminus C_{p_2}$. Thus we get a contradiction, and consequently $|C_{p_1}, C_{p_2}, C_{p_3}|$. □

Proposition 2.3. *Let M be a parallelizable region of $\{f^t : t \in \mathbb{R}\}$. Let $r \in M$ and let H be a component of $\mathbb{R}^2 \setminus C_r$. Then for all distinct orbits C_{p_1}, C_{p_2} contained in $\operatorname{fr} M \cap H$, the relation $|C_{p_1}, C_{p_2}, C_r|$ holds.*

Proof. By Proposition 1.1, the points r, p_1 and r, p_2 are contained in the same component of $\mathbb{R}^2 \setminus C_{p_2}$ and in the same component of $\mathbb{R}^2 \setminus C_{p_1}$, respectively. Hence, by assumption that p_1 and p_2 are contained in the same component of $\mathbb{R}^2 \setminus C_r$, we obtain $|C_{p_1}, C_{p_2}, C_r|$. □

Proposition 2.4. *Let $q_1, q_2 \in J(p)$, $C_{q_1} \neq C_{q_2}$. Then $|C_{q_1}, C_{q_2}, C_r|$ for every $r \in D_{q_1,q_2} \setminus C_p$ holds (cf. Figure 2.1).*

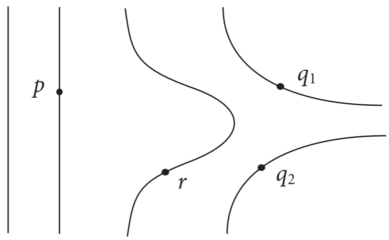

Figure 2.1. The first prolongational limit set of p containing two orbits.

Proof. First we show that $p \in D_{q_1,q_2}$. Suppose, on the contrary, that p belongs to the component of $\mathbb{R}^2 \setminus C_{q_1}$ which does not contain q_2. Denote this component by H_0. Then $J(p) \subset \mathrm{cl}\, H_0 = H_0 \cup C_{q_1}$. Hence $q_2 \notin J(p)$, which is a contradiction. In the same way, we can show that p cannot belong to the component of $\mathbb{R}^2 \setminus C_{q_2}$ which does not contain q_1. Fix a point $r \in D_{q_1,q_2} \setminus C_p$. Then either $|C_{q_1}, C_{q_2}, C_r|$ or $C_{q_1} \mid C_r \mid C_{q_2}$ holds. We show that the second possibility cannot hold. Suppose that $C_{q_1} \mid C_r \mid C_{q_2}$ holds. Then either $p \in D_{q_1,r}$ or $p \in D_{r,q_2}$ since $p \notin C_r$ and $p \in D_{q_1,q_2}$. The first case contradicts the assumption that $q_2 \in J(p)$, and the second one contradicts the assumption that $q_1 \in J(p)$ since $J(p) \subset \mathrm{cl}\, H_1$, where H_1 is the component of $\mathbb{R}^2 \setminus C_r$ which contains p . Thus $|C_{q_1}, C_{q_2}, C_r|$ holds. □

Corollary 2.5. *Let M be a parallelizable region of $\{f^t : t \in \mathbb{R}\}$. Let $p \in \mathrm{fr}\, M$ and $q_1, q_2 \in M$. Assume that $q_1, q_2 \in J(p)$. Then $C_{q_1} = C_{q_2}$.*

Proof. Suppose, on the contrary, that $C_{q_1} \neq C_{q_2}$. Since $q_1, q_2 \in M$ and M is arcwise connected, there exists a point $r \in M \cap D_{q_1,q_2}$. Hence by the parallelizability of M, we get $C_{q_1} \mid C_r \mid C_{q_2}$. By Proposition 2.1, we have $C_p \subset \mathrm{fr}\, M$. Hence $r \notin C_p$ since $r \in M$ and M is open. Thus on account of Proposition 2.4, we have $|C_{q_1}, C_{q_2}, C_r|$, which is a contradiction. □

Remark 2.6. From Corollary 2.5, we get that for every parallelizable region M and every $p \in \mathrm{fr}\, M$, the set $M \cap J(p)$ is either an orbit (in case $p \in J(M)$) or empty (in case $p \in \mathrm{fr}\, M \setminus J(M)$). In the case where M is a maximal parallelizable region such that $M \neq \mathbb{R}^2$ (i.e., $\mathrm{fr}\, M \neq \emptyset$), the existence of such an orbit for each $p \in \mathrm{fr}\, M$ follows from the fact that $J(M) = \mathrm{fr}\, M$ (see [8]). In this case, for each $p \in \mathrm{fr}\, M$ the set $J(p)$ can contain also orbits from $\mathrm{fr}\, M$ and orbits from the component of $\mathbb{R}^2 \setminus C_p$ which does not contain M. By Proposition 1.2, the last possibility can hold only if the equivalence class containing p consists of just one orbit.

3. First prolongational limit set of the boundary of a parallelizable region

In this section, we study properties of orbits contained in a parallelizable region M by using the set $J(\mathrm{fr}\, M) \cap M$.

Proposition 3.1. *Let $p \in J(q)$. Then $|C_p, C_q, C_r|$ for every $r \in D_{pq}$ holds.*

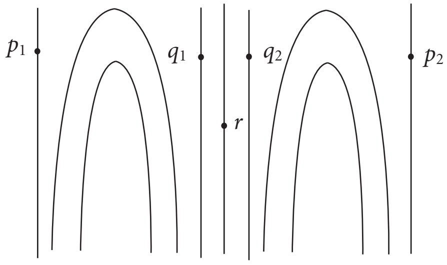

Figure 3.1. A parallelizable region with two boundary orbits.

Proof. Since $r \in D_{pq}$, the points r and q belong to the same component of $\mathbb{R}^2 \setminus C_p$ and r and p belong to the same component of $\mathbb{R}^2 \setminus C_q$. Now we prove that the points p and q are elements of the same component of $\mathbb{R}^2 \setminus C_r$. Denote by H_0 the component of $\mathbb{R}^2 \setminus C_r$ which contains q. Then, by the definition of $J(q)$, we have $J(q) \subset \operatorname{cl} H_0$. Hence $p \in H_0$ since $p \notin C_r$. Therefore $|C_p, C_q, C_r|$ holds since each orbit of C_p, C_q, C_r divides the plane in such a way that the other two orbits are contained in the same component. □

Corollary 3.2. *Let M be a parallelizable region of $\{f^t : t \in \mathbb{R}\}$, $p \in \operatorname{fr} M$ and $q \in M \cap J(p)$. Let $r \in M$ be contained in the component of $\mathbb{R}^2 \setminus C_q$ which contains p. Then $|C_p, C_q, C_r|$ holds.*

Proof. On account of Proposition 1.1, the point r is contained in the component of $\mathbb{R}^2 \setminus C_p$ which contains q. Thus $r \in D_{pq}$. Hence by Proposition 3.1, we have $|C_p, C_q, C_r|$. □

Proposition 3.3. *Let M be a parallelizable region of $\{f^t : t \in \mathbb{R}\}$. Let $p_1, p_2 \in \operatorname{fr} M$, $q_1, q_2 \in M$, $q_1 \in J(p_1)$, $q_2 \in J(p_2)$, and $C_{q_1} \neq C_{q_2}$. Then there exists $r \in M$ such that $C_{q_1}|C_r|C_{p_2}$, $C_{p_1}|C_r|C_{q_2}$, and $C_{p_1}|C_r|C_{p_2}$ hold (cf. Figure 3.1).*

Proof. Since $q_1, q_2 \in M$, $C_{q_1} \neq C_{q_2}$, and M is arcwise connected, there exists $r \in M \cap D_{q_1 q_2}$. Then $C_{q_1}|C_r|C_{q_2}$ holds since M is parallelizable. Denote by H_1 the component of $\mathbb{R}^2 \setminus C_r$ which contains q_1 and by H_2 the component of $\mathbb{R}^2 \setminus C_r$ which contains q_2. Then by the definition of the first prolongational limit set, we have $p_1 \in \operatorname{cl} H_1$ and $p_2 \in \operatorname{cl} H_2$, since $p_1 \in J(q_1)$, $p_2 \in J(q_2)$. By the fact that $p_1, p_2 \notin M$, we have $p_1, p_2 \notin C_r$. Thus $p_1 \in H_1$ and $p_2 \in H_2$. Since each component of $\mathbb{R}^2 \setminus C_r$ is invariant, we have $C_{q_1} \subset H_1$ and $C_{q_2} \subset H_2$. Consequently $C_{q_1}|C_r|C_{p_2}$, $C_{p_1}|C_r|C_{q_2}$, and $C_{p_1}|C_r|C_{p_2}$ hold. □

Remark 3.4. From Proposition 3.3, we get that if C_{p_1} and C_{p_2} are boundary orbits of a maximal parallelizable region M such that the only orbit C_{q_1} contained in $M \cap J(p_1)$ and the only orbit C_{q_2} contained in $M \cap J(p_2)$ are different, then there exists an orbit $C_r \subset M$ such that C_{p_1} and C_{p_2} belong to the different components of $\mathbb{R}^2 \setminus C_r$. However, in the case where the boundary orbits C_{p_1} and C_{p_2} have the same orbit C_q contained in $M \cap J(p_1)$ and in $M \cap J(p_2)$, we get from Proposition 2.4 that for every $r \in M \setminus C_q$, the orbits C_{p_1} and C_{p_2} belong to the same component of $\mathbb{R}^2 \setminus C_r$ (the assumptions of Proposition 2.4 are satisfied, since on account of Proposition 1.1 $M \subset D_{p_1,p_2}$). Moreover, by Corollary 3.2

for $i \in \{1,2\}$, we have $|C_{p_i}, C_q, C_r|$ if p_i and r are contained in the same component of $\mathbb{R}^2 \setminus C_q$.

4. Properties of components of parallelizable regions

In this section, we will consider orbits C_r of a parallelizable region M having the property that at least one of the components of $\mathbb{R}^2 \setminus C_r$ does not contain any point from $J(\operatorname{fr} M) \cap M$.

Proposition 4.1. *Let M be a parallelizable region of $\{f^t : t \in \mathbb{R}\}$. Let $r \in M$ and let H be a component of $\mathbb{R}^2 \setminus C_r$. Assume that $H \cap J(\operatorname{fr} M) \cap M = \varnothing$. Then $H \cap M$ is contained in an equivalence class.*

Proof. Suppose, on the contrary, that there exist $p, q \in H \cap M$ such that $p \in G_1$ and $q \in G_2$ for some distinct equivalence classes G_1, G_2. On account of Proposition 1.3, there exists a point $s \in D_{pq}$ such that $s \notin G_1 \cup G_2$. Denote by G_3 the equivalence class which contains s. Now we will show that D_{pq} is contained in an equivalence class.

First we will show that $D_{pq} \subset M$. Suppose, on the contrary, that there exists $x \in D_{pq}$ such that $x \notin M$. Put $A := D_{pq} \cap M$ and $B := D_{pq} \setminus A$. Since D_{pq} is connected, A is open in D_{pq}, $A \neq \varnothing$, and $B \neq \varnothing$, there exists a point $y \in D_{pq}$ such that $y \in \operatorname{fr} A$. Hence $y \in \operatorname{fr} M$.

Let M_1 be a maximal parallelizable region such that $M \subset M_1$. Now we prove that $M_1 \cap \operatorname{cl} D_{pq} = M \cap \operatorname{cl} D_{pq}$. Let $z \in M_1 \cap D_{pq}$. Then $C_p|C_z|C_q$ holds since M_1 is parallelizable. Hence $C_z \cap M \neq \varnothing$ since M is arcwise connected and $p, q \in M$. Thus by the fact that M is invariant, we have $z \in M$.

Take a ball $B(y,\varepsilon)$ centered at y with radius $\varepsilon > 0$. Without loss of generality, we can assume that $B(y,\varepsilon) \subset D_{pq}$ (such a ball exists since D_{pq} is an open set). From the fact that $y \in \operatorname{fr} M$, we obtain that there exist $z_1 \in B(y,\varepsilon) \cap M$ and $z_1 \in B(y,\varepsilon) \setminus M$. Then by the equality $M_1 \cap \operatorname{cl} D_{pq} = M \cap \operatorname{cl} D_{pq}$, we have $z_1 \in M_1$ and $z_2 \notin M_1$. Consequently $y \in \operatorname{fr} M_1$.

Since M_1 is a maximal parallelizable region, we have $J(M_1) = \operatorname{fr} M_1$ (see [8]). Thus $y \in J(M_1)$. Hence $J(y) \cap M_1 \neq \varnothing$. By the definition of the first prolongational limit set, we have $J(y) \subset \operatorname{cl} D_{pq}$ since $y \in D_{pq}$. Hence $J(y) \cap M \neq \varnothing$ since $M_1 \cap \operatorname{cl} D_{pq} = M \cap \operatorname{cl} D_{pq}$. Thus by the fact that $\operatorname{cl} D_{pq} \subset H$, the set $J(y) \cap M$ is contained in H, which contradicts the assumption that $H \cap J(\operatorname{fr} M) \cap M = \varnothing$. Consequently $D_{pq} \subset M$.

Fix $p_1, q_1 \in D_{pq}$. Then $p_1, q_1 \in M$. Since M is parallelizable, there exists a homeomorphism $\varphi : M \to \mathbb{R}^2$ such that $f^t(x) = \varphi^{-1}(\varphi(x) + (t,0))$ for $x \in M$ and $t \in \mathbb{R}$. Let K be preimage of the segment with endpoints $\varphi(p_1)$ and $\varphi(q_1)$. Then K is an arc with endpoints p_1 and q_1. We will prove that $f^n(K) \to \infty$ as $n \to \pm\infty$.

Take a ball $B(s,\varepsilon)$ centered at a point $s \in D_{pq}$ with radius $\varepsilon > 0$. Then $\operatorname{cl} B(s,\varepsilon) \cap \operatorname{cl} D_{pq}$ is a compact set. Hence $\varphi(\operatorname{cl} B(s,\varepsilon) \cap \operatorname{cl} D_{pq})$ is compact, since φ is a homeomorphism. Using properties of the flow of translations, we get $(\varphi(K) + (n,0)) \cap \varphi(\operatorname{cl} B(s,\varepsilon) \cap \operatorname{cl} D_{pq}) \neq \varnothing$ only for finitely many $n \in \mathbb{Z}$. Hence $f^n(K) \cap (\operatorname{cl} B(s,\varepsilon) \cap \operatorname{cl} D_{pq}) \neq \varnothing$ only for finitely many $n \in \mathbb{Z}$. Since D_{pq} is invariant and $K \subset D_{pq}$, we have $f^n(K) \cap (\operatorname{cl} B(s,\varepsilon) \setminus \operatorname{cl} D_{pq}) = \varnothing$ for all $n \in \mathbb{Z}$. Hence by the definition of the equivalence relation, p_1 and q_1 belong to the same class. Thus we have shown that D_{pq} is contained in an equivalence class. Since

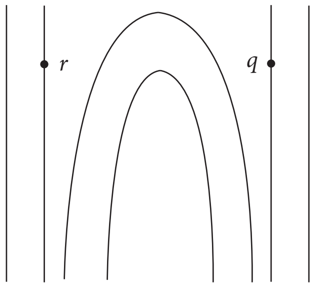

FIGURE 4.1. A maximal parallelizable region containing two classes.

$s \in D_{pq} \cap G_3$, we have $D_{pq} \subset G_3$. Hence by the fact that $p \notin G_3$ and $q \notin G_3$, we get $D_{pq} = G_3$ since each equivalence class is connected.

From the fact that $p \in G_1$, $q \in G_2$, $D_{pq} = G_3$, it follows that $p \in \operatorname{fr} G_1 \cap \operatorname{fr} G_3$ and $q \in \operatorname{fr} G_2 \cap \operatorname{fr} G_3$. Assume without loss of generality that q is contained in the component of $\mathbb{R}^2 \setminus C_p$ which does not contain r. Then $C_r|C_p|C_q$ holds. Let $y \in D_{pq}$. Then $C_p|C_y|C_q$ holds since $\operatorname{cl} D_{pq} \subset M$ and M is parallelizable. Hence $D_{yq} \subset D_{pq}$. On account of Proposition 1.4, there exists a point $z \in D_{yq}$ such that $z \in \operatorname{fr} M$, since $G_2 \cup G_3 \subset M$ and $\operatorname{fr} G_2 \cap \operatorname{fr} G_3 \neq \emptyset$. Hence $z \in D_{pq}$ and $z \notin M$ since $D_{yq} \subset D_{pq}$ and M is an open set, respectively. But this contradicts the fact that $D_{pq} \subset M$. Thus $H \cap M$ is contained in an equivalence class. □

COROLLARY 4.2. *Let M be a parallelizable region of $\{f^t : t \in \mathbb{R}\}$. Let $r \in M$ and let H be a component of $\mathbb{R}^2 \setminus C_r$. Assume that $H \cap \operatorname{fr} M = \emptyset$. Then $H \subset M$ and H is contained in an equivalence class.*

Proof. Let $H' = \mathbb{R}^2 \setminus (C_r \cup H)$. From the assumption $H \cap \operatorname{fr} M = \emptyset$, we obtain that $\operatorname{fr} M \subset H'$ since $C_r \subset M$. Thus by the definition of the first prolongational limit set, $J(\operatorname{fr} M) \subset \operatorname{cl} H' = H' \cup C_r$. Hence $H \cap J(\operatorname{fr} M) = \emptyset$. Thus on account of Proposition 4.1, $H \cap M$ is contained in an equivalence class. Put $H_1 = H \cap M$ and $H_2 = H \setminus H_1$. Then H_1 is an open set in H. Suppose, on the contrary, that $H_2 \neq \emptyset$. Then H_2 cannot be an open set in H since H is connected, $H_1 \cap H_2 = \emptyset$, and $H = H_1 \cup H_2$. Hence there exists a point $p \in H_2 \cap \operatorname{fr} H_2$. Take a ball $B(p,\varepsilon)$ centered at p with radius $\varepsilon > 0$ such that $B(p,\varepsilon) \subset H$. Then there exist $q \notin H_2$ such that $q \in B(p,\varepsilon)$, since $p \in \operatorname{fr} H_2$. Hence $q \in H_1$. Thus $p \in \operatorname{fr} H_1$ and consequently $p \in \operatorname{fr} M$, which contradicts the assumption that $H \cap \operatorname{fr} M = \emptyset$. Hence $H_2 = \emptyset$ and consequently $H \subset M$. Thus $H \cap M = H$ and H is contained in an equivalence class. □

Remark 4.3. From Proposition 4.1, we do not obtain that H is contained in an equivalence class. Let us consider the case where $J(\mathbb{R}^2) = C_r \cup C_q$ for some $r, q \in \mathbb{R}^2$ such that $r \notin C_q$ (cf. Figure 4.1). Let H be the component of $\mathbb{R}^2 \setminus C_r$ which contains q. Let $H' = \mathbb{R}^2 \setminus (C_r \cup H)$ and let M be a maximal parellelizable region containing r. Then $M = H' \cup C_r \cup D_{rq}$, $\operatorname{fr} M = C_q$, and $H \cap J(\operatorname{fr} M) \cap M = \emptyset$. The only equivalence class containing $H \cap M$ is the strip D_{rq}, and D_{rq} is a proper subset of H since $q \notin D_{rq}$.

References

[1] Z. Leśniak, "On an equivalence relation for free mappings embeddable in a flow," *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, vol. 13, no. 7, pp. 1911–1915, 2003.

[2] S. A. Andrea, "On homeomorphisms of the plane which have no fixed points," *Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg*, vol. 30, pp. 61–74, 1967.

[3] W. Kaplan, "Regular curve-families filling the plane—I," *Duke Mathematical Journal*, vol. 7, pp. 154–185, 1940.

[4] W. Kaplan, "Regular curve-families filling the plane—II," *Duke Mathematical Journal*, vol. 8, pp. 11–46, 1941.

[5] Z. Leśniak, "On maximal parallelizable regions of flows of the plane," *International Journal of Pure and Applied Mathematics*, vol. 30, no. 2, pp. 151–156, 2006.

[6] Z. Leśniak, "On parallelizability of flows of free mappings," *Aequationes Mathematicae*, vol. 71, no. 3, pp. 280–287, 2006.

[7] N. P. Bhatia and G. P. Szegö, *Stability Theory of Dynamical Systems*, vol. 161 of *Die Grundlehren der mathematischen Wissenschaften*, Springer, New York, NY, USA, 1970.

[8] R. C. McCann, "Planar dynamical systems without critical points," *Funkcialaj Ekvacioj*, vol. 13, pp. 67–95, 1970.

[9] Z. Leśniak, "On parallelizable regions of flows of the plane," *Grazer Mathematische Berichte*, vol. 350, pp. 175–183, 2006.

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