T. Ge
A.Y.T. Leung
School of Engineering, University of Manchester, M13 9PL, UK

Toeplitz Jacobian Method for Nonlinear Double-Periodic Excitations

The Toeplitz Jacobian matrix method is an efficient algorithm for computing the steady state solutions of nonlinear periodic vibration. In this paper, the method is generalized by using multiple time scales to double-periodic solutions in a multi-frequency excited system. The method is combined with a standard multi-dimensional FFT algorithm to accurately simulate the nonlinear oscillators with widely separated frequencies. The continuation technique can also be incorporated with the Newton–Raphson iteration to further increase its efficiency, and to achieve the complete frequency response characteristics.

INTRODUCTION

The prediction of quasi-periodic nonlinear vibration is important in the nonlinear engineering dynamics. Firstly, there is a number of applications including mechanical, electrical and electronic systems subjected to excitations with widely different oscillating periods. If the exciting frequencies are incommensurable, then the period of the solution will be too large to reveal the periodicity within a finite length of observation time. Secondly, routes to chaos through quasi-periodic motions have been intensively studied. The complete understanding of different motion patterns and their transition boundaries in the parameter space is essential for system design and control. The steady state analysis of a quasi-periodic system using direct numerical integration techniques such as Runge–Kutta algorithm, is difficult since the integration time step must be smaller than twice the smallest period while the number of iterations toward steady state is determined by the largest period in the system.

An alternative time domain simulation with the fixed-point algorithms (FPA) was developed, in which a Poincaré map is often constructed by shooting techniques and the original flow of an nth-order continuous time system is compacted into an $(n-1)$th-order discrete time system. Kaas-Petersen (1985) reformulated the problem of seeking quasi-periodic solutions of a forced system with two incommensurable forcing periods as a fixed point problem of the second-order Poincaré map. An interpolation technique was applied to locate a fixed point in the second-order Poincaré map. More recently, Ling (1991) presented a modified version of Kaas-Petersen's method (1985) for forced double-periodic systems by using analytical derivatives instead of the difference approximation and by using all iterative points in the interpolation rather than a few as in Kaas-Petersen's formulation. Although these algorithms are more efficient than direct numerical integration, especially in weakly damped nonlinear systems, they still require considerable computation time in the case of strongly nonlinear oscillation (Ushida and Chua, 1984). An alternative time-frequency hybrid nonlinear analysis method was proposed by Ushida and Chua (1984) and later modified by Kundert et al. (1988) together with least square

Received 10 October 1996; Revised 24 July 1997.
Shock and Vibration, Vol. 4, No. 5,6, pp. 351–359 (1997)
ISSN 1070-9622/97/$8.00 © 1997 IOS Press

approximation and generalized discrete Fourier transform techniques. These methods are more efficient for general multi-frequency excitations. However, since the least square technique has been involved in these multi-frequency methods, they need the numerical integration to obtain the initial conditions and have poor convergent rate near stability points.

An efficient way for obtaining quasi-periodic solutions is to exact the period for each individual frequency. The introduction of multiple time scales can provide a more straightforward description to quasi-periodic motions. Lau et al. (1983) use an incremental harmonic balance method with multiple time scales to obtain steady state quasi-periodic solution of a buckled beam, by approximating the solution in a finite trigonometric series and by balancing all terms having identical frequency components through Galerkin procedure. However, due to the limitation of enormous analytical work involved, the multiple time scales IHB method is quite difficult in general.

The method proposed in this paper is a natural development of our previous contribution. It is shown by Leung and Ge (1995) that the introduction of the Toeplitz Jacobian matrix into harmonic balance method enables us to obtain an efficient algorithm for nonlinear periodic vibration. The basic idea will be extended to a multi-frequency excited system with double periodicity.

BASIC FORMULATION

Consider a nonlinear oscillator governed by the equation

$$f\left(u, \frac{du}{dt}, \frac{d^2u}{dt^2}, \omega_1, \omega_2, t\right) \equiv \frac{d^2u}{dt^2} + N\left(u, \frac{du}{dt}, \omega_1, \omega_2, t\right) = 0 \quad (1)$$

where $\omega_1 = 2\pi/T_1$, $\omega_2 = 2\pi/T_2$; T_1, T_2 are incommensurable periods existed in the quasi-periodic motion of the system. After applying a two-dimensional time transformation

$$\begin{cases} \tau_1 = \omega_1 t, & 0 \leqslant \tau_1 \leqslant 2\pi, \\ \tau_2 = \omega_2 t, & 0 \leqslant \tau_2 \leqslant 2\pi, \end{cases} \quad (2)$$

to Eq. (1), we can write the responses of the system in the form of

$$\begin{cases} u(t) = u(\tau_1, \tau_2), \\ \dfrac{du}{dt} = \omega_1 \dfrac{\partial u}{\partial \tau_1} + \omega_2 \dfrac{\partial u}{\partial \tau_2} = \omega_1 u_1' + \omega_2 u_2', \\ \dfrac{d^2u}{dt^2} = \omega_1^2 \dfrac{\partial^2 u}{\partial \tau_1^2} + 2\omega_1\omega_2 \dfrac{\partial^2 u}{\partial \tau_1 \partial \tau_2} + \omega_2^2 \dfrac{\partial^2 u}{\partial \tau_2^2} \\ \quad = \omega_1^2 u_{11}'' + 2\omega_1\omega_2 u_{12}'' + \omega_2^2 u_{22}''. \end{cases} \quad (3)$$

The corresponding two-dimensional Fourier series expansion in complex exponential form is

$$\begin{pmatrix} u(\tau_1, \tau_2) \\ f(\tau_1, \tau_2) \end{pmatrix} = \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} \begin{pmatrix} U_{k_1,k_2} \\ F_{k_1,k_2} \end{pmatrix} \times \exp\left[i(k_1\tau_1 + k_2\tau_2)\right], \quad (4)$$

where

$$\begin{pmatrix} U_{k_1,k_2} \\ F_{k_1,k_2} \end{pmatrix} = \frac{1}{4\pi^2} \int_0^{2\pi} d\tau_1 \int_0^{2\pi} d\tau_2 \cdot \begin{pmatrix} u(\tau_1, \tau_2) \\ f(\tau_1, \tau_2) \end{pmatrix} \times \exp\left[-i(k_1\tau_1 + k_2\tau_2)\right], \quad (5)$$

and N_1 and N_2 are the highest orders of harmonic terms in T_1 and T_2, respectively. The discrete form of Eq. (5) can be achieved by sampling two continuous periods into M_1 and M_2 divisions with distinct uniformly spaced intervals $\Delta_1 = 2\pi/M_1$ and $\Delta_2 = 2\pi/M_2$, such that

$$\tau_1 = s_1\Delta_1 \quad \text{and} \quad \tau_2 = s_2\Delta_2, \quad s_1 = 0, 1, 2, \ldots, M_1 - 1; \quad s_2 = 0, 1, 2, \ldots, M_2 - 1, \quad (6)$$

and

$$\begin{pmatrix} u(\tau_1, \tau_2) \\ f(\tau_1, \tau_2) \end{pmatrix} = \begin{pmatrix} u_{s_1,s_2} \\ f_{s_1,s_2} \end{pmatrix}.$$

Consequently, Eq. (5) can be represented by

$$\begin{pmatrix} U_{k_1,k_2} \\ F_{k_1,k_2} \end{pmatrix} = \frac{1}{M_1 M_2} \sum_{s_1=0}^{M_1-1} \sum_{s_2=0}^{M_2-1} \begin{pmatrix} u_{s_1,s_2} \\ f_{s_1,s_2} \end{pmatrix} \times \exp\left[-i2\pi\left(\frac{k_1 s_1}{M_1} + \frac{k_2 s_2}{M_2}\right)\right], \quad k_1 = -N_1, \ldots, 0, \ldots, N_1; \quad k_2 = -N_2, \ldots, 0, \ldots, N_2. \quad (7)$$

Similar to the case of the one-dimensional Fourier transform (Leung and Ge, 1995; Ge and Leung, 1995), the two-dimensional Fourier transform defined by

Eq. (7) can be exactly inverted to recover u_{s_1,s_2}. The inverse discrete Fourier transform (DFT) is

$$\begin{pmatrix} u_{s_1,s_2} \\ f_{s_1,s_2} \end{pmatrix} = \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} \begin{pmatrix} U_{k_1,k_2} \\ F_{k_1,k_2} \end{pmatrix} \times \exp\left[\mathrm{i}\left(\frac{k_1 s_1}{M_1} + \frac{k_2 s_2}{M_2} \right) \right], \quad s_1 = 0, 1, 2, \ldots, M_1 - 1; \quad s_2 = 0, 1, 2, \ldots, M_2 - 1. \tag{8}$$

in which $M_1 \geqslant 2N_1 + 1$, $M_2 \geqslant 2N_2 + 1$ are required. For two dimensions, the upper limits of N_1 and N_2 on k_1 and k_2 limit the maximum frequencies (*Nyquist frequencies*) of the orthogonal functions in the series expansion,

$$\Omega_1 = \frac{1}{2\Delta_1}, \quad \Omega_2 = \frac{1}{2\Delta_2} \text{ cycles/interval.} \tag{9}$$

Consider components in the τ_1 direction, to which the subscript k_1 relates, by virtue of Eq. (7) and Eq. (8). For the case $N_1 \leqslant k_1 \leqslant 2N_1 - 1$, the frequency coefficients $\{U_{k_1,k_2}, F_{k_1,k_2}\}^{\mathrm{T}}$ are related to values in range $0 \leqslant k_1 \leqslant N_1$. The value $k_1 = N_1$ corresponds to a frequency given by

$$\frac{k_1}{T_1} = \frac{N_1}{M_1 \Delta_1} \leqslant \frac{1}{2\Delta_1} = \Omega_1. \tag{10}$$

Similarly, in the τ_2 direction, it gives

$$\frac{k_2}{T_2} = \frac{N_2}{M_2 \Delta_2} \leqslant \frac{1}{2\Delta_2} = \Omega_2. \tag{11}$$

In case the system response array $\{u_{s_1,s_2}, f_{s_1,s_2}\}^{\mathrm{T}}$ includes components with frequencies above the *Nyquist frequencies* in Eq. (9), then these components introduce errors by folding the corresponding values of $\{U_{k_1,k_2}, F_{k_1,k_2}\}^{\mathrm{T}}$ to the frequencies below the Nyquist frequencies.

Assume that the current state of vibration is represented by u, ω_1, ω_2 and the corresponding increments are $\Delta u, \Delta\omega_1, \Delta\omega_2$. Then a neighboring state can be written as

$$\tilde{u} = u\Delta u, \quad \tilde{\omega}_1 = \omega_1 + \Delta\omega_1, \quad \tilde{\omega}_2 = \omega_2 + \Delta\omega_2. \tag{12}$$

Inserting Eq. (12) into Eq. (1) and neglecting all the terms containing higher order increments, we have the resulting homogeneous incremental equation

$$\omega_1^2 \Delta u''_{11} + 2\omega_1\omega_2 \Delta u''_{12} + \omega_2^2 \Delta u''_{22} + \alpha(\omega_1 \Delta u'_1 + \omega_2 \Delta u'_2) + \beta \Delta u = r - \psi_1 \Delta\omega_1 - \psi_2 \Delta\omega_2 \tag{13}$$

where r is a residual term

$$r = -\left[\omega_1^2 u''_{11} + 2\omega_1\omega_2 u''_{12} + \omega_2^2 u''_{22} + N(u, u'_1, u'_2, \omega_1, \omega_2, \tau_1, \tau_2) \right] \tag{14}$$

and the parametric functions α, β, ψ_1, ψ_2 are also given at the state $\{u, \omega_1, \omega_2\}$ by

$$\alpha = \frac{\partial N}{\partial \dot{u}}, \quad \beta = \frac{\partial N}{\partial u}, \quad \psi_1 = 2\omega_1 u''_{11} + 2\omega_2 u''_{12} + \frac{\partial N}{\partial \omega_1}, \quad \psi_2 = 2\omega_2 u''_{22} + 2\omega_1 u''_{12} + \frac{\partial N}{\partial \omega_2}, \tag{15}$$

respectively. It is obvious that r will vanish if the current solution is exact. In numerical analysis, the residual term in Eq. (13) is very important as it ensures the Newton–Raphson iteration converging to a correct solution.

Substitutions of

$$\begin{aligned} \begin{pmatrix} u'_1 \\ \Delta u'_1 \end{pmatrix} &= \mathrm{i} \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} k_1 \begin{pmatrix} U_{k_1,k_2} \\ \Delta U_{k_1,k_2} \end{pmatrix} \times \exp\left[\mathrm{i}\left(\frac{k_1 s_1}{M_1} + \frac{k_2 s_2}{M_2} \right) \right], \\ \begin{pmatrix} u'_2 \\ \Delta u'_2 \end{pmatrix} &= \mathrm{i} \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} k_2 \begin{pmatrix} U_{k_1,k_2} \\ \Delta U_{k_1,k_2} \end{pmatrix} \times \exp\left[\mathrm{i}\left(\frac{k_1 s_1}{M_1} + \frac{k_2 s_2}{M_2} \right) \right], \end{aligned} \tag{16}$$

and

$$\begin{aligned} \begin{pmatrix} u''_{11} \\ \Delta u''_{11} \end{pmatrix} &= - \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} k_1^2 \begin{pmatrix} U_{k_1,k_2} \\ \Delta U_{k_1,k_2} \end{pmatrix} \times \exp\left[\mathrm{i}\left(\frac{k_1 s_1}{M_1} + \frac{k_2 s_2}{M_2} \right) \right], \\ \begin{pmatrix} u''_{22} \\ \Delta u''_{22} \end{pmatrix} &= - \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} k_2^2 \begin{pmatrix} U_{k_1,k_2} \\ \Delta U_{k_1,k_2} \end{pmatrix} \times \exp\left[\mathrm{i}\left(\frac{k_1 s_1}{M_1} + \frac{k_2 s_2}{M_2} \right) \right], \\ \begin{pmatrix} u''_{12} \\ \Delta u''_{12} \end{pmatrix} &= - \sum_{k_1=-N_1}^{N_1} \sum_{k_2=-N_2}^{N_2} k_1 k_2 \begin{pmatrix} U_{k_1,k_2} \\ \Delta U_{k_1,k_2} \end{pmatrix} \times \exp\left[\mathrm{i}\left(\frac{k_1 s_1}{M_1} + \frac{k_2 s_2}{M_2} \right) \right] \end{aligned} \tag{17}$$

into Eq. (13) yields

$$\sum_{k_1=-N_1}^{N_1}\sum_{k_2=-N_2}^{N_2}\Big\{\big[\beta_{s_1,s_2}-(k_1^2\omega_1^2+2k_1k_2\omega_1\omega_2+k_2^2\omega_2^2)+\mathrm{i}(k_1\omega_1+k_2\omega_2)\alpha_{s_1,s_2}\big]\times\Delta U_{k_1,k_2}\exp\Big[\mathrm{i}2\pi\Big(\frac{k_1s_1}{M_1}+\frac{k_2s_2}{M_2}\Big)\Big]\Big\}=r_{s_1,s_2}+\psi_{1s_1,s_2}\Delta\omega_1+\psi_{2s_1,s_2}\Delta\omega_2. \tag{18}$$

After performing the inverse DFT to Eq. (18), we obtain

$$\begin{aligned}&\sum_{s_1=0}^{M_1-1}\sum_{s_2=0}^{M_2-1}\sum_{k_1=-N_1}^{N_1}\sum_{k_2=-N_2}^{N_2}\Big\{\big[\beta_{s_1,s_2}-(k_1^2\omega_1^2+2k_1k_2\omega_1\omega_2+k_2^2\omega_2^2)+\mathrm{i}(k_1\omega_1+k_2\omega_2)\alpha_{s_1,s_2}\big]\\&\quad\times\Delta U_{k_1,k_2}\exp\Big[-\mathrm{i}2\pi\Big(\frac{s_1(l_1-k_1)}{M_1}+\frac{s_2(l_2-k_2)}{M_2}\Big)\Big]\Big\}\\&=\sum_{s_1=0}^{M_1-1}\sum_{s_2=0}^{M_2-1}r_{s_1,s_2}\exp\Big[-\mathrm{i}2\pi\Big(\frac{s_1l_1}{M_1}+\frac{s_2l_2}{M_2}\Big)\Big]\\&+\Delta\omega_1\sum_{s_1=0}^{M_1-1}\sum_{s_2=0}^{M_2-1}\psi_{1s_1,s_2}\exp\Big[-\mathrm{i}2\pi\Big(\frac{s_1l_1}{M_1}+\frac{s_2l_2}{M_2}\Big)\Big]\\&+\Delta\omega_2\sum_{s_1=0}^{M_1-1}\sum_{s_2=0}^{M_2-1}\psi_{2s_1,s_2}\exp\Big[-\mathrm{i}2\pi\Big(\frac{s_1l_1}{M_1}+\frac{s_2l_2}{M_2}\Big)\Big]\\&=R_{l_1l_2}+\Delta\omega_1\Psi_{1l_1l_2}+\Delta\omega_2\Psi_{2l_1l_2},\\&\quad l_1=-N_1,\dots,0,\dots,N_1,\\&\quad l_2=-N_2,\dots,0,\dots,N_2.\end{aligned}\tag{19}$$

Similar to Eq. (11), Eq. (19) can also be equivalently expressed in a matrix form as

$$\mathbf{J}\cdot\Delta\mathbf{U}=\mathbf{R}+\Delta\omega_1\vec{\Psi}_1+\Delta\omega_2\vec{\Psi}_2 \tag{20}$$

where $\Delta\mathbf{U}=\{\Delta U_{k_1,k_2}\}$ is the unknown vector of the incremental equation; $\mathbf{R}=\{R_{l_1,l_2}\}$, $\vec{\Psi}_1=\{\Psi_{1l_1,l_2}\}$ and $\vec{\Psi}_2=\{\Psi_{2l_1,l_2}\}$ are evaluated by the two-dimensional inverse FFT. $\mathbf{J}=[J_{l_1,l_2,k_1,k_2}]$ is the Jacobian matrix which is obtained efficiently by the two-dimensional Toeplitz matrix formulations,

$$\begin{aligned}J_{l_1,l_2,k_1,k_2}&=B_{l_1-k_1,l_2-k_2}\\&-\big(k_1^2\omega_1^2+2k_1k_2\omega_1\omega_2+k_2^2\omega_2^2\big)\delta_{l_1-k_1,l_2-k_2}\\&+\mathrm{i}(k_1\omega_1+k_2\omega_2)A_{l_1-k_1,l_2-k_2}\end{aligned}\tag{21}$$

in which $A_{l_1-k_1,l_2-k_2}$, $B_{l_1-k_1,l_2-k_2}$ correspond to the Toeplitz matrices $\mathbf{A}$, $\mathbf{B}$, respectively,

$$\begin{aligned}A_{p,q}&=\sum_{s_1=0}^{M_1-1}\sum_{s_2=0}^{M_2-1}\alpha_{s_1,s_2}\exp\Big[-\mathrm{i}2\pi\Big(\frac{ps_1}{M_1}+\frac{qs_2}{M_2}\Big)\Big],\\B_{p,q}&=\sum_{s_1=0}^{M_1-1}\sum_{s_2=0}^{M_2-1}\beta_{s_1,s_2}\exp\Big[-\mathrm{i}2\pi\Big(\frac{ps_1}{M_1}+\frac{qs_2}{M_2}\Big)\Big],\end{aligned}\tag{22}$$

and

$$p=\mathrm{mod}\Big(\frac{M_1+l_1-k_1}{M_1}\Big),\qquad q=\mathrm{mod}\Big(\frac{M_2+l_2-k_2}{M_2}\Big);$$

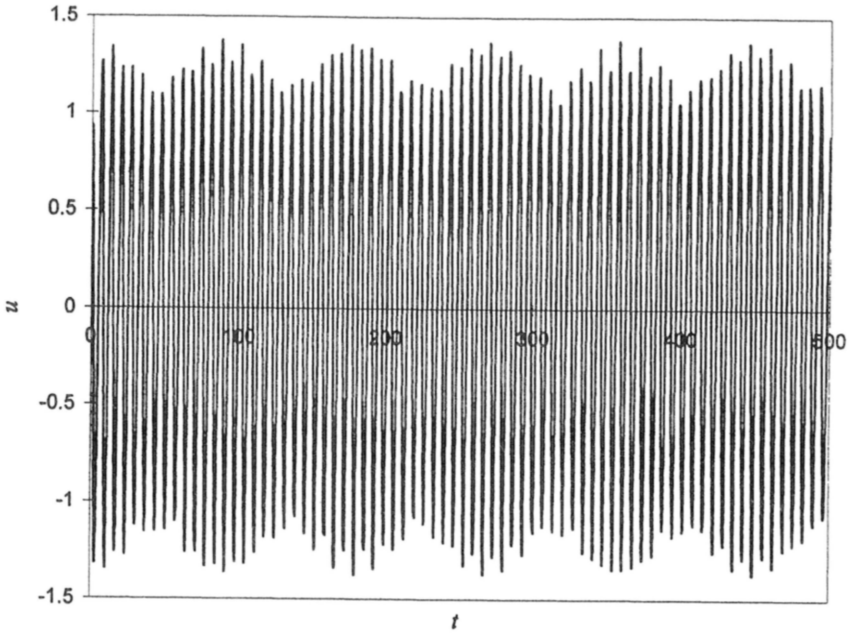
δ denotes the Kronecker function

$$\delta_{l_1-k_1,l_2-k_2}=\begin{cases}1 & \text{if } l_1=k_1 \text{ and } l_2=k_2,\\0 & \text{otherwise.}\end{cases}\tag{23}$$

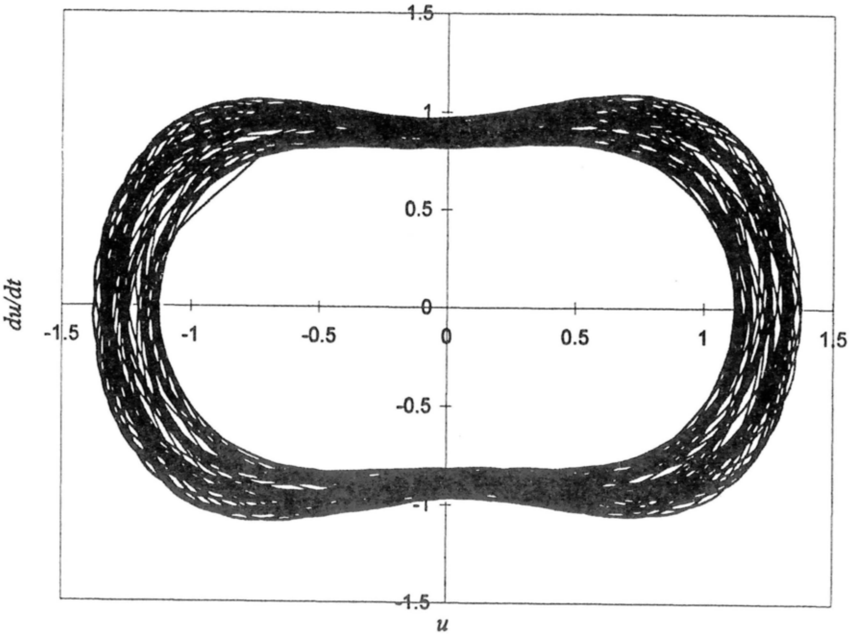
The Toeplitz Jacobian formulations of Eq. (20) to Eq. (23) are also valid in trigonometric forms. Instead of Eq. (4), we can construct the system functions and its solution in the form of

$$\begin{aligned}\begin{Bmatrix}u_{s_1,s_2}\\f_{s_1,s_2}\end{Bmatrix}&=\frac{1}{2}\begin{Bmatrix}X_{0,0}\\C_{0,0}\end{Bmatrix}\\&+\sum_{k_2=1}^{N_2}\Big\{\begin{Bmatrix}X_{0,k_2}\\C_{0,k_2}\end{Bmatrix}\cos\Big[2\pi\Big(\frac{k_2s_2}{M_2}\Big)\Big]+\begin{Bmatrix}Y_{0,k_2}\\D_{0,k_2}\end{Bmatrix}\sin\Big[2\pi\Big(\frac{k_2s_2}{M_2}\Big)\Big]\Big\}\\&+\sum_{k_1=1}^{N_1}\sum_{k_2=-N_2}^{N_2}\Big\{\begin{Bmatrix}X_{k_1,k_2}\\C_{k_1,k_2}\end{Bmatrix}\times\cos\Big[2\pi\Big(\frac{k_1s_1}{M_1}+\frac{k_2s_2}{M_2}\Big)\Big]\\&\quad+\begin{Bmatrix}Y_{k_1,k_2}\\D_{k_1,k_2}\end{Bmatrix}\sin\Big[2\pi\Big(\frac{k_1s_1}{M_1}+\frac{k_2s_2}{M_2}\Big)\Big]\Big\},\\&s_1=0,1,2,\dots,M_1-1;\\&s_2=0,1,2,\dots,M_2-1.\end{aligned}\tag{24}$$

It should be noted that the symmetry properties of U_{k_1,k_2} in Eq. (4) permit the negative integers of k_1 removed in the expansion of Eq. (24) without loss of information. Thus the number of frequency components has been reduced and the possible linear dependence existing in the basis functions can be removed. More-



(a) Time history



(b) Phase diagram

FIGURE 1 $\lambda = 0.93, \omega_1 = 1.5$.

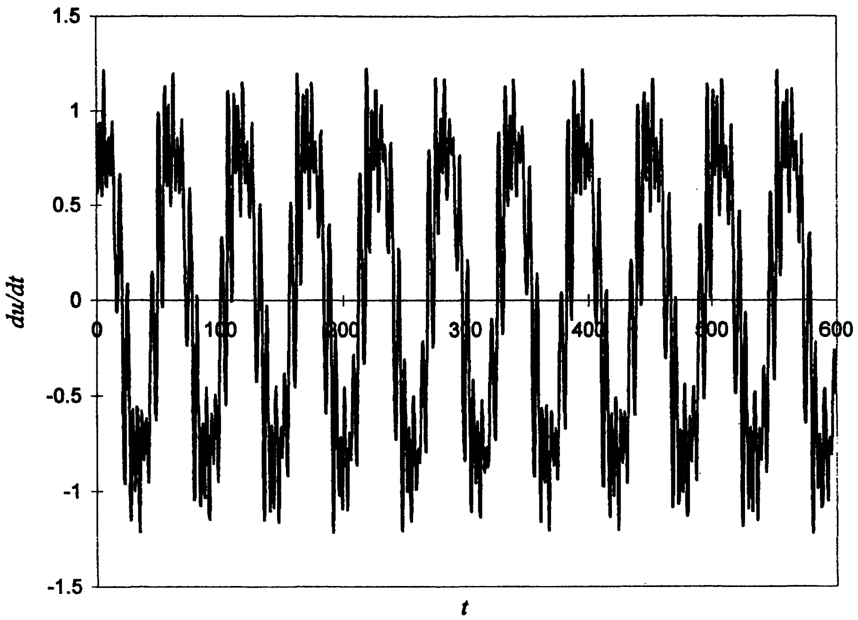
over, together with the identities

$$\begin{cases} C_{k_1,k_2} = F_{k_1,k_2} + F_{-k_1,-k_2} \\ D_{k_1,k_2} = \mathrm{i}(F_{k_1,k_2} - F_{-k_1,-k_2}) \end{cases},$$

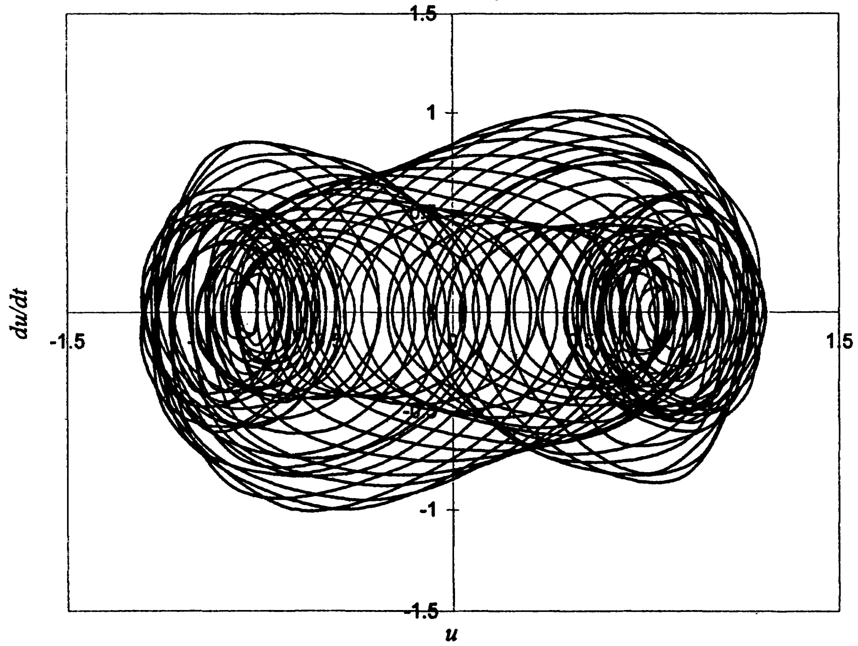
$$\begin{cases} U_{k_1,k_2} = \frac{1}{2}(X_{k_1,k_2} - \mathrm{i}X_{k_1,k_2}) \\ U_{-k_1,-k_2} = \frac{1}{2}(X_{k_1,k_2} - \mathrm{i}X_{k_1,k_2}) \end{cases} \tag{25}$$

and the chain rule of differentiation, we obtain the transformation matrix for the trigonometric forms of TJM formulation,

$$\begin{Bmatrix} \partial C_{l_1,l_2}/\partial X_{k_1,k_2} \\ \partial C_{l_1,l_2}/\partial Y_{k_1,k_2} \\ \partial D_{l_1,l_2}/\partial X_{k_1,k_2} \\ \partial D_{l_1,l_2}/\partial Y_{k_1,k_2} \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -\mathrm{i} & \mathrm{i} & -\mathrm{i} & \mathrm{i} \\ \mathrm{i} & \mathrm{i} & -\mathrm{i} & -\mathrm{i} \\ 1 & -1 & -1 & 1 \end{bmatrix}$$



(a) Time history



(b) Phase diagram

FIGURE 2 $\lambda = 0.113, \omega_1 = 1.5$.

$$\times \begin{Bmatrix} J_{l_1,l_2,k_1,k_2} \\ J_{l_1,l_2,-k_1,-k_2} \\ J_{-l_1,-l_2,k_1,k_2} \\ J_{-l_1,-l_2,-k_1,-k_2} \end{Bmatrix},$$

$$l_1, k_1 = 0, 1, \ldots, N_1;$$

$$l_2, k_2 = -N_2, \ldots, 0, \ldots, N_2. \tag{26}$$

EXAMPLE OF APPLICATION

In order to show the advantage of the proposed algorithm, we consider a nonlinear Duffing's equation subjected to two-frequency excitations,

$$\ddot{u} + 0.1\dot{u} + u + u^3 = 0.3\cos\omega_1 t + 1.5\cos\omega_2 t. \tag{27}$$

Table 1. List of the Generalized Harmonic Terms

Harmonic terms	Case 1 ($\omega_1 = \omega, \omega_2 = 0.93\omega$)			Case 2 ($\omega_1 = \omega, \omega_2 = 0.113\omega$)		
	k_1	k_2	$k_1\omega_1 + k_2\omega_2$	k_1	k_2	$k_1\omega_1 + k_2\omega_2$
1	−4	5	0.65ω	0	1	0.113ω
2	−3	4	0.72ω	1	−6	0.322ω
3	−2	3	0.79ω	0	3	0.339ω
4	−1	2	0.86ω	1	−4	0.548ω
5	0	1	0.93ω	0	5	0.565ω
6	1	0	ω	1	−2	0.774ω
7	2	−1	1.07ω	1	0	ω
8	3	−2	1.14ω	1	2	1.226ω
9	4	−3	1.21ω	2	−5	1.435ω
10	5	−4	1.28ω	1	4	1.452ω
11	−2	5	2.65ω	2	−3	1.661ω
12	−1	4	2.72ω	1	6	1.678ω
13	0	3	2.79ω	2	−1	1.887ω
14	1	2	2.86ω	2	1	2.113ω
15	2	1	2.93ω	2	3	2.339ω
16	3	0	3ω	2	5	2.565ω

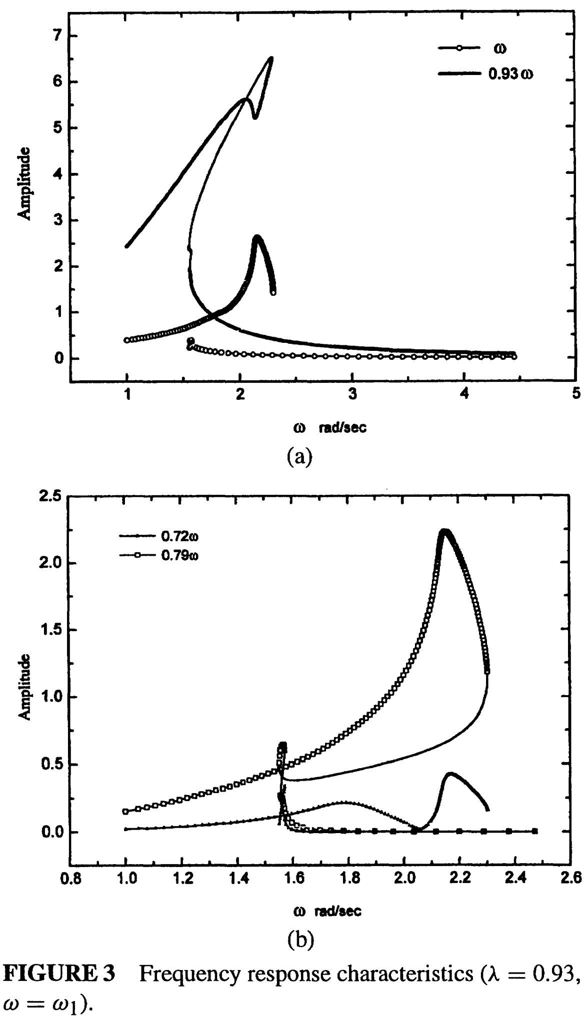


FIGURE 3 Frequency response characteristics ($\lambda = 0.93$, $\omega = \omega_1$).

The corresponding incremental equation is

$$\begin{aligned}&\omega_1^2\left(\Delta u''_{11} + 2\lambda\Delta u''_{12} + \lambda^2\Delta u''_{22}\right)\\&\quad + 0.1\omega_1\left(\Delta u'_1 + \lambda\Delta u'_2\right)\\&\quad + \left(1 + 3u^2\right)\Delta u = r - \psi\Delta\omega_1 \qquad (28)\end{aligned}$$

where $\lambda = \omega_2/\omega_1$ is assumed to be nearly incommensurable,

$$\begin{aligned}r &= \omega_1^2\left(u''_{11} + 2\lambda u''_{12} + \lambda^2 u''_{22}\right)\\&\quad + 0.1\omega_1\left(\Delta u'_1 + \lambda\Delta u'_2\right)\\&\quad + u + u^3 - 0.3\cos\omega_1 t - 1.5\cos\lambda\omega_1 t,\\ \psi &= 2\omega_1\left(u''_{11} + \lambda^2 u''_{22}\right)\\&\quad + 4\lambda\omega_1 u''_{12} + 0.1\left(u'_1 + \lambda u'_2\right). \qquad (29)\end{aligned}$$

The computations are performed for the cases: (1) $\lambda = 0.93$ and (2) $\lambda = 0.113$. Figure 1(a,b) and Fig. 2(a,b) show the exemplified portraits of time history and phase diagram of quasi-periodic motion for Case 1 and Case 2, respectively. It may be necessary to prescribe the *Nyquist frequencies* beforehand for τ_1, τ_2, respectively. According to the magnitudes of λ, we define $N_1 = N_2 = 5$ and $\Omega_1 = 6\omega_1$ for the first case and $N_1 = 2$, $N_2 = 6$ and $\Omega_1 = 6\omega_1$ for the second case. The resulting generalized harmonic components are listed in the Table 1. The frequency response characteristics are presented individually in Fig. 3(a,b) and Fig. 4(a,b,c) where both the difference and the summation type of resonance are found in addition to two primary resonances near the frequency $\omega = 2$. It is

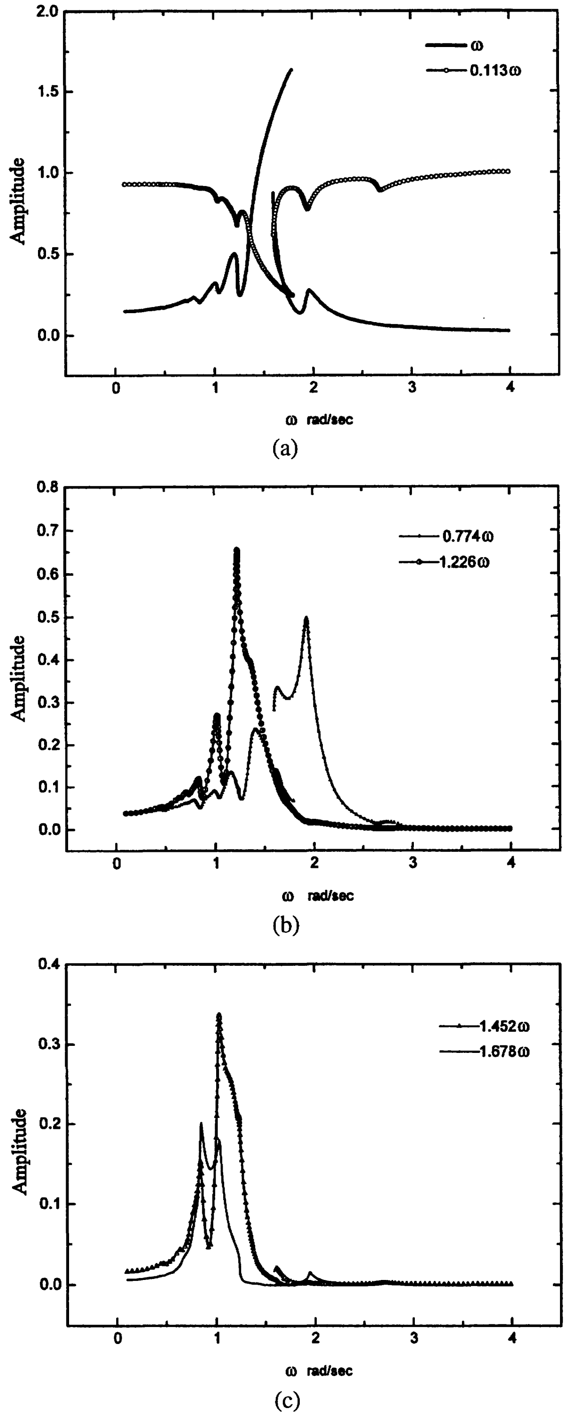


FIGURE 4 Frequency response characteristics ($\lambda = 0.113, \omega = \omega_1$).

seen from Fig. 3(a,b) when two exciting frequencies are close to each other ($\omega_1 = \omega$, $\omega_2 = 0.93\omega$), the largest difference component in response is

$$0.79\omega = (-2 + 3 \times 0.93)\omega$$

rather than

$$0.86\omega = (-1 + 2 \times 0.93)\omega$$

given in the least square harmonic analysis (Ushida and Chua, 1984). On the other hand, the largest difference and summation components in the system excited by two sparsely distributed exciting frequencies ($\omega_1 = \omega$, $\omega_2 = 0.113\omega$) are illustrated in Fig. 4(b) to be

$$0.774\omega = (1 - 2 \times 0.113)\omega,$$
$$1.226\omega = (1 + 2 \times 0.113)\omega$$

and in Fig. 4(c) to be

$$1.452\omega = (1 + 4 \times 0.113)\omega,$$
$$1.678\omega = (1 + 6 \times 0.113)\omega,$$

respectively. The indeterminacy of combinational resonances in solutions suggests the importance of the completeness of basis functions. The dot lines in Fig. 3 and Fig. 4 represent the unstable solution resulting from the occurrence of fold bifurcations.

The nonlinear oscillator does possess periodic solution but its period is too large to be handled by any normal method of harmonic balance. On the contrary, the current method could be very efficient since it produces an analytical Toeplitz Jacobian matrix for the Newton–Raphson iteration. In the numerical computations, it significantly reduces the number of multiplications and provides a compatible way to incorporate the continuations to pass through the stability and bifurcation point. It is worth mentioning that although Ushida and Chua (1984) and Kundert et al. (1988) studied examples of an almost periodic circuit using various least-square methods of harmonic balance, their method is inapplicable in the neighborhood of the stability point. Therefore, they cannot obtain the complete frequency response characteristics.

CONCLUSIONS

The Toeplitz Jacobian matrix method is an efficient algorithm for computing the steady state solutions of nonlinear periodic vibration. In this paper, the method is generalized with multiple time scales to double-periodic solutions in a multi-frequency excited system. The method is combined with a standard multi-dimensional FFT algorithm to accurately simulate the nonlinear oscillators with widely separated frequencies. The continuation technique can also be incorporated with the Newton–Raphson iteration to further increase its efficiency, and to achieve the complete frequency response characteristics.

REFERENCES

Ge, T., and Leung, A. Y. T., 1995, "A Toeplitz Jacobian matrix/fast Fourier transformation method for steady-state analysis of discontinuous oscillators," *Shock and Vibration*, Vol. 2, pp. 205–218.

Kaas-Petersen, C., 1985, "Computation of quasi-periodic solutions of forced dissipative systems," *J. of Computational Phys.*, Vol. 58, pp. 395–403.

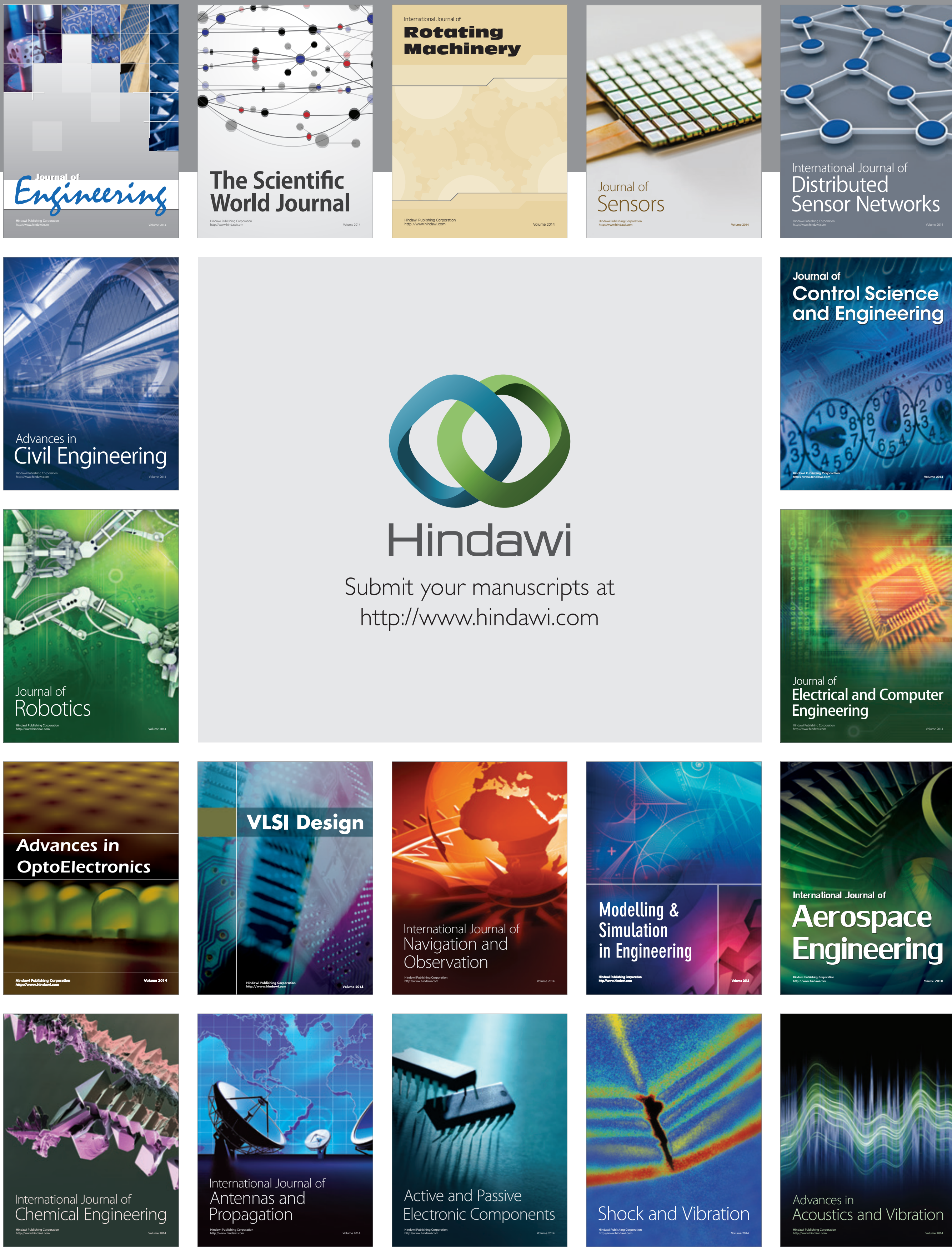
Kundert, K. S., Sorkin, G. B., and Alberto, S. V., 1988, "Applying harmonic balance to almost periodic circuits," *IEEE Trans. Microwave Theory Tech.*, Vol. MTT-36, pp. 366–378.

Lau, S. L., Cheung, Y. K., and Wu, S. Y., 1983, "Incremental harmonic balance method with multiple time scales for aperiodic vibration of nonlinear systems," *ASME J. of Applied Mechanics*, Vol. 49, pp. 849–853.

Leung, A. Y. T., and Ge, T., 1995, "Toeplitz Jacobian matrix for nonlinear periodic vibration," *ASME J. of Applied Mechanics*, Vol. 62, pp. 709–717.

Ling, F. H., 1991, "Quasi-periodic solutions calculated with the simple shooting technique," *J. of Sound and Vibration*, Vol. 144, pp. 293–304.

Ushida, A., and Chua, L. O., 1984, "Frequency-domain analysis of nonlinear circuits driven by multi-ton signals," *IEEE Trans. Circuits Syst.*, Vol. CAS-31, pp. 766–778.


Journal of Engineering
The Scientific World Journal
International Journal of Rotating Machinery
Journal of Sensors
International Journal of Distributed Sensor Networks
Advances in Civil Engineering
Journal of Control Science and Engineering
Hindawi
Submit your manuscripts at
http://www.hindawi.com
Journal of Robotics
Journal of Electrical and Computer Engineering
Advances in OptoElectronics
VLSI Design
International Journal of Navigation and Observation
Modelling & Simulation in Engineering
International Journal of Aerospace Engineering
International Journal of Chemical Engineering
International Journal of Antennas and Propagation
Active and Passive Electronic Components
Shock and Vibration
Advances in Acoustics and Vibration