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Group flow, complex flow, unit vector flow, and the $(2 + \epsilon)$-flow conjecture

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February 22, 2014

Abstract

If F is a (possibly infinite) subset of an abelian group Γ, then we define $f(F, \Gamma)$ as the smallest natural number such that every $f(F, \Gamma)$-edge-connected (finite) graph G has a flow where all flow values are elements in F. We prove that $f(F, \Gamma)$ exists if and only if some odd sum of elements in F equals some even sum. We discuss various instances of this problem. We prove that every 6-edge-connected graph has a flow whose flow values are the three roots of unity in the complex plane. If the edge-connectivity 6 can be reduced, then it can be reduced to 4, and the 3-flow conjecture follows. We prove that every 14-edge-connected graph has flow whose flow values are the five roots of unity in the complex plane. Any such flow is balanced modulo 5. So, if the edge-connectivity 14 can be reduced to 9, then the 5-flow conjecture follows, as observed by F. Jaeger. We use vector flow to prove that, for each odd natural number $k \geq 3$, every $(3k - 1)$-edge-connected graph has a collection of k even subgraphs such that every edge is in precisely $k - 1$ of them. Finally, the flow result gives a considerable freedom to prescribe the flow values in the $(2 + \epsilon)$-flow conjecture by L. Goddyn and P. Seymour. For example, if k is a natural number and G is a $6k$-edge-connected graph, then G has a flow with flow values $1, 1 + 1/k$. It also has, for any irrational number ϵ, a flow with flow values $1, 1 + \epsilon, 1 + \epsilon + 1/k$.

1 Introduction

Jaeger [7], [8] proposed the following conjecture which he called the *the circular flow conjecture*:

If k is an odd natural number, and G is a $(2k - 2)$-edge-connected graph, then G has an orientation such that each vertex is balanced modulo k, that is, each vertex has the same indegree and outdegree modulo k.

This conjecture does not extend to the case where k is an even number (because a vertex of odd degree cannot be balanced modulo an even number) also not in the weak version where we replace the edge-connectivity $2k - 2$ by a larger function of k. However, the weak version becomes true, also when k is odd, in the following version proved in [15]: Let k be a natural number, and let G be a $(2k^2 + k)$-edge-connected graph with n vertices $v_1, v_2, \ldots, v_n$. Let d_i be an integer for each $i = 1, 2, \ldots, n$ such that the sum of all d_i is congruent $(mod \ k)$ to the number of edges of G. Then each edge of G can be directed such that each v_i has outdegrees d_i modulo k. Recently, the bound $(2k^2 + k)$ has been reduced to $3k - 3$ for k odd [10]. It was reduced to $14k$ for k even in [18].

In this paper we consider a (possibly infinite) subset F of an additive abelian group Γ. If possible, we define $f(F, \Gamma)$ as the smallest natural number such that every $f(F, \Gamma)$-edge-connected (finite) graph G has a flow where all flow values are elements in F. That is, each edge is assigned an orientation and an element of F such that, for each vertex v, the in-sum of v equals the out-sum of v. We prove that $f(F, \Gamma)$ exists if and only if there exist elements $a_1, a_2, \ldots a_{2p}, b_1, b_2, \ldots, b_{2q+1}$ in F such that

$$a_1 + a_2 + \ldots + a_{2p} = b_1 + b_2 + \ldots + b_{2q+1}.$$

We refer to this as *the odd sum condition*. Clearly, the odd sum condition is satisfied at every vertex of odd degree.

Here p, q are nonnegative integers. Repetition is allowed, and possibly $p = 0$ in which case the first sum is 0.

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We discuss various instances of this problem. We prove that every 6-edge-connected graph has a flow whose flow values are the three roots of unity in the complex plane. If the edge-connectivity 6 can be reduced, then it can be reduced to 4, and the 3-flow conjecture follows. We prove that every 14-edge-connected graph has flow whose flow values are the five roots of unity in the complex plane. If the edge-connectivity 14 can be reduced to 9, then the 5-flow conjecture follows.

These results are related to conjectures of Kamal Jain on S^d-flow where S^d is the set of unit vectors in $\mathbb{R}^{d+1}$, see [22]. Jain conjectured that every 2-edge-connected graph has an S^2-flow and that every 4-edge-connected graph has an S^1-flow, as mentioned by Matt DeVos in [22]. As pointed out by DeVos, a graph has an S^1-flow if it has a nowhere zero 3-flow, that is, a flow with flow values $1, 2$ in Z. He suggested that the converse might be true. We provide infinitely many contraction-critical counterexamples, but show that the suggested equivalence holds when S^1 is replaced by its subset consisting of the three roots of unity in the complex plane.

The known graph vector flow results are derived from graph covering results. We show that there are many possibilities for applications in the other direction as well and illustrate with the following example: For each odd natural number $k \geq 3$, every $(3k - 1)$-edge-connected graph has a collection of k even subgraphs such that every edge is in precisely $k - 1$ of them.

A $(2 + \epsilon)$-*flow* is a flow whose flow values are in the real interval from 1 to $(1 + \epsilon)$. The $(2 + \epsilon)$-flow conjecture by Goddyn and Seymour says that, for each positive ϵ, every graph of sufficiently large edge-connectivity (depending on ϵ only) has a $(2 + \epsilon)$-flow, see [21] and Section 9.2 in [20]. Our flow result shows that we even have considerable freedom to prescribe the flow values in the $(2 + \epsilon)$-flow conjecture by Goddyn and Seymour. In fact, it is possible to use any fixed set of flow values close to $1 + \epsilon$ as long as they satisfy the odd sum condition. If these flow values form a finite set which is minimal linearly dependent in the vector space of real numbers over the field of rational numbers, then any odd cut in the graph must use all the flow values. It would be interesting to explore structural consequences of this fact.

2 Flows with flow values in a prescribed set

The terminology and notation are the same as in [15] which are essentially the same as in [2], [4], [12]. In the present paper, however, a *graph* may have multiple edges (but no loops). A graph is *even* if every vertex has even degree.

Theorem 1. *Let $a_1, a_2, \ldots, a_{2p}, b_1, b_2, \ldots, b_{2q+1}$ be elements in an additive abelian group Γ such that*

$$a_1 + a_2 + \ldots + a_{2p} = b_1 + b_2 + \ldots + b_{2q+1}.$$

Put $k = 2p + 2q + 1$.

If G is a graph with edge-connectivity at least $3k - 1$, then G has a flow whose flow values are in $\{a_1, a_2, \ldots, a_{2p}, b_1, b_2, \ldots, b_{2q+1}\}$.

Proof: The proof is by induction on the number of edges of the graph G. If G has two vertices and $3k - 1$ edges, the statement is trivial. So we proceed to the induction step.

If G has a vertex v with only one neighbor, then we first apply induction to $G - v$ and then dispose of the edges incident with v. If v has even degree, we need only one flow value for the edges incident with v. If v has odd degree, then we first use all flow values for k edges incident with v and then one flow value for the remaining edges incident with v. So assume that each vertex has at least two neighbors.

If G has a vertex v of degree at least $3k + 1$, then we use Mader's lifting theorem [11] to lift two edges incident with v, that is, we replace two edges vu, vw by just one edge uw such that the resulting graph G' has edge-connectivity at least $3k - 1$. Then we apply induction to G'. If the edge uw is directed from u to w, then we replace uw by the directed path uvw and we give these two edges the same flow value as uw. So assume that each vertex has degree at most $3k$.

If some vertex v has degree precisely $3k - 1$, then we successively lift edges incident with v. Either all edges are lifted, or we reach a situation where v has only one neighbor. In either case we complete the proof by induction.

So we are left with the case where each vertex has degree precisely $3k$. We now apply the result in [15] and the extension on the bound in [10]. As G has edge-connectivity at least $3k - 3$, the edges of G can be orientated such that all outdegrees are divisible by k. Then also all indegrees are divisible by k. Each

vertex of G is first split onto two vertices, a source and a sink, respectively. The resulting graph is bipartite. As each source has outdegree divisible by k each source is further split into vertices (sources) of degree precisely k. Similarly for the sinks. We may loose the high edge-connectivity, and we shall not need it. The resulting graph H is bipartite and k-regular. Hence it has a proper k-edge-coloring with colors $1, 2, \ldots, k$. We replace these colors by $\{-a_1, -a_2, \ldots, -a_{2p}, b_1, b_2, \ldots, b_{2q+1}\}$, respectively. For each vertex in H the sum of indegrees is 0, and the sum of outdegrees is 0. The same holds when we identify vertices in H in order to get G back. When we use a color $-a_i$ for an edge in G, we reverse the direction of the edge and replace $-a_i$ by a_i. This proves Theorem 1. □

Below we discuss various instances of Theorem 1.

3 Nowhere zero Γ-flow and the circular flow conjecture

We begin with the two extreme cases for the set F of prescribed flow values, namely $|F| = 1$, and $F = \Gamma \setminus \{0\}$, respectively. (If F contains 0, then any graph has an F-flow.)

If $|F| = 1$ and F satisfies the odd sum condition, then the element of F must have finite odd order. In other words, we may assume that F consists of the element 1 in $\mathbb{Z}_k$, the integers modulo the odd natural number k. Moreover, an F-flow is an orientation which is balanced modulo k.

Theorem 2. *Let k be an odd natural number. Then*

$$2k - 2 \leq f(\{1\}, \mathbb{Z}_k) \leq 3k - 3.$$

Proof: The upper bound was proved in [10]. Jaeger's circular flow conjecture asserts that the lower bound is an equality. □

A $\Gamma \setminus \{0\}$-flow is also called a *nowhere zero Γ-flow*. Known results can be summarized as follows.

Theorem 3. *Let Γ be an abelian (possibly infinite) group. Put $F = \Gamma \setminus \{0\}$.*
(a) If $|\Gamma| \geq 6$, then $f(F, \Gamma) = 2$.

(b) If $|\Gamma| = 5$, then $f(F, \Gamma) = 2$ or $f(F, \Gamma) = 4$.
(c) If $|\Gamma| = 4$, then $f(F, \Gamma) = 4$.
(d) If $|\Gamma| = 3$, then $f(F, \Gamma) = 4$ or $f(F, \Gamma)) = 6$.

Proof: Tutte [16] proved that for any abelian group Γ of finite order k, a graph has a nowhere zero Γ-flow if and only if it has a nowhere zero k-flow, that is, an integer flow with flow values $1, 2, \ldots, k - 1$. Thus statement (a) is Seymour's 6-flow theorem [14]. (If Γ is infinite, it contains either a finite subgroup of order at least 6, or it contains a subgroup isomorphic to the group of integers.)

Tutte [17] proved that every 4-edge-connected graph has two edge-disjoint spanning trees. This implies, as observed by Jaeger [6], that it is the union of two even subgraphs which easily implies the existence of a nowhere zero 4-flow. As the Petersen graph has no nowhere zero 4-flow, statement (c) follows.

If the 5-flow conjecture holds, then the first alternative in statement (b) holds. If the 5-flow conjecture is false, then a smallest counterexample is cubic and 3-connected. In that case statement (c) implies the second alternative in (b).

If the 3-flow conjecture holds, then the first alternative in statement (d) holds. If the 3-flow conjecture is false, then there is a 5-edge-connected counterexample, as proved by Kochol [9]. Then the bound in [10] implies that the second alternative in (d) holds. □

4 S^1-flow

An S^1-flow is the same as a flow whose flow values are complex numbers of modulus 1. Let R_k denote the k roots of unity, that is, the solutions to the equation $z^k = 1$. The first observation is well known and easy. For the sake of completeness we repeat the well-known arguments.

Proposition 1. *If G be a graph. Then (a) and (b) below are equivalent, and they imply (c) where*
   (a) G has a nowhere zero 3-flow.
   (b) G has an R_3-flow.
   (c) G has an S^1-flow.
   If G is cubic the three statements are equivalent, and G satisfies (a),(b),(c) if and only if G is bipartite.

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Proof: We first prove that (a) implies (b). Assume that G has a nowhere zero 3-flow, If some vertex has an incoming and outgoing edge of the same flow, then we lift those two edges. If all outgoing edges have flow 2 and all incoming edges have flow 1, then we split the vertex into vertices of degree 3 such that there is one outgoing edge of flow 2 and two incoming edges of flow 1. Then all edges of flow 2 form a perfect matching, and all edges of flow 1 form a collection of even cycles. We replace the flow values 2 by the flow value 1. And each even cycle receives the flow values $e^{2\pi i/3}, e^{4\pi i/3}$ alternately. This proves that (a) implies (b). By a similar argument, (b) implies (a).

If z_1, z_2, z_3 are complex numbers of modulus 1 such that $z_1 + z_2 + z_3 = 0$, and one of z_1, z_2, z_3 is one of the three roots of unity $1, e^{2\pi i/3}, e^{4\pi i/3}$, then z_1, z_2, z_3 are the three roots of unity. This shows that an S^1-flow of a cubic graph such that some flow value is 1 is an R_3-flow (possibly after reversal of some edge-directions). Each vertex of a cubic graph with an R_3-flow has indegree or outdegree zero, and hence the graph is bipartite. This completes the proof. □

Theorem 3 and Proposition 1 imply

Theorem 4. $4 \leq f(S^1, \mathbb{R}^2) \leq f(R_3, \mathbb{C}) \leq 6$.

□

Matt DeVos [22] suggested that the three statements in Proposition 1 are equivalent for all graphs. We show that this is not so. We shall refer to the *unit distance graph*, that is, the graph whose vertices are the points in the Euclidean plane $\mathbb{R}^2$ such that two vertices are neighbors if and only if they have distance 1.

Proposition 2. *Let G be a planar graph. Then G has an S^1-flow if and only if the dual graph G^* is homomorphic to a subgraph of the unit distance graph, that is, there exists a map from the vertex set of G^* to the vertex set of the unit distance graph such that neighbors are mapped to neighbors.*

Proof: Assume first that G^* is homomorphic to a subgraph of the unit distance graph. Let f denote the homomorphism. Then every edge xy in the dual graph corresponds to a unit vector from $f(x)$ to $f(y)$. For any cycle in the dual graph, the sum of vectors $f(x)f(y)$ around the cycle is the zero

vector. So, if we associate the vector $f(x)f(y)$ to the dual edge of xy, we obtain an S^1-flow in G.

Suppose conversely that we have an S^1-flow in G. For any edge xy in the dual graph we associate the vector of the dual edge to xy. Then the sum of vectors around any cycle in G^* is the zero vector. Take any vertex x of G^* and map it to $(0,0)$ in the Euclidean plane $\mathbb{R}^2$. For any other vertex y in G^* we consider a path P from x to y and map y into the point obtained by adding the vectors in P. The resulting map is a homomorphism from G^* to the unit distance graph. $\qquad\square$

It is well-known that a planar graph G has a nowhere zero 3-flow if and only if G^* has chromatic number at most 3. So, if H is a planar 4-chromatic graph which is also a subgraph of the unit distance graph, then the dual graph of H is a planar graph which has an S^1-flow but not a nowhere zero 3-flow. There is a great variety of such graphs. The smallest 4-chromatic subgraph of the unit distance graph is the graph obtained by applying Hajos' construction to two copies of K_4. Its dual graph is obtained by pasting two copies of K_4 together along an edge. If we apply successively Hajos' construction using copies of K_4, then we obtain 4-color-critical planar graphs which are also subgraphs of the unit distance graphs. The dual graphs are obtained by copies of K_4 by pasting them together along edges. They all have an S^1-flow but not a nowhere zero 3-flow, and they are contraction-critical with this property.

We conclude this section with R_5-flows where R_5 consists of the five roots of unity in the complex plane.

Theorem 5. $8 \leq f(R_5, \mathbb{C}) \leq 14$.
If $f(R_5, \mathbb{C}) \leq 9$, then the 5-flow conjecture holds.

Proof: As the sum of the five roots of unity is zero, it follows from Theorem 1 that $f(R_5, \mathbb{C}) \leq 14$.

The five roots of unity form a minimal linearly dependent set when we view $\mathbb{C}$ as a vector space over the field of rational numbers. In other words, if

$$m_1 + m_2 e^{2\pi i/5} + m_3 e^{4\pi i/5} + m_4 e^{6\pi i/5} + m_5 e^{8\pi i/5} = 0,$$

then
$$m_1 = m_2 = m_3 = m_4 = m_5.$$

To see this we consider the real parts. Using that $cos(2\pi/5) = (-1 + \sqrt{5})/4$ and $cos(4\pi/5) = (-1 - \sqrt{5})/4$ and that $\sqrt{5}$ is irrational, we conclude that
$$m_1 = (m_2 + m_3 + m_4 + m_5)/4.$$

By rotating the coordinate system, we conclude that each of m_i ($i = 1, 2, 3, 4, 5$) is 1/4 times the sum of the others. Those equations are satisfied only if all m_i are equal.

Consider now a graph G which has an R_5-flow. If some incoming edge to a vertex x has the same flow value as some outgoing edge, then we lift these two edges from x and replace them by one edge. By the above remark about minimal dependence, we end up with a graph which has outdegree (or indegree) zero and indegree (or outdegree) divisible by 5. Hence the directions of the edges of G form an orientation which is balanced modulo 5. Jaeger [8] observed that K_8 does not have such an orientation. Hence $8 \leq f(R_5, \mathbb{C})$. If $f(R_5, \mathbb{C}) \leq 9$, then every 9-edge-connected graph has an R_5-flow and hence also a flow which is balanced modulo 5. Jaeger [8] observed that the last statement implies the 5-flow conjecture.

$\square$

5 S^2-flow

Let T denote the vectors $(1, 1, 0), (0, 1, 1), (1, 0, 1), (1, -1, 0), (0, 1, -1), (1, 0, -1)$ in $\mathbb{R}^3$. As the sum of two of them equals a third, it follows from Theorem 1 that every 8-edge-connected graph has a T-flow. However, by combining known results we conclude that $f(T, \mathbb{R}^3) = 4$. Again, the arguments are well-known but we repeat them for the sake of completeness.

Proposition 3. *Let G be a graph. Then (a) and (b) below are equivalent, where*

(a) G has a T-flow.

(b) G is the union of three even subgraphs such that each edge is in precisely two of these subgraphs.

If G is cubic, then G satisfies (a),(b) if and only if G is has a proper 3-edge-coloring.

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Proof: . Assume first that G is the union of three even subgraphs H_1, H_2, H_3 such that each edge in in precisely two of these subgraphs. We orient the edges of H_i such that the indegree equals the outdegree for each vertex. We also orient the edges of G. To each edge e of G we associate the vector (c_1, c_2, c_3) where $c_i = 0$ if H_i does not contain e, and c_i is equal to 1 or -1 otherwise. We choose -1 when the orientation of e in G and H_i disagree. (If an edge e receives the vector $(-1, -1, 0)$ which is not in T, then we reverse the direction of e so that $(-1, -1, 0)$ is replaced by $(1, 1, 0)$ which is in T.) This proves that G has a T-flow.

Conversely, if G has a T-flow, then we define H_i as the graph whose edges have i'th coordinate distinct from 0.

It is well-known (and easy to see) that a cubic graph satisfies (b) if and only if it has a proper 3-edge-coloring. □

There is a more direct way to obtain a 3-edge-coloring from a T-flow in a cubic graph. We may think of the vectors of T as the edges of a tetrahedron whose edges have length $\sqrt{2}$. This tetrahedron has a proper 3-edge-coloring where two edges have the same color if and only if they are perpendicular. If a cubic graph has a T-flow, then the 3-edge-coloring of the tetrahedron results in a 3-edge-coloring of G. That coloring must be proper because three vector of T cannot add up to zero if two of them are perpendicular.

Theorem 6. $f(S^2, \mathbb{R}^3) \leq f(T, \mathbb{R}^3) = 4$.

Proof: Every cubic 3-connected graph which has no proper 3-edge-coloring shows that $f(T, \mathbb{R}^3) \geq 4$. To prove that $f(T, \mathbb{R}^3) \leq 4$, we consider a 4-edge-connected graph G. By a result of Tutte, G has two edge-disjoint spanning trees T_1, T_2. By deleting some edges of T_i we obtain an even graph H_i, $i = 1, 2$. The symmetric difference of H_1, H_2 is an even graph H_3, and H_1, H_2, H_3 satisfy the statement (b) in Proposition 3. □

Kamal Jain conjectured that $f(S^2, \mathbb{R}^3) = 2$, that is, every 2-edge-connected graph has a 3-dimensional unit distance flow.

Bermond, Jackson, and Jaeger [1] showed that every 2-edge-connected-graph has a collection of seven even subgraphs so that every edge is in exactly four of these subgraphs. This easily implies that every 2-edge-connected graph has an 7-dimensional unit distance flow.

Fulkerson [5] conjectured that every cubic 3-connected graph has six perfect matchings such that every edge in in precisely two of them. Hence it has six 2-factors such that every edge is in four of them. If true, this would imply that every 2-edge-connected-graph has a collection of six even subgraphs so that every edge is in exactly four of these subgraphs. This would imply that every 2-edge-connected graph has an 6-dimensional unit distance flow.

Celmins [3] and Preissmann [13] conjectured that every 2-edge-connected-graph has a collection of five even subgraphs so that every edge is in exactly two of these subgraphs. If true, that would imply that every 2-edge-connected graph has an 5-dimensional unit distance flow.

Thus covering results can be used to vector flows. We shall here point out that there are applications in the other direction as well. Recall that every 4-edge-connected graph has a collection of three even subgraphs such that every edge is in precisely two of them. Below is an analogue where 3 is replaced by k.

Theorem 7. *Let k be an odd natural number, $k \geq 3$. Every $(3k-1)$-edge-connected graph has a collection of k even subgraphs such that every edge in G is in precisely $k-1$ of them.*

Proof: Consider the k-dimensional vector space over the field of two elements. Let F be the set of vectors that have precisely one zero. Then F has cardinality k and sum zero. Consider now a $(3k-1)$-edge-connected graph G. By Theorem 1, G has an F-flow. Now let H_i be the subgraph consisting of those edges whose i'th coordinate is 1. Then H_i is even, and each edge of G is in precisely $k-1$ of the graphs $H_1, H_2, \ldots, H_k$. $\qquad \square$

Note that many other variations of Theorem 7 are possible. For example, if k, p are natural numbers such that $k > p > 1$, and $\binom{n}{p}$ is odd and $\binom{n-1}{p-1}$ is even, and G is a $(3\binom{k}{p} - 1)$-edge-connected graph, then G has a collection of k even subgraphs such that every edge in G is in precisely p of them. To verify this we repeat the proof of Theorem 7 by letting F be the set of vectors that have precisely $n - p$ zeros.

6   The $(2 + \epsilon)$-flow conjecture

We considered first the case where the set F of flow values has only one element. We conclude with the case where it has two elements.

Theorem 8. *Let k be a natural number. If G is $6k$-edge-connected, then G has a flow with flow values $1, 1 + 1/k$.*

Proof: Since

$$(k + 1) \cdot 1 = k \cdot (1 + 1/k)$$

the odd sum condition is satisfied, and hence Theorem 8 follows from Theorem 1 $\square$.

This implies the $(2 + \epsilon)$-flow conjecture by Goddyn and Seymour [21].

There is another ways to derive the $(2 + \epsilon)$-flow from the weak circular flow conjecture, as pointed out in [20]. If a graph has an orientation which is balanced modulo k where k is odd, then we can think of this as a flow where each flow value is $(k - 1)/2$ in the integers reduced modulo k. A result of Younger [19] then says that this flow can be replaced by an integer flow where each flow value is $(k - 1)/2$ or $(k + 1)/2$, possibly after reversing some edge directions. If we divide by k, then all flow values are $1 - 1/(2k)$ or $1 + 1/(2k)$.

There are many other possible variations of Theorem 8. By choosing F (and possibly also the vector space) differently, you may derive other flow results of this nature. We mention but one such example.

Theorem 9. *Let k be a natural number, and let α be an irrational positive number. If G is $6k$-edge-connected, then G has a flow with flow values $1, \alpha, \alpha + 1/k$.*

Proof: $1 \cdot 1 + k \cdot \alpha = k \cdot (\alpha + 1/k)$ $\square$

The flow in Theorem 9 has the property that any odd edge-cut contains all three flow values because the flow values form a minimal dependent set in the vector space $\mathbb{R}$ over the field $\mathbb{Q}$. An analogous result holds for any such minimal dependent set. Maybe the fact that any odd cut contains all flow values has some structural consequences.

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