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Limit cases for rotor theories with Betz optimization

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Abstract. A complete analytical formulation of the vortex approach for the rotor with an ideal load distribution under Betz optimal condition needs some additional assumption about a correct choice of the helical pitch for vortex sheets in the rotor wake. An examination of the three evident assumptions (the pitch is independent from velocities induced by the wake; the pitch depends on the induced velocities in the far wake; the pitch depends on the induced velocities in the rotor plane) was considered by a comparison with the main restriction of the actuator disk theory - the Betz-Joukowsky limit. In the present investigation an analytical solution for the limit case of an infinite number of blades was used to re-examine the choice of the wake pitch.

1. Introduction

For planar wings of a finite span with a planar vortex wake in a uniform flow it's well-known [1] that an elliptic distribution of the load along the lifting line corresponds to the lowest drag of the trailing vortex (Fig. 1a). Betz [2] generalized this result for rotor and formulated an analogical condition for the optimum of a rotating propeller: the distribution of circulation along the lifting line replacing the blade should be such that the free vortex sheet trailing from it has an exact helical shape and moves uniformly along the rotor axis in the direction of the main flow (Fig. 1b). His formulation looks like an obvious consequence of the wing theory because a rotating blade in a uniform flow should produce a helical shape of the vortex sheet leaving the trailing edge as an ideal image of the rotor wake.

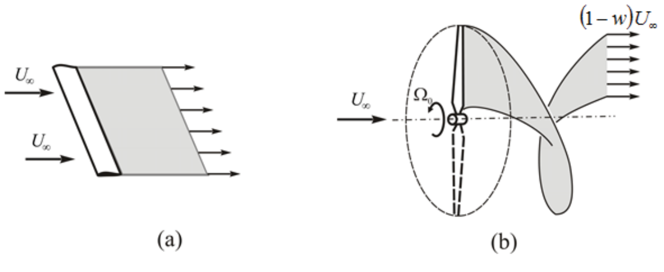

Figure 1. (a) Prandtl vortex model of the wing with a finite span and the elliptic distribution of the load along the span [2]; (b) vortex model of the propeller proposed by Betz [1].

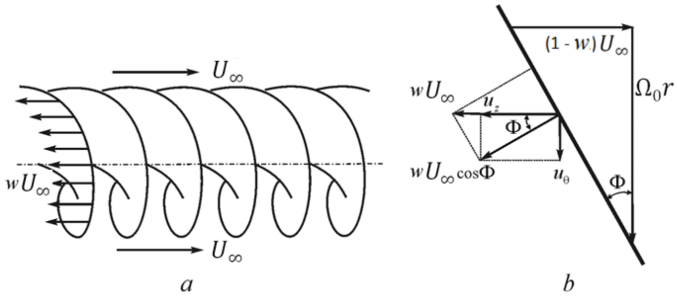

Figure 2. (a) Schematic representation of the associated wake structure; (b) triangles of velocities: for the (i) wake model - $w = 0$; for the (ii) model - $w = 1$, and for the (iii) model - $w = \frac{1}{2}$

The next correlation with the wing theory is the uniform moving of the vortex sheet in the axial direction with a constant velocity $(1-w)U_\infty$. In the case of wind turbines we introduce an induction coefficient - w of the self-induction motion of the vortices in the wake which defines a dimensionless slowing-down of the motion of the vortex sheet with respect to the wind speed U_∞. Those semi-infinite helical vortex sheets are usually replaced by an associating vortex system, which extends on both sides to infinity and moves in axial direction in equilibrium (Fig. 2a). In this case the circulation changes from the elliptic distribution along the wing to asymmetrical function along the blade and the task to find it was a challenge which Betz could not solve. In addition, a unique solution does not follow from his proof of the minimum drag for the trailing vortex with the exact helical shape because it is still necessary to set a value of a pitch (or angle Φ on Fig. 2b) for the helical sheets from supplementary reason. Three reasonable variants for the determination are based on different inclusion of an additional velocity induced by the vortex wake itself: (i) to neglect this velocity entirely; (ii) to use the velocity in the far wake; (iii) to use the velocity in the rotor plane (Fig. 2b). In the table, we have listed the well-known rotor theories in which a different pitch of the vortex wake was selected with a reference to their authors [4–8]. This paper and our analysis have a purpose to investigate which pitch in the mentioned theories gives a more accurate result. With this in mind a method of the rotor optimization for the (iii) model from [8, 9] with discretization of the wake by a superposition of single helical vortex filaments [10] was extended for optimizations of the (i) and (ii) models. In the second paragraph we repeat the lifting line theory as a common part for all derivations but in the third paragraph the optimal power and thrust coefficients were originally derived for the (i) and (ii) models to compare with the solution from [8, 9] for the (iii) model. The fourth paragraph contains results of a comparison of all rotor optimizations.

Table Assumptions of the different rotor theories

theorie	*number of blades*	*definition of the pitch of the vortex wake*	*w*
Betz–Joukowsky limit [4]	actuator disk	not relevant	-
BEM calculation of Glauert [5]	not relevant	induction factors defined in rotor plane	-
(i) Goldstein's model [6]	finite	without correction to the wake induction	0
(ii) Theodorsen's model [7]	finite	correction by the induction in far wake	1
(iii) model of [8, 9]	finite	correction by the induction in rotor plane	$\frac{1}{2}$

2. Lifting line vortex theory for the Betz optimization

An operating mode of wind turbine is defined by tip speed ratio or, otherwise, the dimensionless circumferential velocity of blade tips referred to the wind speed:

$$\lambda_0 \equiv \frac{\Omega_0 R_0}{U_\infty}, \quad (1)$$

where U_∞ is wind speed and Ω_0 is angular velocity of the rotor which radius is R_0. Using an analogy with the vortex theory of wings, we replace rotating blades by a distribution of the bounded vortices along the lifting line on the blade, and the rotor wake is replaced by a system of free vortices with the fixed form of regular helical vortex sheets leaving from the trailing edges of blades (Fig. 1b).

The load distribution along the blade can be found on the basis of the Kutta–Joukowsky theorem

$$d\vec{L} = \rho \vec{U}_0 \times \vec{\Gamma} dr, \quad (2)$$

where dL is the lift acting on an blade element with the running size of dr, U_0 is the relative velocity of the incoming flow, and Γ is the circulation of the bounded vortices. The free vortex sheet of the wake induces additional velocities u_{z_0} and u_{θ_0} in the rotor plane; i.e., the components of relative velocity U_0 in Eq. (2) take the form $U_{\theta_0} = \Omega r + u_{\theta_0}$ and $U_{z_0} = U_\infty - u_{z_0}$.

For a simplification [8], the semi-infinite wake from free helical vortex sheets is replaced by the associated vortex system, which extends on both sides to infinity and moves in an equilibrium along rotor axis (Fig. 2a). According to the Helmholtz' vortex theorems, the bounded circulation Γ of a blade element is unambiguously related to the circulation of a corresponding free vortex of the wake. It is clear that the velocities u_z and u_θ induced in an arbitrary cross section of the associated vortex system precisely describe the properties of the semi-infinite wake with the same distribution of circulation only in the Treftz plane which is located far from the rotor plane. We can pass from the Trefftz plane to the rotor plane by symmetry properties of the associated vortex system. We noted that, because of the symmetry, the velocity induced by a semi-infinite wake (half of the associated vortex system) is equal to half of the velocity induced at the corresponding point on the Trefftz plane [1, 3]:

$$u_{\theta_0} = \tfrac{1}{2} u_\theta \quad \text{and} \quad u_{z_0} = \tfrac{1}{2} u_z. \quad (3)$$

Integrating of Eq. (2) along the lifting vortex line and summing the contribution from each blade, on the basis of Eq. (3), gives formulas for the power and thrust coefficients:

$$C_P = \frac{N_b \Omega_0}{\pi R_0^2 U_\infty^3} \int_0^{R_0} \Gamma\left(U_\infty - \tfrac{1}{2} u_z\right) r dr \quad \text{and} \quad C_T = \frac{N_b}{\pi R_0^2 U_\infty^2} \int_0^{R_0} \Gamma\left(\Omega_0 r + \tfrac{1}{2} u_\theta\right) dr, \quad (4)$$

which were made dimensionless with respect to the kinetic energy of wind in the rotor area and where N_b is the number of blades of the wind turbine. Because we neglected the wake expansion, the sheet radius coincides with the rotor radius R_0. In Eq. (4), the distribution of circulation Γ along the blade and the velocities u_z and u_θ induced by the vortex sheet are unknown. Thus, the first step of our solution is to determine a radial distribution of the circulation providing the equilibrium axial motion of the associated vortex system with some pitch $h = 2\pi l$. The wake displacement can be decomposed into normal $wU_\infty \cos\Phi$ and tangential $wU_\infty \sin\Phi$ components (Fig. 2b). The tangential component corresponds to a displacement of vortex elements along the sheet and does not change position of the sheet. Only the normal component should be taken into account for the sheet displacement. The axial

u_z and azimuthal u_θ components of the induced velocity after decomposition can be written through the induction factor

$$u_\theta = wU_\infty \cos\Phi \sin\Phi \quad \text{and} \quad u_z = wU_\infty \cos^2\Phi . \tag{5}$$

From simple geometrical reasons, these formulas can be rewritten:

$$u_\theta = wU_\infty \frac{x\bar{l}}{\bar{l}^2 + x^2} \quad \text{and} \quad u_z = wU_\infty \frac{x^2}{\bar{l}^2 + x^2} , \tag{6}$$

where $x = r/R_0$ and $\bar{l} = l/R_0$ are the dimensionless radius and pitch.

For each values of an arbitrarily pitch l, we will try to find the radial circulation distribution Γ providing the equilibrium motion of the associated vortex sheets with a constant relative velocity -wU_∞. We introduce a dimensionless circulation $G(x,l)$ by the formula

$$N_b\Gamma = 2\pi l\, wU_\infty G(x,l). \tag{7}$$

The $G(x,l)$ in Eq. (7) was called as "Goldstein's function" or "Goldstein's circulation" because he was the first who analytically determined it for two cases N_b = 2 and 4 [4]. Later the Goldstein function at arbitrary N_b and l was calculated in [10] by discretization of each vortex sheet by 100 uniformly distributed single helical filaments. A condition of the uniform motion with a relative velocity wU_∞ was satisfied in the middle of the two nearest filaments. The efficiency of the algorithm was confirmed in [8, 9] by good correlation with tabulated data which was calculated by direct simulation of the Goldstein function [11].

3. Assumptions of the different rotor theories

As it was mentioned above to obtain a final form of power and thrust coefficients it is necessary to set the value of pitch l. We consider the three variants of the determination in accordance with the table:

(i) the pitch is independent from velocities induced by the wake: $\dfrac{l_B}{r} = \tan\Phi_B = \dfrac{U_\infty}{\Omega_0 r}$

(ii) the pitch depends on the velocities induced in the far wake: $\dfrac{l_T}{r} = \tan\Phi_T = \dfrac{U_\infty - u_z}{\Omega_0 r + u_\theta}$

(iii) the pitch depends on the velocities induced on the rotor: $\dfrac{l_O}{r} = \tan\Phi_O = \dfrac{U_\infty - \frac{1}{2}u_z}{\Omega_0 r + \frac{1}{2}u_\theta}$

The simplest model (i) was used in the first calculations of the rotor [1, 3, 6]. It was considered as a good approximation for weakly loaded rotors but in fact that was applied for simplification only because in this solution the helical pitch l_B becomes independent from the induced velocities u_z and u_θ, which depend on the pitch themselves. The pitch of the second model (ii) introduced by Theodorsen [7] were defined via induction velocity in far wake. At his time the near wake was considered as a changeable structure which becomes frozen in far wake. The third model was applied in [8, 9]. With help of the appendix of [12] all formulas for the helical pitch can be rewritten in form

$$l_B = \frac{U_\infty}{\Omega_0} \text{ or } \frac{\Omega_0 l_B}{U_\infty} = 1; \quad l_T = \frac{U_\infty(1-w)}{\Omega_0} \text{ or } \frac{\Omega_0 l_T}{U_\infty} = 1 - w; \quad l_O = \frac{U_\infty\left(1-\frac{1}{2}w\right)}{\Omega_0} \text{ or } \frac{\Omega_0 l_O}{U_\infty} = 1 - \tfrac{1}{2}w . \tag{8}$$

For all cases after the substitution of Eqs (6), (7), and (8) in Eq. (4) and identical transformations the power and thrust coefficients can be defined:

(i) for the theories of Betz and Goldstein:

$$C_{P_B} = w\left(2I_1\left(\bar{l}_B\right) - wI_3\left(\bar{l}_B\right)\right) \quad \text{and} \quad C_{T_B} = w\left(2I_1\left(\bar{l}_B\right) + wI_2\left(\bar{l}_B\right)\right) \equiv w\left(I_1\left(\bar{l}_B\right)\left(2+w\right) - wI_3\left(\bar{l}_B\right)\right); \tag{9}$$

(ii) for the theory of Theodorsen:

$$C_{P_T} = w(1-w)\left(2I_1\left(\bar{l}_T\right) - wI_3\left(\bar{l}_T\right)\right) \quad \text{and} \quad C_{T_T} = w\left(2(1-w)I_1\left(\bar{l}_T\right) + wI_2\left(\bar{l}_T\right)\right) \equiv w\left(I_1\left(\bar{l}_T\right)\left(2+w\right) - wI_3\left(\bar{l}_T\right)\right); \tag{10}$$

(iii) for the rotor solution from [8, 9]:

$$C_{P_O} = 2w\left(1-\tfrac{1}{2}w\right)\left(I_1\left(\bar{l}_O\right) - \tfrac{1}{2}wI_3\left(\bar{l}_O\right)\right) \quad \text{and} \quad C_{T_O} = 2w\left(I_1\left(\bar{l}_O\right) - \tfrac{1}{2}wI_3\left(\bar{l}_O\right)\right); \tag{11}$$

where $I_1\left(\bar{l}\right) = \int_0^1 G\left(x,\bar{l}\right)x\,dx$; $I_2\left(\bar{l}\right) = \int_0^1 \frac{G\left(x,\bar{l}\right)\bar{l}^2 x}{\bar{l}^2 + x^2}dx$; $I_3\left(\bar{l}\right) = \int_0^1 \frac{G\left(x,\bar{l}\right)x^3}{\bar{l}^2 + x^2}dx$ and $I_1 - I_2 = I_3$. (12)

The relation $l_O\left(\Omega_0 r + \tfrac{1}{2}u_\theta\right) = \left(U_\infty - \tfrac{1}{2}u_z\right)r$ following from the pitch definition (iii) was used when the thrust coefficient C_{T_O} was obtained in Eq. (11). It is very important that power and thrust coefficients (9)–(11) for all models of rotors under a fixed value of the helical pitch are functions of one variable - the induction factor *w*, which is the same for all points of the sheets. It is convenient to be used for the rotor optimizations in all cases. Taking into consideration that the thrust coefficient is not principal for efficiency of wind turbines we find a value of the induction factor *w* coinciding to a maximum of the power coefficient. For each rotor model differentiating of C_P with respect to *w* yields the maximum value of $C_{P\max}$, resulting in:

$$w\left(\bar{l}_B\right) = \frac{I_1}{I_3}; \; w\left(\bar{l}_T\right) = \frac{2I_1 + I_3 - \sqrt{4I_1^2 - 2I_1I_3 + I_3^2}}{3I_3}; \; w\left(\bar{l}_O\right) = \frac{2}{3I_3}\left(I_1 + I_3 - \sqrt{I_1^2 - I_1I_3 + I_3^2}\right). \tag{13}$$

For the limiting case, the rotor with an infinite number of blades $N_b = \infty$, the value of the Goldstein circulation takes a simple form $G_\infty(x,l) = x^2/\left(x^2 + l^2\right)$. I_1 and I_3 from Eq. (12) can be presented in a simple analytical form [9]:

$$I_1^\infty = 1 - \bar{l}^2 \ln\frac{1+\bar{l}^2}{\bar{l}^2} \quad \text{и} \quad I_3^\infty = 1 + \frac{\bar{l}^2}{1+\bar{l}^2} - 2l^2 \ln\frac{1+\bar{l}^2}{\bar{l}^2}, \tag{14}$$

i.e., the analytical forms of the solutions for all rotor theories at $N_b = \infty$ are written by a combination of conventional functions. This limiting case is of great importance for a comparison of the rotor theories because it corresponds to the most possible value of the power coefficient for an ideal rotor.

4. Results

For a correct comparison it is necessary to replace an abstract pitch to a real parameter of the operating regimes Eq. (1) for wind turbines which for each theory according to Eq. (8) transform to the form:

$$\lambda_0 = \frac{1}{\bar{l}_B}; \qquad \lambda_0 = \frac{1 - w\left(\bar{l}_T\right)}{\bar{l}_T}; \quad \text{и} \quad \lambda_0 = \frac{1 - \tfrac{1}{2}w\left(\bar{l}_O\right)}{\bar{l}_O} \tag{15}$$

where *w* defines by Eq. (13). The results of the optimization for all cases are shown in Fig. 3.

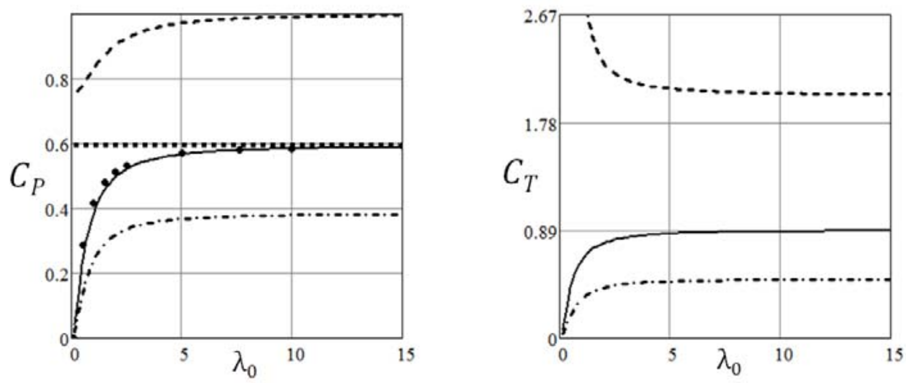

Figure 3. The largest C_P and the corresponding C_T for the different rotor models: Goldstein theory (dashed line); Theodorsen theory (dash-dotted line); correct calculation by [8, 9] (solid line); Betz–Joukowski limit (dotted line); and Glauert calculation (symbols).

There is an exact restriction for the power coefficient by a value of the Betz–Joukowsky limit [4] which concluded from actuator disk theory and should not be exceeded when the rotor operates as wind turbine. An additional well- examined result was found in Glauert's calculation [5] by the blade element momentum method. Our comparison of both classical results with the power coefficients obtained for the (i) and (ii) rotor theories show their inadmissibility (Fig. 3). The (i) model [6] gives an absurd prediction for a possibility of 100% utilization of wind energy, and the (ii) model [7] shows an underestimate of the limiting value. Only the (iii) model is well correlated with both Betz–Joukowsky limit and Glauert's calculation [5]. That means the (iii) rotor theory [8, 9] is correct and the choice of (iii) wake model completely closed the problem of the optimal rotor.

We will try to see why the former erroneous models of (i) and (ii) [6, 7] exist so long time. It may be related to their initial creation to describe the propeller modes only, where the ratio C_T/C_P should be analysed instead of C_P with the limit value for wind turbine case. If we rule out an anomalous behaviour of the thrust coefficient C_T for the (i) model at small λ_0 (Fig. 3), the ratio C_T/C_P takes about the same value for the three wake models. It means for the propeller operating mode the choice of the pitch is not important as for wind turbines. This fact for a long time prevented establishing the mistake in the Goldstein and Theodorsen theories of rotors.

5. Conclusions

Thus, in this study, we obtained for the first time analytical solutions for the optimization of wind turbines with the rotor models of Goldstein and Theodorsen to compare both ones with the optimization for the rotor model from [8, 9] in the case of an infinite number of blades. Our analysis of these solutions on operating modes of wind turbine enabled to establish the correct rotor theory in which the vortex pitch depends on the velocities induced by the wake in the rotor plane [8, 9]. Our running solutions for the outdated rotor models by Goldstein [6] and Theodorsen [7] indicate incorrect values for the limit power extracting from wind. This conclusion completely agrees with our conclusions made on the basis of preliminary computations for rotor with a finite number of blades [13].

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