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Research Article

Output Regulation of the Pan System

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We solve the problem of regulating the output of the Pan system (2010), which is one of the recently discovered three-dimensional chaotic attractors. Pan system has many interesting complex dynamical behaviours, and it has potential applications in secure communication. In this paper, we construct explicit state feedback control laws for regulating the output of the Pan system so as to track constant reference signals. The state feedback control laws are derived using the regulator equations of Byrnes and Isidori (1990). The simulation results are provided to illustrate the effectiveness of the regulation schemes derived for the output regulation of the Pan system.

1. Introduction

Regulating the output of nonlinear control systems is one of the central problems in nonlinear control theory. Essentially, the output regulation problem is the problem of controlling a fixed linear or nonlinear plant so that the output of the plant tracks the reference signals produced by some external input generator (the *exosystem*). For linear control systems, the output regulation problem was solved by Francis and Wonham [1]. For nonlinear control systems, the output regulation problem was solved by Byrnes and Isidori [2], who generalized the internal model principle obtained by Francis and Wonham and proved their results using Centre Manifold Theory for flows [3]. Byrnes and Isidori made an important assumption in their work [2] that the exosystem is a *neutrally stable* system (Lyapunov stable system in both forward and backward time). This class of exosystem signals includes the important special cases of constant reference signals as well as periodic reference signals.

The output regulation problem for linear and nonlinear control systems has been the focus of many important studies in the last two decades [4–14]. In [4], Mahmoud and Khalil derived results for the problem of asymptotic regulation of minimum phase nonlinear systems using output feedback. In [5], Fridman derived results on the output regulation problem for nonlinear control systems with delay. In [6, 7], Chen and Huang derived results on the robust output regulation problem for output feedback systems with nonlinear exosystems. In [8], Liu and Huang derived results on the global robust output regulation problem for lower triangular nonlinear systems with unknown control direction. In [9], Immonen derived results on the practical output regulation for bounded linear infinite-dimensional state space systems. In [10], Pavlov et al. derived results on the global nonlinear output regulation using convergence-based controller design. In [11], Xi and Ding studied the global adaptive output regulation of a class of nonlinear systems with nonlinear exosystems. In [12–14], Serrani et al. derived results on the semiglobal and global output regulation problem for minimum phase nonlinear systems.

In this paper, we solve the output regulation problem for the chaotic Pan system (2010). The Pan system [15] is an important model of three-dimensional chaotic system discovered by Pan et al. From the control engineering point of view, the Pan system provides an interesting framework for advanced control techniques since it is a very complex system than the Lorenz's system. In this paper, we use the regulator equations for nonlinear control systems [2] to derive state feedback control laws for regulating the output of the Pan system so as to track constant reference input signals (set-point signals).

This paper is organized as follows. In Section 2, we present a review of the output regulation problem for nonlinear control systems and its solution using the regulator equations derived by Byrnes and Isidori [2]. In Section 3, we detail our solution of the output regulation problem for the Pan system. In Section 4, we discuss the simulation results illustrating the effectiveness of the state feedback control laws derived in Section 3. In Section 5, we summarize the main results obtained in this paper.

2. Review of the Output Regulation for Nonlinear Control Systems

In this section, we consider a multivariable nonlinear control system given by

$$\dot{x} = f(x) + g(x)u + p(x)\omega, \tag{2.1a}$$

$$\dot{\omega} = s(\omega), \tag{2.1b}$$

$$e = h(x) - q(\omega). \tag{2.2}$$

Here, the differential equation (2.1a) describes the plant dynamics with state x defined in a neighbourhood X of the origin of R^n, and the input u takes values in R^m subject to the effect of a disturbance represented by the vector field $p(x)\omega$. The differential equation (2.1b) describes the autonomous system, known as the exosystem, defined in a neighbourhood W of the origin of R^k, which models the class of disturbance and reference signals taken into consideration.

We also assume that all the constituent mappings of the system (2.1a), and (2.1b) and the error equation (2.2), namely, $f,\ g,\ p,\ s,\ h$, and q are C^1 mappings vanishing at the origin, that is,

$$f(0) = 0, \qquad g(0) = 0, \qquad p(0) = 0, \qquad s(0) = 0, \qquad h(0) = 0, \qquad q(0) = 0. \tag{2.3}$$

Thus, for $u = 0$, the composite system (2.1a), and (2.1b) has an equilibrium state $(x, \omega) = (0, 0)$ with zero error (2.2).

A *state feedback controller* for the composite system (2.1a), and (2.1b) has the form

$$u = \alpha(x, \omega), \tag{2.4}$$

where α is a C^1 mapping defined on $X \times W$ such that $\alpha(0,0) = 0$.

Upon the substitution of the feedback law (2.4) into the composite system (2.1a), and (2.1b), we get the closed-loop system given by

$$\begin{aligned} \dot{x} &= f(x) + g(x)\alpha(x, \omega) + p(x)\omega, \\ \dot{\omega} &= s(\omega). \end{aligned} \tag{2.5}$$

The purpose of designing the state feedback controller (2.4) is to achieve both *internal stability* and *output regulation*. Internal stability means that when the input is disconnected from (2.5) (i.e., when $\omega = 0$), the closed-loop system (2.5) has an exponentially stable equilibrium at $x = 0$. Output regulation means that for the closed-loop system (2.5), for all initial states $(x(0), \omega(0))$ sufficiently close to the origin, $e(t) \to 0$ asymptotically as $t \to \infty$. Formally, we can summarize the requirements as follows.

2.1. State Feedback Regulator Problem [2]

Find, if possible, a state feedback control law $u = \alpha(x, \omega)$ such that the following two conditions are satisfied.

(OR1) (*Internal Stability*) The equilibrium $x = 0$ of the dynamics

$$\dot{x} = f(x) + g(x)\alpha(x, 0) \tag{2.6}$$

is locally exponentially stable.

(OR2) (*Output Regulation*) There exists a neighbourhood U of $(x, \omega) = (0, 0)$ contained in $X \times W$ such that for each initial condition $(x(0), \omega(0)) \in U$, the solution $(x(t), \omega(t))$ of the closed-loop system (2.5) satisfies

$$\lim_{t \to \infty} \left[h(x(t)) - q(\omega(t)) \right] = 0. \tag{2.7}$$

Byrnes and Isidori [2] solved this problem under the following assumptions.

(H1) The exosystem dynamics $\dot{\omega} = s(\omega)$ is *neutrally stable* at $\omega = 0$, that is, the system is Lyapunov stable in both forward and backward time at $\omega = 0$.

(H2) The pair $(f(x), g(x))$ has a stabilizable linear approximation at $x = 0$, that is, if

$$A = \left[\frac{\partial f}{\partial x} \right]_{x=0}, \qquad B = \left[\frac{\partial g}{\partial x} \right]_{x=0}, \tag{2.8}$$

then (A, B) is stabilizable, which means that we can find a gain matrix K so that $A + BK$ is Hurwitz.

Next, we state the solution of the output regulation problem derived by Byrnes and Isidori [2].

Theorem 2.1 (see [2]). *Under the assumptions (H1) and (H2), the state feedback regulator problem is solvable if and only if there exist C^1 mappings $x = \pi(\omega)$ with $\pi(0) = 0$ and $u = \varphi(\omega)$ with $\varphi(0) = 0$, both defined in a neighbourhood $W^0 \subset W$ of $\omega = 0$ such that following equations (called the regulator equations) are satisfied:*

(1) $(\partial\pi/\partial\omega)s(\omega) = f(\pi(\omega)) + g(\pi(\omega))\varphi(\omega) + p(\pi(\omega))\omega$,

(2) $h(\pi(\omega)) - q(\omega) = 0$.

When the regulator equations (2.1a), (2.1b), *and* (2.2) *are satisfied, a control law solving the state feedback regulator problem is given by*

$$u = \varphi(\omega) + K[x - \pi(\omega)], \tag{2.9}$$

where K is any gain matrix such that $A + BK$ is Hurwitz.

3. Output Regulation of the Pan System

The Pan system is a new three-dimensional chaotic system discovered by Pan et al. [15] and described by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1), \\ \dot{x}_2 &= cx_1 - x_1x_3 + u, \\ \dot{x}_3 &= -bx_3 + x_1x_2, \end{aligned} \tag{3.1}$$

where $a > 0,\ b > 0,\ c > 0$ are the parameters of the system and u is the control.

The unforced Pan system (i.e. with $u = 0$) is chaotic when $a = 10,\ b = 8/3$, and $c = 16$. The chaotic state portrait of the unforced Pan system is illustrated in Figure 1.

In this paper, we solve the problem of output regulation for Pan system (3.1) for the tracking of constant reference signals (*set-point signals*).

The constant reference signals are generated by the exosystem dynamics

$$\dot{\omega} = 0. \tag{3.2}$$

It is important to note that the exosystem given by (3.2) is neutrally stable.

This follows simply because the differential equation (3.2) admits only constant solutions, that is,

$$\omega(t) \equiv \omega(0) = \omega_0, \quad \forall t \in R. \tag{3.3}$$

Thus, the assumption (H1) of Theorem 2.1 holds trivially.

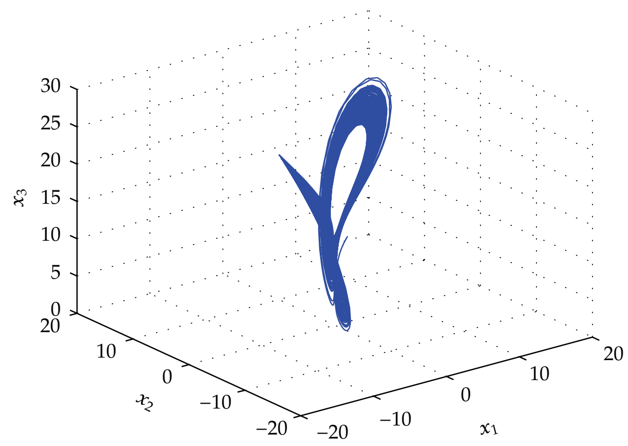

Figure 1: The Chaotic state portrait of the unforced pan system.

Linearizing the dynamics of the Pan system (3.1) at the equilibrium $(x_1, x_2, x_3) = (0, 0, 0)$, we get the following system matrices:

$$A = \begin{bmatrix} -a & a & 0 \\ c & 0 & 0 \\ 0 & 0 & -b \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}. \tag{3.4}$$

Using Kalman's rank test for controllability [16], it can be easily seen that the pair (A, B) is not controllable. However, we shall show that the pair (A, B) is stabilizable, that is, we can find a gain matrix K such that $A + BK$ is Hurwitz.

Suppose that

$$K = \begin{bmatrix} k_1 & k_2 & k_3 \end{bmatrix}, \tag{3.5}$$

where k_1, k_2, and k_3 are real numbers to be determined so that $A + BK$ is Hurwitz.

The characteristic equation of the matrix

$$A + BK = \begin{bmatrix} -a & a & 0 \\ c + k_1 & k_2 & k_3 \\ 0 & 0 & -b \end{bmatrix} \tag{3.6}$$

is easily obtained as

$$(\lambda + b)\left[\lambda^2 + \lambda(a - k_2) - a(c + k_1 + k_2)\right] = 0. \tag{3.7}$$

Since $b > 0$, we note that $\lambda = -b$ is a stable eigenvalue of $A+BK$, and the other two eigenvalues of $A + BK$ will be stable if and only if

$$a - k_2 > 0, \qquad -a(c + k_1 + k_2) > 0. \tag{3.8}$$

Since $a > 0$, the above conditions are equivalent to

$$k_2 < a, \quad c + k_1 + k_2 < 0. \tag{3.9}$$

Since k_3 does not play any role in the above eigenvalue calculations, we can take $k_3 = 0$.

Thus, we shall assume that $K = [\, K_1 \; K_2 \; 0 \,]$, where k_1 and k_2 are chosen so that the inequalities (3.9) are satisfied. Thus, $A + BK$ is Hurwitz. Hence, (A, B) is stabilizable. Hence, the assumption (H2) of Theorem 2.1 also holds.

Hence, we can apply Theorem 2.1 to completely solve the output regulation problem for the Pan system for the tracking of constant reference signals (*set-point signals*).

3.1. Constant Tracking Problem for x_1

Here, we consider the problem of the state x_1 of the Pan system tracking the constant signal ω. Thus, the nonlinear system under consideration is

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1),\\
\dot{x}_2 &= cx_1 - x_1x_3 + u,\\
\dot{x}_3 &= -bx_3 + x_1x_2,\\
e &= x_1 - \omega.
\end{aligned} \tag{3.10}$$

The regulator equations for the system (3.10) are obtained by Theorem 2.1 as

$$\begin{gathered}
\pi_2(\omega) - \pi_1(\omega) = 0,\\
c\pi_1(\omega) - \pi_1(\omega)\pi_3(\omega) + \varphi(\omega) = 0,\\
-b\pi_3(\omega) + \pi_1(\omega)\pi_2(\omega) = 0,\\
\pi_1(\omega) - \omega = 0.
\end{gathered} \tag{3.11}$$

Solving the regulator equations (3.11), we obtain the unique solution

$$\pi_1(\omega) = \omega, \quad \pi_2(\omega) = \omega, \quad \pi_3(\omega) = \frac{\omega^2}{b}, \quad \varphi(\omega) = -c\omega + \frac{\omega^3}{b}. \tag{3.12}$$

Using Theorem 2.1 and the solution (3.12) of the regulator equations (3.11) for the nonlinear system (3.10), we obtain the following result which provides a state feedback control law solving the output regulation problem for the system (3.10).

Theorem 3.1. *A state feedback control law solving the output regulation for the system* (3.10) *is given by*

$$u = \varphi(\omega) + K[x - \pi(\omega)], \tag{3.13}$$

where $\varphi(\omega)$ *and* $\pi(\omega)$ *are as given in* (3.12), *and the gain matrix* $K = [\, K_1 \; K_2 \; 0 \,]$ *is chosen so as to satisfy the inequalities* (3.9).

3.2. Constant Tracking Problem for x_2

Here, we consider the problem of the state x_2 of the Pan system tracking the constant signal ω. Thus, the nonlinear system under consideration is

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1), \\ \dot{x}_2 &= cx_1 - x_1x_3 + u, \\ \dot{x}_3 &= -bx_3 + x_1x_2, \\ e &= x_2 - \omega. \end{aligned} \tag{3.14}$$

The regulator equations for the system (3.14) are obtained by Theorem 2.1 as

$$\begin{gathered} \pi_2(\omega) - \pi_1(\omega) = 0, \\ c\pi_1(\omega) - \pi_1(\omega)\pi_3(\omega) + \varphi(\omega) = 0, \\ -b\pi_3(\omega) + \pi_1(\omega)\pi_2(\omega) = 0, \\ \pi_2(\omega) - \omega = 0. \end{gathered} \tag{3.15}$$

Solving the regulator equations (3.15), we obtain the unique solution

$$\pi_1(\omega) = \omega, \quad \pi_2(\omega) = \omega, \quad \pi_3(\omega) = \frac{\omega^2}{b}, \quad \varphi(\omega) = -c\omega + \frac{\omega^3}{b}. \tag{3.16}$$

Using Theorem 2.1 and the solution (3.16) of the regulator equations (3.15) for the nonlinear system (3.14), we obtain the following result which provides a state feedback control law solving the output regulation problem for the system (3.14).

Theorem 3.2. *A state feedback control law solving the output regulation for the system* (3.14) *is given by*

$$u = \varphi(\omega) + K[x - \pi(\omega)], \tag{3.17}$$

where $\varphi(\omega)$ *and* $\pi(\omega)$ *are as given in* (3.16), *and the gain matrix* $K = [\, K_1 \; K_2 \; 0 \,]$ *is chosen so as to satisfy the inequalities* (3.9).

3.3. Constant Tracking Problem for x_3

Here, we consider the problem of the state x_3 of the Pan system tracking the constant signal ω. Thus, the nonlinear system under consideration is

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1), \\
\dot{x}_2 &= cx_1 - x_1x_3 + u, \\
\dot{x}_3 &= -bx_3 + x_1x_2, \\
e &= x_3 - \omega.
\end{aligned} \tag{3.18}$$

The regulator equations for the system (3.18) are obtained by Theorem 2.1 as

$$\begin{gathered}
\pi_2(\omega) - \pi_1(\omega) = 0, \\
c\pi_1(\omega) - \pi_1(\omega)\pi_3(\omega) + \varphi(\omega) = 0, \\
-b\pi_3(\omega) + \pi_1(\omega)\pi_2(\omega) = 0, \\
\pi_3(\omega) - \omega = 0.
\end{gathered} \tag{3.19}$$

Solving the regulator equations (3.19), we obtain the unique solution

$$\pi_1(\omega) = \sqrt{b\omega}, \quad \pi_2(\omega) = \sqrt{b\omega}, \quad \pi_3(\omega) = \omega, \quad \varphi(\omega) = \sqrt{b\omega}(\omega - c). \tag{3.20}$$

Using Theorem 2.1 and the solution (3.20) of the regulator equations (3.19) for the nonlinear system (3.18), we obtain the following result which provides a state feedback control law solving the output regulation problem for the system (3.18).

Theorem 3.3. *A state feedback control law solving the output regulation for the system* (3.18) *is given by*

$$u = \varphi(\omega) + K[x - \pi(\omega)], \tag{3.21}$$

where $\varphi(\omega)$ *and* $\pi(\omega)$ *are as given in* (3.20), *and the gain matrix* $K = [\, k_1 \; k_2 \; 0 \,]$ *is chosen so as to satisfy the inequalities* (3.9).

4. Numerical Results

For numerical simulations, the parameters are chosen as in the chaotic case of Pan's system, namely,

$$a = 10, \qquad b = \frac{8}{3}, \qquad c = 16. \tag{4.1}$$

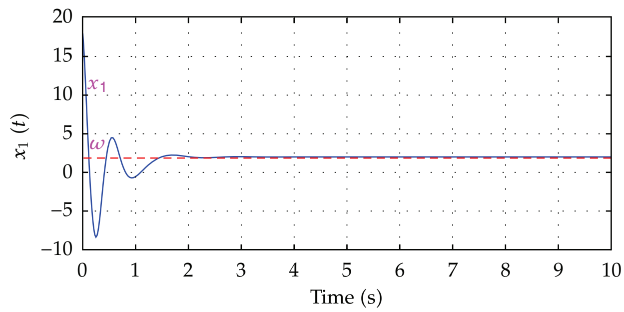

Figure 2: Constant tracking problem for x_1.

For achieving internal stability of the state feedback regulator problem, a gain matrix K must be chosen so that $A+BK$ is Hurwitz. Since $b = 8/3$, one eigenvalue of $A+BK$ is always given by $\lambda_1 = -b = -8/3$. The other two eigenvalues of $A+BK$ are given by the quadratic equation (see (3.7))

$$\lambda^2 + \lambda(a - k_2) - a(c + k_1 + k_2) = \lambda^2 + \lambda(10 - k_2) - 10(16 + k_1 + k_2) = 0. \tag{4.2}$$

Suppose that we wish to choose K so that it has two stable eigenvalues $\lambda_2 = \lambda_3 = -3$.

Then, we must have

$$\lambda^2 + \lambda(10 - k_2) - 10(16 + k_1 + k_2) = \lambda^2 + 6\lambda + 9 = 0. \tag{4.3}$$

Comparing the coefficients of the two quadratic equations, we have

$$10 - k_2 = 6, \qquad k_1 + k_2 + 16 = -0.9. \tag{4.4}$$

A simple calculation yields

$$K = \begin{bmatrix} k_1 & k_2 & 0 \end{bmatrix} = \begin{bmatrix} -20.9 & 4 & 0 \end{bmatrix}. \tag{4.5}$$

4.1. Constant Tracking Problem for x_1

Here, the initial conditions are taken as

$$x_1(0) = 18, \qquad x_2(0) = 10, \qquad x_3(0) = 12, \qquad \omega = 2. \tag{4.6}$$

The simulation graph is depicted in Figure 2 from which it is clear that the state trajectory $x_1(t)$ tracks the constant reference signal $\omega = 2$ in 2 seconds.

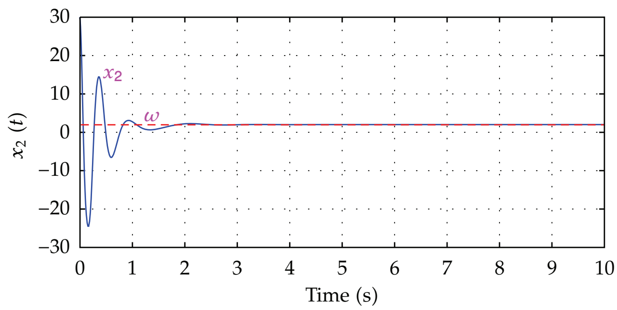

Figure 3: Constant tracking problem for x_2.

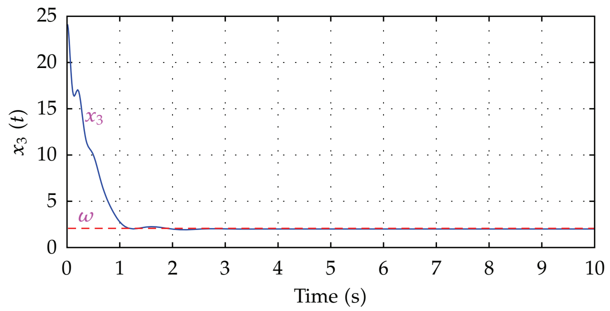

Figure 4: Constant tracking problem for x_3.

4.2. Constant Tracking Problem for x_2

Here, the initial conditions are taken as

$$x_1(0) = 20, \qquad x_2(0) = 30, \qquad x_3(0) = 6, \qquad \omega = 2. \tag{4.7}$$

The simulation graph is depicted in Figure 3 from which it is clear that the state trajectory $x_2(t)$ tracks the constant reference signal $\omega = 2$ in 3 seconds.

4.3. Constant Tracking Problem for x_3

Here, the initial conditions are taken as

$$x_1(0) = 12, \qquad x_2(0) = 8, \qquad x_3(0) = 24, \qquad \omega = 2. \tag{4.8}$$

The simulation graph is depicted in Figure 4 from which it is clear that the state trajectory $x_3(t)$ tracks the constant reference signal $\omega = 2$ in 2 seconds.

5. Conclusions

In this paper, we have derived new results for the output regulation problem of the Pan chaotic system (2010). Basically, we have applied the regulator equations of Byrnes and

Isidori (1990) to derive state feedback control laws for regulating the output of the Pan system (2010) so as to track constant reference signals. We have provided simulation results to illustrate the effectiveness of the regulation schemes for the three cases of the constant regulation problem for the chaotic Pan system.

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