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Homoclinic orbits for damped vibration systems with asymptotically quadratic or subquadratic potentials

Huiwen Chen^{1*}, Zhimin He², Zigen Ouyang¹ and Maoxin Liao¹

*Correspondence:

huiwchen@163.com

¹School of Mathematics and Physics, University of South China, Hengyang, Hunan 421001, China
Full list of author information is available at the end of the article**Abstract**

In this paper, we study a class of damped vibration systems,

$$\ddot{u}(t) + B\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad \forall t \in \mathbb{R},$$

where $W(t, u)$ is of indefinite sign. By using a critical point theorem of Ding, we establish a new criterion to guarantee that the above system has infinitely many nontrivial homoclinic orbits under the assumption that $W(t, u)$ is asymptotically quadratic or subquadratic as $|u| \rightarrow \infty$. Recent results in the literature are generalized and significantly improved.**MSC:** 34C37; 35A15; 37J45**Keywords:** damped vibration systems; homoclinic orbits; variational methods; asymptotically quadratic theorem; subquadratic

1 Introduction

In this paper, we consider the following damped vibration system:

$$\ddot{u}(t) + B\dot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad \forall t \in \mathbb{R}, \quad (1.1)$$

where $u \in \mathbb{R}^N$, B is an antisymmetric $N \times N$ constant matrix, $L \in C(\mathbb{R}, \mathbb{R}^{N \times N})$ is a symmetric matrix-valued function, and $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$. As usual, we say that a solution u of system (1.1) is homoclinic to zero if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, $u(t) \rightarrow 0$, and $\dot{u}(t) \rightarrow 0$ as $|t| \rightarrow \infty$. In addition, if $u(t) \not\equiv 0$, then $u(t)$ is called a nontrivial homoclinic solution.

Homoclinic orbits have been found in various models of continuous dynamical systems and play an important role in the study of the behavior of dynamical systems; see [1]. Thus, the study of homoclinic orbits has become one of the most important directions in the research of dynamical systems.

When $B = 0$, system (1.1) reduces to the following second order Hamiltonian system:

$$\ddot{u}(t) - L(t)u(t) + \nabla W(t, u(t)) = 0, \quad \forall t \in \mathbb{R}. \quad (1.2)$$

As a special case of dynamical systems, Hamiltonian systems play an important role in practical problems concerning relativistic mechanics, gas dynamics, nuclear physics, fluid mechanics. With the aids of the variational methods, the existence and multiplicity of homoclinic orbits for (1.2) have been extensively investigated in many recent papers; see [2–24].

Compared with the case where $B = 0$, the case where $B \neq 0$ is more difficult. The study of homoclinic orbits for system (1.1) has attracted a lot of attention by many researchers; see [25–33]. This work is mainly based on variational methods. Some of the authors considered the superquadratic case [27, 29, 30, 32, 33]; the authors of [25, 28, 31] considered the subquadratic case; for the asymptotically quadratic case, except for [26], few researchers have investigated this case. More precisely, in [25], Chen studied system (1.1) under the assumption that $W(t, u)$ is subquadratic as $|u| \rightarrow \infty$. In detail, he obtained the following result.

Theorem 1.1 ([25]) *Assume that the following conditions hold.*

(L₁) *There exists a constant $\nu > 1$ such that*

$$\text{meas}\{t \in \mathbb{R} : |t|^{-\nu}L(t) < TI_N\}, \quad \forall T > 0,$$

where I_N denotes the $N \times N$ identity matrix.

(L₂) *There exists a constant $\beta \geq 0$ such that*

$$l(t) := \inf_{|u|=1} (L(t)u, u) \geq -\beta, \quad t \in \mathbb{R}.$$

(H₁) $W(t, u) \geq 0, \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N$, and there exist constants $\kappa \in (0, 2)$ and $R_0 > 0$ such that

$$(\nabla W(t, u), u) \leq \kappa W(t, u), \quad \forall t \in \mathbb{R} \text{ and } |u| \geq R_0$$

and

$$(\nabla W(t, u), u) \leq 2W(t, u), \quad \forall t \in \mathbb{R} \text{ and } |u| \leq R_0.$$

(H₂) *There exists $a > 0$ such that*

$$W(t, u) \leq a|u|, \quad \forall t \in \mathbb{R} \text{ and } |u| \leq R_0.$$

(H₃) $\liminf_{|u| \rightarrow \infty} \frac{W(t, u)}{|u|} \geq b$ uniformly in $t \in \mathbb{R}$, where $b > 0$ is a constant.

(H₄) $\lim_{|u| \rightarrow 0} \frac{W(t, u)}{|u|^2} = +\infty$ uniformly in $t \in \mathbb{R}$.

(H₅) $W(t, -u) = W(t, u), \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N$.

Then system (1.1) has infinitely many nontrivial homoclinic orbits.

In [26], $W(t, u)$ being asymptotically quadratic as $|u| \rightarrow \infty$, by using the variant fountain theorem, Chen obtained the following result.

Theorem 1.2 ([26]) *Assume that (L₁), (L₂), and (H₅) hold. Moreover, we assume that the following conditions are satisfied:*

(H₆) $\tilde{W}(t, u) = W(t, u) - (1/2)(\nabla W(t, u), u) \rightarrow +\infty$ as $|u| \rightarrow \infty$ uniformly in $t \in \mathbb{R}$.

(H₇) There are constants $\tau \in (1, 2)$ and $a_1, a_2, a_3 > 0$ such that

$$a_3|u|^\tau \leq W(t, u) \leq a_1|u|, \quad \forall t \in \mathbb{R} \text{ and } |u| \leq a_2.$$

(H₈) $\limsup_{|u| \rightarrow 0} \frac{|\nabla W(t, u)|^{\tau/(\tau-1)}}{W(t, u)} = P(t)$ uniformly in t , $|P(t)| < \infty$.

(H₉) $W(t, u) \geq (1/2)(\nabla W(t, u), u) \geq 0, \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N$.

(H₁₀) $\lim_{|u| \rightarrow \infty} \frac{W(t, u)}{|u|^2} = f(t)$ uniformly in t , where $\inf_{t \in \mathbb{R}} f(t) \leq \sup_{t \in \mathbb{R}} f(t) < +\infty$.

Then system (1.1) has infinitely many nontrivial homoclinic orbits.

Motivated by the above facts, in this paper, our aim is to generalize some results in [25, 26]. Moreover, our approach is different from [25, 26].

We will use the following conditions:

(W₁) $\nabla W(t, u) = S(t)u + \nabla G(t, u)$, where $S : \mathbb{R} \rightarrow \mathbb{R}^{N \times N}$ is bounded symmetric $N \times N$ matrix-valued function.

(W₂) $W(t, 0) = G(t, 0) = 0$ and there exist $\bar{a}, \bar{b} > 0$ and $0 < \nu < 1$ such that

$$|\nabla G(t, u)| \leq \bar{a} + \bar{b}|u|^\nu, \quad \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N.$$

(W₃) $\tilde{W}(n, u) \geq 0, \forall (t, u) \in \mathbb{R} \times \mathbb{R}^N$.

Now, we state our main result.

Theorem 1.3 *Assume that (L₁), (L₂), (H₄)-(H₆), (W₁)-(W₃) hold. Then system (1.1) has infinitely many nontrivial homoclinic orbits.*

Remark 1.1 When $S \not\equiv 0$, Theorem 1.3 generalizes Theorem 1.2. First of all, we remove conditions (H₈) and $W(t, u) \geq 0$ for any $(t, u) \in \mathbb{R} \times \mathbb{R}^N$. Second, condition (H₄) is weaker than (H₇). In fact, condition (H₇) implies (H₄). Third, it is clear that condition (W₃) is weaker than (H₉). Furthermore, there are many functions satisfying our Theorem 1.3 and not satisfying Theorem 1.2. For example, let $W(t, u) = g(t)|u|^2 + |u|$ for all $(t, u) \in \mathbb{R} \times \mathbb{R}^N$, where $g : \mathbb{R} \rightarrow \mathbb{R}$ is bounded continuous function such that $g \not\equiv 0$. It is easy to check that $W(t, u)$ satisfies all conditions of Theorem 1.3, but it does not satisfy condition (H₈) of Theorem 1.2.

Remark 1.2 When $S \equiv 0$, Theorem 1.3 generalizes Theorem 1.1. First, conditions (H₁) and (H₃) imply conditions (H₆) and (W₃). Second, conditions (H₁) and (H₂) imply (W₂). Third, we remove conditions (H₃) and $W(t, u) \geq 0$ for any $(t, u) \in \mathbb{R} \times \mathbb{R}^N$. Furthermore, there are many functions $W(t, u)$ satisfying our Theorem 1.3 and not satisfying Theorem 1.1. For example, set

$$W(t, u) = \begin{cases} \frac{3}{2}|u| - |u|^{\frac{3}{2}}, & |u| < 1, \\ \frac{u^2}{2(|u| + \ln |u|)}, & |u| \geq 1 \end{cases}$$

for all $(t, u) \in \mathbb{R} \times \mathbb{R}^N$. It is easy to check that $W(t, u)$ satisfies all conditions of Theorem 1.3, but it does not satisfy condition (H₁) of Theorem 1.1. Set

$$W(t, u) = \begin{cases} \frac{1}{4}|u|, & |u| < 1, \\ \frac{1}{2}|u| - \frac{u^2}{2(|u| + \ln |u|)}, & |u| \geq 1 \end{cases}$$

for all $(t, u) \in \mathbb{R} \times \mathbb{R}^N$. It is easy to check that $W(t, u)$ satisfies all conditions of Theorem 1.3, but it does not satisfy condition (H_3) of Theorem 1.1.

The remainder of this paper is organized as follows. In Section 2, some preliminary results are presented. In Section 3, we give the proof of Theorem 1.3.

2 Preliminaries

In this section, we give the variational setting for (1.1) and some related preliminary lemmas. Let $X := H^1(\mathbb{R}, \mathbb{R}^N)$ be a Hilbert space with the inner product and the norm given, respectively, by

$$\langle u, v \rangle_X = \int_{\mathbb{R}} [(\dot{u}(t), \dot{v}(t)) + (u(t), v(t))] dt \quad \text{and} \quad \|u\|_X = \langle u, u \rangle_X^{\frac{1}{2}}, \quad \forall u, v \in X.$$

We define an operator $K : X \rightarrow X$ by

$$(Ku, v) = \int_{\mathbb{R}} (Bu(t), \dot{v}(t)) dt, \quad \forall u, v \in X.$$

Since B is an antisymmetric $N \times N$ constant matrix, K is self-adjoint on X . Moreover, we denote by J the self-adjoint extension of the operator $-\frac{d^2}{dt^2} + L(t) + K$ with the domain $D(J) \subset L^2(\mathbb{R}, \mathbb{R}^N)$.

Let $E := D(|J|^{\frac{1}{2}})$ be the domain of $|J|^{\frac{1}{2}}$, which is a Hilbert space equipped with the inner product and norm given by

$$\begin{aligned} \langle u, v \rangle_E &= (|J|^{\frac{1}{2}}u, |J|^{\frac{1}{2}}v)_2 + (u, v)_2, \\ \|u\|_E &= \langle u, u \rangle_E^{\frac{1}{2}} \end{aligned}$$

for $u, v \in E$, where $(\cdot, \cdot)_2$ denotes the inner product in $L^2(\mathbb{R}, \mathbb{R}^N)$. Let $\|\cdot\|_p$ denote the usual norm on $L^p(\mathbb{R}, \mathbb{R}^N)$ ($p \in [1, \infty]$).

Lemma 2.1 ([29]) *Assume that L satisfies (L_1) and (L_2) . Then E is compactly embedded in $L^p(\mathbb{R}, \mathbb{R}^N)$ for any $1 \leq p \leq \infty$.*

By Lemma 2.1, the spectrum $\sigma(J)$ consists of eigenvalues numbered by $\eta_1 \leq \eta_2 \leq \dots \leq \eta_k \leq \dots \rightarrow \infty$ (counted in their multiplicities) and a corresponding system of eigenfunctions $\{e_k\}$ ($Je_k = \eta_k e_k$) which forms an orthogonal basis in $L^2(\mathbb{R}, \mathbb{R}^N)$.

Set

$$n^- = \#\{i | \eta_i < 0\}, \quad n^0 = \#\{i | \eta_i = 0\}, \quad n^+ = n^- + n^0,$$

and

$$E^- = \text{span}\{e_1, \dots, e_{n^-}\}, \quad E^0 = \text{span}\{e_{n^-+1}, \dots, e_{n^0}\}, \quad E^+ = \text{cl}_E(\text{span}\{e_{n^0+1}, \dots\}).$$

Thus, we have the orthogonal decomposition

$$E = E^- \oplus E^0 \oplus E^+$$

with respect to the inner product $\langle \cdot, \cdot \rangle_E$. Now we introduce on E the following inner product:

$$\langle u, v \rangle = (|J|^{\frac{1}{2}}u, |J|^{\frac{1}{2}}v)_2 + (u^0, v^0)_2 \tag{2.1}$$

and the norm

$$\|u\| = \langle u, u \rangle^{\frac{1}{2}}, \tag{2.2}$$

where $u, v \in E$ with $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$. It is easy to verify that $\|\cdot\|_E$ and $\|\cdot\|$ are equivalent; see [8]. Evidently, the aforementioned decomposition is also orthogonal with respect to both inner products $(\cdot, \cdot)_2$ and $\langle \cdot, \cdot \rangle$.

Define the functional Ψ on E by

$$\begin{aligned} \Psi(u) &= \int_{\mathbb{R}} \left[\frac{1}{2} |\dot{u}(t)|^2 + \frac{1}{2} (Bu(t), \dot{u}(t)) + \frac{1}{2} (L(t)u(t), u(t)) - W(t, u(t)) \right] dt \\ &= \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - \int_{\mathbb{R}} W(t, u(t)) dt, \quad \forall u \in E. \end{aligned} \tag{2.3}$$

It follows from the assumptions that Ψ is defined on E and belongs to $C^1(E, \mathbb{R})$, and one can easily check that

$$\Psi'(u)v = \langle u^+, v^+ \rangle - \langle u^-, v^- \rangle - \int_{\mathbb{R}} (\nabla W(t, u(t)), v(t)) dt \tag{2.4}$$

for any $u, v \in E$ with $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$. Furthermore, it is routine to verify that any critical point of Ψ in E is a solution of system (1.1) with $u(\pm\infty) = 0 = \dot{u}(\pm\infty)$ (see [20, 21]). In view of Lemma 2.1, there exists $D_p > 0$ such that

$$\|u\|_p \leq D_p \|u\|, \quad \forall u \in E, \tag{2.5}$$

where $p \in [1, +\infty]$.

Define $E_j = \mathbb{R}e_j$,

$$Y_k = \bigoplus_{j=0}^k E_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} E_j}, \quad k \in \mathbb{N}. \tag{2.6}$$

Lemma 2.2 *Under assumptions (L₁) and (L₂), for $\zeta \in [1, +\infty]$,*

$$\beta_k(\tau) = \sup_{u \in Z_k, \|u\|=1} \|u\|_{\zeta} \rightarrow 0 \quad \text{as } k \rightarrow \infty. \tag{2.7}$$

Proof It is clear that $0 < \beta_{k+1}(\zeta) \leq \beta_k(\zeta)$, so that $\beta_k(\zeta) \rightarrow \tilde{\beta}(\zeta)$, $k \rightarrow \infty$. For every $k \geq 0$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_{\zeta} > \frac{\tilde{\beta}_k}{2}$. For any $v \in X$, let $v = \sum_{i=1}^{\infty} \bar{b}_i e_i$, by the Cauchy-Schwartz inequality, one has

$$\begin{aligned} |\langle u_k, v \rangle| &= \left| \left\langle u_k, \sum_{i=1}^{\infty} \bar{b}_i e_i \right\rangle_0 \right| = \left| \left\langle u_k, \sum_{i=k}^{\infty} \bar{b}_i e_i \right\rangle \right| \\ &\leq \|u_k\| \left\| \sum_{i=k}^{\infty} \bar{b}_i e_i \right\| = \sum_{i=k}^{\infty} \|\bar{b}_i e_i\| \rightarrow 0 \quad \text{as } k \rightarrow \infty, \end{aligned}$$

which implies that $u_k \rightarrow 0$. It follows from Lemma 2.1 that $u_k \rightarrow 0$ in $L^q(\mathbb{R}, \mathbb{R}^N)$. Thus we have proved that $\tilde{\beta}(\zeta) = 0$. □

By Lemma 2.2, we can choose a positive integer $k_0 \geq n^+ + 1$ such that

$$\|u\|_2^2 \leq \frac{1}{4m_0} \|u\|^2, \tag{2.8}$$

where $m_0 = \sup_{t \in \mathbb{R}} [\sup_{x \in \mathbb{R}^N, |x|=1} (S(t)x, x)]$.

In order to prove our main result, we need the following lemma.

Lemma 2.3 ([8]) *Let E be an infinite dimensional Banach space and $\Phi \in C^1(E, \mathbb{R})$ be even, satisfy the (PS) condition, and $\Psi(0) = 0$. If $X = Y \oplus Z$, where Y is finite dimensional, and Ψ satisfies*

- (S₁) Ψ is bounded from below on Z ;
- (S₂) for each finite dimensional subspace $\tilde{E} \subset E$, there are positive constants $\rho = \rho(\tilde{E})$ and $\alpha = \alpha(\tilde{E})$ such that $\Psi|_{B_\rho \cap \tilde{E}} \leq 0$ and $\Psi|_{\partial B_\rho \cap \tilde{E}} \leq -\alpha$, where $B_\rho = \{x \in E : \|x\| \leq \rho\}$.

Then Ψ possesses infinitely many nontrivial critical points.

Remark 2.1 As shown in [34], a deformation lemma can be proved with condition (C) replacing (PS) condition, and it turns out that Lemma 2.3 holds true under condition (C). We say that Ψ satisfies condition (C), i.e. for any $\{u_n\} \subset E$, $\{u_n\}$ has a convergent subsequence if $\Psi(u_n)$ is bounded and $(1 + \|u_n\|)\|\Psi'(u_n)\| \rightarrow 0$ as $n \rightarrow \infty$.

3 Proof of Theorem 1.3

Set $Y = Y_{k_0}, Z = Z_{k_0}$.

Lemma 3.1 *Suppose that (W₁) and (W₂) are satisfied. Then Ψ is bounded from below on Z .*

Proof By virtue of (W₁), (W₂), (2.3), (2.5), and (2.8), we have

$$\begin{aligned} \Psi(u) &= \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 - \sum_{n \in \mathbb{Z}} W(n, u(n)) \\ &= \frac{1}{2} \|u\|^2 - \sum_{n \in \mathbb{Z}} W(n, u(n)) \\ &\geq \frac{1}{2} \|u\|^2 - m_0 \|u\|_2^2 - \bar{a} \|u\|_1 - \bar{b} \|u\|_{v+1}^{v+1} \\ &\geq \frac{1}{4} \|u\|^2 - \bar{a} D_1 \|u\| - \bar{b} D_{v+1}^{v+1} \|u\|^{v+1} \rightarrow +\infty \end{aligned} \tag{3.1}$$

as $\|u\| \rightarrow \infty$ and $u \in Z_{k_0}$. The proof is completed. □

Lemma 3.2 *Assume that (H₄) holds. Then for each finite dimensional subspace $\tilde{E} \subset E$, there are positive constants $\rho = \rho(\tilde{E})$ and $\alpha = \alpha(\tilde{E})$ such that $\Psi|_{B_\rho \cap \tilde{E}} \leq 0$ and $\Psi|_{\partial B_\rho \cap \tilde{E}} \leq -\alpha$, where $B_\rho = \{x \in E : \|x\| \leq \rho\}$.*

Proof Let $\tilde{E} \subset E$ be any finite dimensional subspace. Then there exists $M_0 > 0$ such that

$$\|u\|^2 \leq M_0 \|u\|_2^2, \quad \forall u \in \tilde{E}. \tag{3.2}$$

By virtue of (H₄), for M_0 given above, there exists a constant $\sigma > 0$,

$$W(t, u) \geq M_0 |u|^2, \quad \forall t \in \mathbb{R} \text{ and } |u| \leq \sigma. \tag{3.3}$$

In view of (2.8), for any $u \in \tilde{E}$ with $\|u\| \leq \frac{\sigma}{D_\infty}$, we have

$$\|u\|_\infty \leq \sigma. \tag{3.4}$$

By (3.2)-(3.4), we have

$$\begin{aligned} \Psi(u) &= \frac{1}{2} \|u^+\|^2 - \frac{1}{2} \|u^-\|^2 + \int_{\mathbb{R}} W(t, u(t)) dt \\ &\leq \frac{1}{2} \|u^+\|^2 - \int_{\mathbb{R}} W(t, u(t)) dt \\ &\leq \frac{1}{2} \|u^+\|^2 - M_0 \|u\|_2^2 \\ &\leq -\frac{1}{2} \|u\|^2 \end{aligned} \tag{3.5}$$

for any $u = u^- + u^0 + u^+ \in \tilde{E}$ with $\|u\| \leq \frac{\sigma}{D_\infty}$. Then there exist $\rho = \rho(\tilde{E}) > 0$ and $\alpha = \alpha(\tilde{E}) > 0$ such that

$$\Psi(u) \leq 0, \quad \forall u \in B_\rho \cap \tilde{E}; \quad \Psi(u) \leq -\alpha, \quad \forall u \in \partial B_\rho \cap \tilde{E}.$$

The proof is completed. □

Lemma 3.3 *Under the assumptions of Theorem 1.3, Ψ satisfies condition (C).*

Proof Let $\{u_n\} \subset E$ is a (C) sequence of Ψ , that is, $\{\Psi(u_n)\}$ is bounded and

$$(1 + \|u_n\|) \|\Psi'(u_n)\| \rightarrow 0 \quad \text{as } n \rightarrow \infty, \tag{3.6}$$

then there exists a constant $M_1 > 0$ such that

$$|\Phi(u_m)| \leq M_1, \quad (1 + \|u_n\|) \|\Psi'(u_n)\| \leq M_1 \tag{3.7}$$

for every $n \in \mathbb{N}$. We choose $k \geq n^+ + 1$ large enough such that

$$\|u\|^2 \geq 2m_0 \|u\|_2^2, \quad \forall u \in Z_k, \tag{3.8}$$

where $m_0 = \sup_{t \in \mathbb{R}} [\sup_{x \in \mathbb{R}^N, |x|=1} (S(t)x, x)]$. We now prove that $\{u_n\}$ is bounded in E . In fact, if not, we may assume by contradiction that $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$. Let $u_n = \tilde{w}_n + \tilde{v}_n$, $z_n = \frac{u_n}{\|u_n\|}$, then $\|z_n\| = 1$, $z_n = w_n + v_n \in E$, where $w_n = \frac{\tilde{w}_n}{\|u_n\|}$, $v_n = \frac{\tilde{v}_n}{\|u_n\|}$, $\tilde{w}_n \in Y_k$, $\tilde{v}_n \in Z_k$. After passing to a subsequence, we have $z_n \rightharpoonup z$, $w_n \rightarrow w$, and $\gamma = \lim_{n \rightarrow \infty} \|v_n\|$ exists.

Case 1. $\gamma = 0$. Since $\dim Y_k < \infty$, we obtain $\|w_n\| \rightarrow \|w\| = 1$. It follows from (3.7) that

$$\frac{3}{2}M_1 \geq \Psi(u_n) - \frac{1}{2}\Psi'(u_n)u_n \geq \int_{\mathbb{R}} \left(W(t, u_n(t)) - \frac{1}{2}(\nabla W(t, u_n(t)), u_n(t)) \right) dt. \tag{3.9}$$

By virtue of (H₆), for any $\eta > 0$, there exists $M_2 > 0$ such that

$$\tilde{W}(t, u) = W(t, u) - \frac{1}{2}(\nabla W(t, u), u) \geq \eta, \quad \forall t \in \mathbb{R}, |u| \geq M_2. \tag{3.10}$$

For any $\varepsilon > 0$, define $\Lambda_\varepsilon := \{t \in \mathbb{R} : |w(t)| \geq \varepsilon\}$ and $\Lambda_{n\varepsilon} := \{t \in \mathbb{R} : |v_n(t)| \geq \frac{\varepsilon}{2}\}$. First, we claim that there exists $\varepsilon_0 > 0$ such that

$$\text{meas}\{t \in \mathbb{R} : |u(t)| \geq \varepsilon_0\} \geq \varepsilon_0, \quad \forall u \in Y_k \text{ with } \|u\| = 1.$$

Otherwise, for any positive integer m , there exists $w_m \in Y_k$ with $\|w_m\| = 1$ such that

$$\text{meas}\left\{t \in \mathbb{R} : |w_m(t)| \geq \frac{1}{m}\right\} < \frac{1}{m}. \tag{3.11}$$

Passing to a subsequence if necessary, we may assume $w_m \rightarrow w_0$ in E for some $w_0 \in Y_k$ since $\dim Y_k < \infty$. Evidently, $\|w_0\| = 1$. By the equivalence of the norms on the finite dimensional space Y_k , we have

$$\int_{\mathbb{R}} |w_m(t) - w_0(t)|^2 dt \rightarrow 0 \quad \text{as } m \rightarrow \infty. \tag{3.12}$$

Thus there exists $\varepsilon_1 > 0$ such that

$$\text{meas}\{t \in \mathbb{R} : |w_0(t)| \geq \varepsilon_1\} \geq \varepsilon_1. \tag{3.13}$$

In fact, if not, then, for all positive integers m , we have

$$\text{meas}\left\{t \in \mathbb{R} : |w_0(t)| \geq \frac{1}{m}\right\} = 0. \tag{3.14}$$

It implies that

$$0 \leq \int_{\mathbb{R}} |w_0(t)|^4 dt < \frac{1}{m^2} \|w_0\|_2^2 \rightarrow 0 \quad \text{as } m \rightarrow \infty.$$

Hence $w_0 = 0$, which contradicts that $\|w_0\|_0 = 1$. Thus, (3.13) holds.

Now set

$$\Upsilon_0 = \{t \in \mathbb{R} : |w_0(t)| \geq \varepsilon_1\}, \quad \Upsilon_m = \left\{t \in \mathbb{R} : |w_m(t)| < \frac{1}{m}\right\},$$

and $\Upsilon_m^c = \mathbb{R} \setminus \Upsilon_m = \{t \in \mathbb{R} : |w_m(t)| \geq \frac{1}{m}\}$. By virtue of (3.11) and (3.13), we obtain

$$\begin{aligned} \text{meas}(\Upsilon_m \cap \Upsilon_0) &= \text{meas}(\Upsilon_0 \setminus \Upsilon_m^c \cap \Upsilon_0) \\ &\geq \text{meas}(\Upsilon_0) - \text{meas}(\Upsilon_m^c \cap \Upsilon_0) \\ &\geq \varepsilon_1 - \frac{1}{m} \end{aligned}$$

for all positive integers m . Let m be large enough such that $\frac{1}{2}\varepsilon_1 - \frac{1}{m} > \frac{1}{4}\varepsilon_1$. Thus, for m large enough,

$$\begin{aligned} \int_{\mathbb{R}} |w_m(t) - w_0(t)|^2 dt &\geq \int_{t \in \Upsilon_0 \cap \Upsilon_m} |w_m(t) - w_0(t)|^2 dt \\ &\geq \frac{1}{2} \int_{t \in \Upsilon_0 \cap \Upsilon_m} |w_0(t)|^2 dt - \int_{t \in \Upsilon_0 \cap \Upsilon_m} |w_m(t)|^2 dt \\ &\geq \left(\frac{1}{2}\varepsilon_1 - \frac{1}{m}\right) \text{meas}(\Upsilon_m \cap \Upsilon_0) \\ &\geq \frac{\varepsilon_1^2}{16} > 0, \end{aligned}$$

which is a contradiction to (3.12). Thus, there exists $\varepsilon > 0$ such that $\text{meas}(\Lambda_\varepsilon) \geq \varepsilon$.

In view of (2.5), we obtain

$$\text{meas}(\Lambda_{n\varepsilon}) \leq \frac{4}{\varepsilon^2} \int_{\mathbb{R}} |v_n(t)|^2 dt \leq \frac{4D_2^2}{\varepsilon^2} \|v_n\|^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then we have $\text{meas}(\Lambda_\varepsilon \setminus \Lambda_{n\varepsilon}) \rightarrow \text{meas}(\Lambda_\varepsilon)$ as $n \rightarrow \infty$. Therefore, there exists $N_0 > 0$ such that $|z_n(t)| \geq \frac{\varepsilon}{3}, \forall t \in \Lambda_\varepsilon \setminus \Lambda_{n\varepsilon}$ and $n \geq N_0$, then we have $|u_n(t)| \geq \frac{\varepsilon}{3} \|u_n\|, \forall t \in \Lambda_\varepsilon \setminus \Lambda_{n\varepsilon}$ and $n \geq N_0$. By (W₃), (3.9), and (3.10), there exists $N_1 > 0$ such that

$$\frac{3}{2}M_1 \geq \int_{\mathbb{R}} \tilde{W}(t, u_n(t)) dt \geq \int_{t \in \Lambda_\varepsilon \setminus \Lambda_{n\varepsilon}} \eta dt \geq \eta \text{meas}(\Lambda_\varepsilon \setminus \Lambda_{n\varepsilon}), \quad \forall n \geq N_1,$$

which gives a contradiction due to the arbitrariness of η .

Case 2. $\gamma > 0$. In view of (W₁), (W₂), (2.5), (3.7), (3.8), and Hölder’s inequality, we get

$$\begin{aligned} M_1 &\geq \Psi'(u_n)\tilde{v}_n = \|\tilde{v}_n\|^2 - \int_{\mathbb{R}} (\nabla W(t, u_n(t)), \tilde{v}_n(t)) dt \\ &\geq \|\tilde{v}_n\|^2 - \int_{\mathbb{R}} [(S(t)u_n(t), \tilde{v}_n(t)) + (\bar{a} + \bar{b}|u_n(t)|^v)|\tilde{v}_n(t)|] dt \\ &\geq \|\tilde{v}_n\|^2 - m_0\|\tilde{v}_n\|_2^2 - \bar{a}\|\tilde{v}_n\|_1 - \bar{b}\|u_n\|_{2v}^v\|\tilde{v}_n\|_2 \\ &\geq \frac{1}{2}\|\tilde{v}_n\|^2 - \bar{a}D_1\|\tilde{v}_n\| - \bar{b}D_2D_{2v}^v\|u_n\|^v\|\tilde{v}_n\| \\ &\geq \frac{1}{2}\|\tilde{v}_n\|^2 - \bar{a}D_1\|u_n\| - \bar{b}D_2D_{2v}^v\|u_n\|^{v+1}. \end{aligned} \tag{3.15}$$

Dividing by $\|u_n\|^2$ on both sides of (3.15), we obtain

$$0 \geq \frac{\gamma^2}{2} > 0,$$

which gives a contradiction.

Thus, $\{u_n\}$ is bounded. Next, we show that $\{u_n\}$ has a convergent subsequence. In view of the boundedness of $\{u_n\}$, without loss of generality, we may assume that

$$u_n \rightharpoonup u, \quad u_n^+ \rightharpoonup u^+, \quad u_n^- \rightharpoonup u^-, \quad u_n^0 \rightarrow u^0. \tag{3.16}$$

It follows from (2.4) that

$$\begin{aligned} \|u_n^+ - u^+\|^2 &= (\Psi'(u_n) - \Psi'(u))(u_n^+ - u^+) \\ &\quad + \int_{\mathbb{R}} (\nabla W(t, u_n) - \nabla W(t, u), u_n^+ - u^+) dt. \end{aligned} \tag{3.17}$$

It is clear that

$$(\Psi'(u_n) - \Psi'(u))(u_n^+ - u^+) \rightarrow 0 \quad \text{as } n \rightarrow \infty. \tag{3.18}$$

By (2.5), Lemma 2.1, (W_1) , (W_2) , and Hölder’s inequality, we have

$$\begin{aligned} &\int_{\mathbb{R}} (\nabla W(t, u_n) - \nabla W(t, u), u_n^+ - u^+) dt \\ &\leq \bar{a} \|u_n^+ - u^+\|_1 + (m_0 \|u_n\|_2 + \bar{b} \|u_n\|_{2\nu}^\nu) \|u_n^+ - u^+\|_2 \\ &\quad + \bar{a} \|u_n^+ - u^+\|_1 + (m_0 \|u\|_2 + \bar{b} \|u\|_{2\nu}^\nu) \|u_n^+ - u^+\|_2 \\ &\leq 2\bar{a} \|u_n^+ - u^+\|_1 + (m_0 D_2 \|u_n\| + \bar{b} D_{2\nu}^\nu \|u_n\|^\nu) \|u_n^+ - u^+\|_2 \\ &\quad + (m_0 D_2 \|u\| + \bar{b} D_{2\nu}^\nu \|u\|^\nu) \|u_n^+ - u^+\|_2 \rightarrow 0 \end{aligned} \tag{3.19}$$

as $n \rightarrow \infty$. Therefore, by (3.17)-(3.19), we get $\|u_n^+ - u^+\| \rightarrow 0$ as $n \rightarrow \infty$. Consequently, $u_n \rightarrow u$. The proof is completed. □

Proof of Theorem 1.3 Obviously, $\Psi \in C^1(X, \mathbb{R})$ is even and $\Phi(0) = 0$. It follows from Lemmas 3.1-3.3 that all conditions of Lemma 2.3 are satisfied. By Lemma 2.3, we see that Ψ possesses infinitely many nontrivial critical points, that is, system (1.1) possesses infinitely many nontrivial homoclinic orbits. □

Competing interests

The authors declare that they have no competing interests.

Authors’ contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Author details

¹School of Mathematics and Physics, University of South China, Hengyang, Hunan 421001, China. ²School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, China.

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