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On Jensen's inequality, Hölder's inequality, and Minkowski's inequality for dynamically consistent nonlinear evaluations

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Abstract

In this paper, the dynamically consistent nonlinear evaluations that were introduced by Peng are considered in probability space $L^2(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. We investigate the n-dimensional ($n \geq 1$) Jensen inequality, Hölder inequality, and Minkowski inequality for dynamically consistent nonlinear evaluations in $L^1(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. Furthermore, we give four equivalent conditions on the n-dimensional Jensen inequality for g-evaluations induced by backward stochastic differential equations with non-uniform Lipschitz coefficients in $L^p(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ $(1 < p \leq 2)$. Finally, we give a sufficient condition on g that satisfies the non-uniform Lipschitz condition under which Hölder's inequality and Minkowski's inequality for the corresponding g-evaluation hold true. These results include and extend some existing results.

Keywords: dynamically consistent nonlinear evaluation; g-evaluation; g-expectation; Jensen's inequality; Hölder's inequality; Minkowski's inequality

1 Introduction

It is well known that (see Peng [1, 2]) a dynamically consistent nonlinear evaluation in probability space $L^2(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a given filtration, is a system of operators:

$$\mathcal{E}_{s,t}[X] : X \in L^2(\Omega, \mathcal{F}_t, P) \mapsto L^2(\Omega, \mathcal{F}_s, P), \quad 0 \leq s \leq t < \infty,$$

which satisfies the following properties:

(i) $\mathcal{E}_{s,t}[X_1] \geq \mathcal{E}_{s,t}[X_2]$, if $X_1 \geq X_2$;
(ii) $\mathcal{E}_{t,t}[X] = X$;
(iii) $\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[X]] = \mathcal{E}_{r,t}[X]$, if $0 \leq r \leq s \leq t < \infty$;
(iv) $1_A \mathcal{E}_{s,t}[X] = 1_A \mathcal{E}_{s,t}[1_A X]$, $\forall A \in \mathcal{F}_s$.

Of course, we can define this notion in $L^1(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$.

In a financial market, the evaluation of the discounted value of a derivative is often treated as a dynamically consistent nonlinear evaluation (expectation). The well-known g-evaluation (g-expectation) induced by backward stochastic differential equations (BSDEs for short), which was put forward by Peng, is a special case of a dynamically consistent nonlinear evaluation (expectation). While nonlinear BSDEs were firstly introduced by

Pardoux and Peng [3], who proved the existence and uniqueness of adapted solutions, when the coefficient g is Lipschitz in (y,z) uniformly in (t,ω), with square-integrability assumptions on the coefficient $g(t,\omega,y,z)$ and terminal condition ξ. Later many researchers developed the theory of BSDEs and their applications in a series of papers (for example see Hu and Peng [4], Lepeltier and San Martin [5], El Karoui *et al.* [6], Pardoux [7, 8], Briand *et al.* [9] and the references therein) under some other assumptions on the coefficients but for a fixed terminal time $T > 0$. In 2000, Chen and Wang [10] obtained the existence and uniqueness theorem for L^2 solutions of infinite time interval BSDEs when $T = \infty$, by the martingale representation theorem and fixed point theorem. Recently, Zong [11] have obtained the result on L^p $(1 < p < 2)$ solutions of infinite time interval BSDEs. One of the special cases is the existence and uniqueness theorem of BSDEs with non-uniformly Lipschitz coefficients.

The original motivation for studying nonlinear evaluation (expectation) and g-evaluation (g-expectation) comes from the theory of expected utility, which is the foundation of modern mathematical economics. Chen and Epstein [12] gave an application of dynamically consistent nonlinear evaluation (expectation) to recursive utility, Peng [1, 2, 13–15] and Rosazza Gianin [16] investigated some applications of dynamically consistent nonlinear evaluations (expectations) and g-evaluations (g-expectations) to static and dynamic pricing mechanisms and risk measures.

Since the notions of nonlinear evaluation (expectation) and g-evaluation (g-expectation) were introduced, many properties of the nonlinear evaluation (expectation) and g-evaluation (g-expectation) have been studied in [1, 2, 6, 10–31]. In [1, 2], Peng obtained an important result: he proved that if a dynamically consistent nonlinear evaluation $\mathcal{E}_{s,t}[\cdot]$ can be dominated by a kind of g-evaluation, then $\mathcal{E}_{s,t}[\cdot]$ must be a g-evaluation. Thus, in this case, many problems on dynamically consistent nonlinear evaluations $\mathcal{E}_{s,t}[\cdot]$ can be solved through the theory of BSDEs.

It is well known that Jensen's inequality for classic mathematical expectations holds in general, which is a very important property and has many important applications. But for nonlinear expectation, even for its special case: g-expectation, by Briand *et al.* [17], we know that Jensen's inequality for g-expectations usually does not hold in general. So under the assumption that g is continuous with respect to t, some papers, such as [18, 19, 25, 27, 28] have been devoted to Jensen's inequality for g-expectations, with the help of the theory of BSDEs, they have obtained the necessary and sufficient conditions under which Jensen's inequality for g-expectations holds in general. Under the assumptions that g does not depend on y and is convex, Chen *et al.* [18, 19] studied Jensen's inequality for g-expectations and gave a necessary and sufficient condition on g under which Jensen's inequality holds for convex functions. Provided g only does not depend on y, Jiang and Chen [28] gave another necessary and sufficient condition on g under which Jensen's inequality holds for convex functions. It was an improved result in comparison with the result that Chen *et al.* found. Later, this result was improved by Hu [25] and Jiang [27], in fact, Jiang [27] showed that g must be independent of y. In addition, Fan [22] studied Jensen's inequality for filtration-consistent nonlinear expectations without domination condition. Jia [26] studied the n-dimensional $(n > 1)$ Jensen's inequality for g-expectations and got the result that the n-dimensional $(n > 1)$ Jensen's inequality holds for g-expectations if and only if g is independent of y and linear with respect to z, in other words, the corresponding g-expectation must be linear. Then the natural question is asked:

For more general dynamically consistent nonlinear evaluation $\mathcal{E}_{s,t}[\cdot]$, what are the sufficient and necessary conditions under which Jensen's inequality for $\mathcal{E}_{s,t}[\cdot]$ holds in general? Roughly speaking, what conditions on $\mathcal{E}_{s,t}[\cdot]$ are equivalent with the inequality

$$\mathcal{E}_{s,t}\bigl[\varphi(\xi)\bigr] \geq \varphi\bigl(\mathcal{E}_{s,t}[\xi]\bigr) \quad \text{a.s.}$$

holding for any convex function $\varphi : \mathcal{R} \mapsto \mathcal{R}$?

One of the objectives of this paper is to investigate this problem. At the same time, this paper will also investigate the sufficient and necessary conditions on $\mathcal{E}_{s,t}[\cdot]$ under which the n-dimensional ($n > 1$) Jensen inequality holds. As applications of these two results, we give four equivalent conditions on the 1-dimensional Jensen inequality and the n-dimensional ($n > 1$) Jensen inequality for g-evaluations induced by BSDEs with non-uniform Lipschitz coefficients in $L^p(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ ($1 < p \leq 2$), respectively.

The remainder of this paper is organized as follows: In Section 2, we study the n-dimensional ($n \geq 1$) Jensen inequality, Hölder inequality, and Minkowski inequality for dynamically consistent nonlinear evaluations in $L^1(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. In Section 3, we give four equivalent conditions on the 1-dimensional Jensen inequality and the n-dimensional ($n > 1$) Jensen inequality for g-evaluations induced by BSDEs with non-uniform Lipschitz coefficients in $L^p(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ ($1 < p \leq 2$), respectively. These results generalize the known results on Jensen's inequality for g-expectation in [18, 19, 22, 25–28, 31]. In Section 4, we give a sufficient condition on g that satisfies the non-uniform Lipschitz condition under which Hölder's inequality and Minkowski's inequality for the corresponding g-evaluation hold true.

2 Jensen's inequality, Hölder's inequality, and Minkowski's inequality for dynamically consistent nonlinear evaluations

Let $(\Omega, \mathcal{F}, P)$ be a probability space carrying a standard d-dimensional Brownian motion $(B_t)_{t\geq 0}$, and let $(\mathcal{F}_t)_{t\geq 0}$ be the σ-algebra generated by $(B_t)_{t\geq 0}$. We always assume that $(\mathcal{F}_t)_{t\geq 0}$ is complete. Let $T > 0$ be a given real number. In this paper, we always work in the probability space $(\Omega, \mathcal{F}_T, P)$, and only consider processes indexed by $t \in [0, T]$. We denote $L^p(\Omega, \mathcal{F}_t, P)$ ($p \geq 1$), the space of $\mathcal{F}_t$-measurable random variables satisfying $E_P[|X|^p] < \infty$, and by $L^p_+(\Omega, \mathcal{F}_t, P)$ the space of non-negative random variables in $L^p(\Omega, \mathcal{F}_t, P)$. Let 1_A denote the indicator of event A. For notational simplicity, we use $L^p(\mathcal{F}_t) := L^p(\Omega, \mathcal{F}_t, P)$ and $L^p_+(\mathcal{F}_t) := L^p_+(\Omega, \mathcal{F}_t, P)$. For the convenience of the reader, we recall the notion of a dynamically consistent nonlinear evaluation, defined in $L^2(\mathcal{F}_T)$ in Peng [1, 2], but defined in $L^1(\mathcal{F}_T)$ in this section.

Definition 2.1 An $\mathcal{F}_t$-consistent nonlinear evaluation in $L^1(\mathcal{F}_T)$ is a system of operators:

$$\mathcal{E}_{s,t}[X] : X \in L^1(\mathcal{F}_t) \mapsto L^1(\mathcal{F}_s), \quad 0 \leq s \leq t \leq T,$$

which satisfies the following properties:

(A.1) monotonicity: $\mathcal{E}_{s,t}[X_1] \geq \mathcal{E}_{s,t}[X_2]$, if $X_1 \geq X_2$;
(A.2) $\mathcal{E}_{t,t}[X] = X$;
(A.3) dynamical consistency: $\mathcal{E}_{r,s}[\mathcal{E}_{s,t}[X]] = \mathcal{E}_{r,t}[X]$, if $0 \leq r \leq s \leq t \leq T$;
(A.4) zero one law: $1_A\mathcal{E}_{s,t}[X] = 1_A\mathcal{E}_{s,t}[1_A X]$, $\forall A \in \mathcal{F}_s$.

First, we consider Jensen's inequality for $\mathcal{F}_t$-consistent nonlinear evaluations. We have the following results.

Theorem 2.1 *Suppose that $\mathcal{E}_{s,t}[\cdot]$, $0 \le s \le t \le T$ is an $\mathcal{F}_t$-consistent nonlinear evaluation in $L^1(\mathcal{F}_T)$, then the following two statements are equivalent*:

(i) *Jensen's inequality for $\mathcal{F}_t$-consistent evaluation $\mathcal{E}_{s,t}[\cdot]$ holds in general, i.e., for each convex function $\varphi : \mathcal{R} \mapsto \mathcal{R}$ and $\xi \in L^1(\mathcal{F}_t)$, if $\varphi(\xi) \in L^1(\mathcal{F}_t)$, then we have*

$$\mathcal{E}_{s,t}\big[\varphi(\xi)\big] \ge \varphi\big(\mathcal{E}_{s,t}[\xi]\big) \quad a.s.;$$

(ii) $\forall (\xi, a, b) \in L^1(\mathcal{F}_t) \times \mathcal{R} \times \mathcal{R}$, $\mathcal{E}_{s,t}[a\xi + b] \ge a\mathcal{E}_{s,t}[\xi] + b$ *a.s.*

Proof First, we prove (i) implies (ii). Suppose (i) holds, for each $(\xi, a, b) \in L^1(\mathcal{F}_t) \times \mathcal{R} \times \mathcal{R}$, let $\varphi(x) := ax + b$. Obviously, $\varphi(x)$ is a convex function and $\varphi(\xi) \in L^1(\mathcal{F}_t)$, then we have

$$\mathcal{E}_{s,t}[a\xi + b] = \mathcal{E}_{s,t}\big[\varphi(\xi)\big] \ge \varphi\big(\mathcal{E}_{s,t}[\xi]\big) = a\mathcal{E}_{s,t}[\xi] + b \quad \text{a.s.}$$

In the following, we prove (ii) implies (i). Suppose (ii) holds, for each $(\xi, a, b) \in L^1(\mathcal{F}_t) \times \mathcal{R} \times \mathcal{R}$, we have

$$\mathcal{E}_{s,t}[a\xi + b] \ge a\mathcal{E}_{s,t}[\xi] + b \quad \text{a.s.} \tag{2.1}$$

But, for any convex function $\varphi : \mathcal{R} \mapsto \mathcal{R}$, there exists a countable set $\mathcal{D} \subseteq \mathcal{R}^2$ such that

$$\varphi(x) = \sup_{(a,b)\in\mathcal{D}} (ax + b). \tag{2.2}$$

In view of (2.1), for any $(a, b) \in \mathcal{D}$, we have

$$\mathcal{E}_{s,t}\big[\varphi(\xi)\big] \ge \mathcal{E}_{s,t}[a\xi + b] \ge a\mathcal{E}_{s,t}[\xi] + b \quad \text{a.s.,}$$

which implies (i) by taking into consideration of (2.2). □

Theorem 2.2 *Suppose that $\mathcal{E}_{s,t}[\cdot]$, $0 \le s \le t \le T$ is an $\mathcal{F}_t$-consistent nonlinear evaluation in $L^1(\mathcal{F}_T)$ and $n > 1$, then the following two statements are equivalent*:

(i) *the n-dimensional Jensen inequality for a $\mathcal{F}_t$-consistent evaluation $\mathcal{E}_{s,t}[\cdot]$ holds in general, i.e., for each convex function $\varphi : \mathcal{R}^n \mapsto \mathcal{R}$ and $\xi_i \in L^1(\mathcal{F}_t)$ $(i = 1, 2, \ldots, n)$, if $\varphi(\xi_1, \xi_2, \ldots, \xi_n) \in L^1(\mathcal{F}_t)$, then we have*

$$\mathcal{E}_{s,t}\big[\varphi(\xi_1, \xi_2, \ldots, \xi_n)\big] \ge \varphi\big(\mathcal{E}_{s,t}[\xi_1], \mathcal{E}_{s,t}[\xi_2], \ldots, \mathcal{E}_{s,t}[\xi_n]\big) \quad a.s.;$$

(ii) *$\mathcal{E}_{s,t}$ is linear, i.e.*,
 (a) $\mathcal{E}_{s,t}[\lambda X] = \lambda\mathcal{E}_{s,t}[X]$ *a.s.*, $\forall (X, \lambda) \in L^1(\mathcal{F}_t) \times \mathcal{R}$;
 (b) $\mathcal{E}_{s,t}[X + Y] = \mathcal{E}_{s,t}[X] + \mathcal{E}_{s,t}[Y]$ *a.s.*, $\forall (X, Y) \in L^1(\mathcal{F}_t) \times L^1(\mathcal{F}_t)$;
 (c) $\mathcal{E}_{s,t}[\mu] = \mu$ *a.s.*, $\forall \mu \in \mathcal{R}$.

Proof We prove (i) implies (ii).

First, we prove (i) implies (ii)(a). For each $(X,\lambda)\in L^1(\mathcal{F}_t)\times\mathcal{R}$, let $\varphi(x_1,x_2,\ldots,x_n):=\lambda x_1$ and $\xi_1:=X$. Obviously, $\varphi(x_1,x_2,\ldots,x_n)$ is a convex function and $\varphi(\xi_1,\xi_2,\ldots,\xi_n)\in L^1(\mathcal{F}_t)$, then we have

$$\mathcal{E}_{s,t}[\lambda X]=\mathcal{E}_{s,t}\bigl[\varphi(\xi_1,\xi_2,\ldots,\xi_n)\bigr]\geq\varphi\bigl(\mathcal{E}_{s,t}[\xi_1],\mathcal{E}_{s,t}[\xi_2],\ldots,\mathcal{E}_{s,t}[\xi_n]\bigr)=\lambda\mathcal{E}_{s,t}[X]\quad\text{a.s.}\tag{2.3}$$

On the other hand, let $\varphi(x_1,x_2,\ldots,x_n):=x_1-(\lambda-1)x_2$, $\xi_1:=\lambda X$, and $\xi_2:=X$. By (i), we can deduce that

$$\begin{aligned}\mathcal{E}_{s,t}[X]&=\mathcal{E}_{s,t}\bigl[\varphi(\xi_1,\xi_2,\ldots,\xi_n)\bigr]\geq\varphi\bigl(\mathcal{E}_{s,t}[\xi_1],\mathcal{E}_{s,t}[\xi_2],\ldots,\mathcal{E}_{s,t}[\xi_n]\bigr)\\&=\mathcal{E}_{s,t}[\lambda X]-(\lambda-1)\mathcal{E}_{s,t}[X]\quad\text{a.s.,}\end{aligned}$$

i.e.,

$$\mathcal{E}_{s,t}[\lambda X]\leq\lambda\mathcal{E}_{s,t}[X]\quad\text{a.s.}\tag{2.4}$$

It follows from (2.3) and (2.4) that (ii)(a) holds true.

Next we prove (ii)(b) holds. For each $(X,Y)\in L^1(\mathcal{F}_t)\times L^1(\mathcal{F}_t)$, let $\varphi(x_1,x_2,\ldots,x_n):=x_1+x_2$, $\xi_1:=X$, and $\xi_2:=Y$, then we have

$$\begin{aligned}\mathcal{E}_{s,t}[X+Y]&=\mathcal{E}_{s,t}\bigl[\varphi(\xi_1,\xi_2,\ldots,\xi_n)\bigr]\geq\varphi\bigl(\mathcal{E}_{s,t}[\xi_1],\mathcal{E}_{s,t}[\xi_2],\ldots,\mathcal{E}_{s,t}[\xi_n]\bigr)\\&=\mathcal{E}_{s,t}[X]+\mathcal{E}_{s,t}[Y]\quad\text{a.s.}\end{aligned}\tag{2.5}$$

On the other hand, let $\varphi(x_1,x_2,\ldots,x_n):=x_1-x_2$, $\xi_1:=X+Y$, and $\xi_2:=Y$. By (i), we have

$$\begin{aligned}\mathcal{E}_{s,t}[X]&=\mathcal{E}_{s,t}\bigl[\varphi(\xi_1,\xi_2,\ldots,\xi_n)\bigr]\geq\varphi\bigl(\mathcal{E}_{s,t}[\xi_1],\mathcal{E}_{s,t}[\xi_2],\ldots,\mathcal{E}_{s,t}[\xi_n]\bigr)\\&=\mathcal{E}_{s,t}[X+Y]-\mathcal{E}_{s,t}[Y]\quad\text{a.s.,}\end{aligned}$$

i.e.,

$$\mathcal{E}_{s,t}[X+Y]\leq\mathcal{E}_{s,t}[X]+\mathcal{E}_{s,t}[Y]\quad\text{a.s.}\tag{2.6}$$

Thus, from (2.5) and (2.6), we can see that (ii)(b) holds.

Finally, we prove (ii)(c) holds. For each $\mu\in\mathcal{R}$, let $\varphi(x_1,x_2,\ldots,x_n):=\mu$, then we have

$$\mathcal{E}_{s,t}[\mu]=\mathcal{E}_{s,t}\bigl[\varphi(\xi_1,\xi_2,\ldots,\xi_n)\bigr]\geq\varphi\bigl(\mathcal{E}_{s,t}[\xi_1],\mathcal{E}_{s,t}[\xi_2],\ldots,\mathcal{E}_{s,t}[\xi_n]\bigr)=\mu\quad\text{a.s.}\tag{2.7}$$

On the other hand, let $\varphi(x_1,x_2,\ldots,x_n):=2x_1-\mu$ and $\xi_1:=\mu$. By (i), we can obtain

$$\mathcal{E}_{s,t}[\mu]=\mathcal{E}_{s,t}\bigl[\varphi(\xi_1,\xi_2,\ldots,\xi_n)\bigr]\geq\varphi\bigl(\mathcal{E}_{s,t}[\xi_1],\mathcal{E}_{s,t}[\xi_2],\ldots,\mathcal{E}_{s,t}[\xi_n]\bigr)=2\mathcal{E}_{s,t}[\mu]-\mu\quad\text{a.s.,}$$

i.e.,

$$\mathcal{E}_{s,t}[\mu]\leq\mu\quad\text{a.s.}\tag{2.8}$$

It follows from (2.7) and (2.8) that (ii)(c) holds true.

In the following, we prove (ii) implies (i). Suppose (ii) holds, for any $(a_1, a_2, \dots, a_n, b) \in \mathcal{R}^{n+1}$ and $\xi_i \in L^1(\mathcal{F}_t)$ $(i = 1, 2, \dots, n)$, we have

$$\mathcal{E}_{s,t}\left[\sum_{i=1}^{n} a_i\xi_i + b\right] = \sum_{i=1}^{n} a_i\mathcal{E}_{s,t}[\xi_i] + b \quad \text{a.s.} \tag{2.9}$$

But, for any convex function $\varphi : \mathcal{R}^n \mapsto \mathcal{R}$, there exists a countable set $\mathcal{D} \subseteq \mathcal{R}^{n+1}$ such that

$$\varphi(x_1, x_2, \dots, x_n) = \sup_{(a_1,a_2,\dots,a_n,b)\in\mathcal{D}} \left(\sum_{i=1}^{n} a_i x_i + b\right). \tag{2.10}$$

In view of (2.9), for any $(a_1, a_2, \dots, a_n, b) \in \mathcal{D}$, we have

$$\mathcal{E}_{s,t}\big[\varphi(\xi_1, \xi_2, \dots, \xi_n)\big] \geq \mathcal{E}_{s,t}\left[\sum_{i=1}^{n} a_i\xi_i + b\right] = \sum_{i=1}^{n} a_i\mathcal{E}_{s,t}[\xi_i] + b \quad \text{a.s.,}$$

which implies (i) by taking into consideration of (2.10). □

The basic version of Hölder's inequality for the classical mathematical expectation E_P defined in $(\Omega, \mathcal{F}_T, P)$ reads

$$E_P[XY] \leq \big(E_P\big[X^p\big]\big)^{\frac{1}{p}} \big(E_P\big[Y^q\big]\big)^{\frac{1}{q}}, \tag{2.11}$$

where X, Y are non-negative random variables in $(\Omega, \mathcal{F}_T, P)$ and $1 < p$, $q < \infty$ is a pair of conjugated exponents, *i.e.*, $\frac{1}{p} + \frac{1}{q} = 1$. One may proceed in the following way (*cf.*, *e.g.*, Krein *et al.* [32], p.43). By elementary calculus, one verifies

$$ab = \inf_{r>0}\left(\frac{r^p}{p}a^p + \frac{r^{-q}}{q}b^q\right)$$

for any constant $a, b \geq 0$. This yields $XY \leq \frac{r^p}{p}X^p + \frac{r^{-q}}{q}Y^q$ a.s. for any $r > 0$. Taking the expectation yields $E_P[XY] \leq \frac{r^p}{p}E_P[X^p] + \frac{r^{-q}}{q}E_P[Y^q]$ for any $r > 0$, and taking the infimum with respect to r again we arrive at (2.11).

By the above argument, we have the following Hölder inequality for $\mathcal{F}_t$-consistent non-linear evaluations.

Theorem 2.3 *Suppose that $\mathcal{E}_{s,t}[\cdot]$, $0 \leq s \leq t \leq T$ is an $\mathcal{F}_t$-consistent nonlinear evaluation in $L^1(\mathcal{F}_T)$. If $\mathcal{E}_{s,t}[\cdot]$ satisfies the following conditions*:

(d) $\mathcal{E}_{s,t}[\xi + \eta] \leq \mathcal{E}_{s,t}[\xi] + \mathcal{E}_{s,t}[\eta]$ *a.s.*, $\forall(\xi, \eta) \in L^1_+(\mathcal{F}_t) \times L^1_+(\mathcal{F}_t)$;

(e) $\mathcal{E}_{s,t}[\lambda\xi] \leq \lambda\mathcal{E}_{s,t}[\xi]$ *a.s.*, $\forall\xi \in L^1_+(\mathcal{F}_t)$, $\lambda \geq 0$,

then, for any $X, Y \in L^1(\mathcal{F}_t)$ and $|X|^p, |Y|^q \in L^1(\mathcal{F}_t)$ $(p, q > 1$ and $1/p + 1/q = 1)$, we have

$$\mathcal{E}_{s,t}\big[|XY|\big] \leq \big(\mathcal{E}_{s,t}\big[|X|^p\big]\big)^{\frac{1}{p}} \big(\mathcal{E}_{s,t}\big[|Y|^q\big]\big)^{\frac{1}{q}} \quad \textit{a.s.}$$

Similarly, we have the following Minkowski inequality for $\mathcal{F}_t$-consistent nonlinear evaluations.

Theorem 2.4 *Suppose that $\mathcal{E}_{s,t}[\cdot]$, $0 \le s \le t \le T$ is an $\mathcal{F}_t$-consistent nonlinear evaluation in $L^1(\mathcal{F}_T)$. If $\mathcal{E}_{s,t}[\cdot]$ satisfies the following conditions*:

(d) $\mathcal{E}_{s,t}[\xi + \eta] \le \mathcal{E}_{s,t}[\xi] + \mathcal{E}_{s,t}[\eta]$ *a.s.*, $\forall(\xi, \eta) \in L^1_+(\mathcal{F}_t) \times L^1_+(\mathcal{F}_t)$;

(e) $\mathcal{E}_{s,t}[\lambda\xi] \le \lambda\mathcal{E}_{s,t}[\xi]$ *a.s.*, $\forall\xi \in L^1_+(\mathcal{F}_t)$, $\lambda \ge 0$,

then, for any $X, Y \in L^1(\mathcal{F}_t)$ *and* $|X|^p, |Y|^p \in L^1(\mathcal{F}_t)$ $(p > 1)$, *we have*

$$\left(\mathcal{E}_{s,t}\left[|X + Y|^p\right]\right)^{\frac{1}{p}} \le \left(\mathcal{E}_{s,t}\left[|X|^p\right]\right)^{\frac{1}{p}} + \left(\mathcal{E}_{s,t}\left[|Y|^p\right]\right)^{\frac{1}{p}} \quad a.s. \tag{2.12}$$

Proof Here $h : [0,\infty) \times [0,\infty) \mapsto [0,\infty)$ is of the form

$$h(x_1, x_2) = \left(x_1^{\frac{1}{p}} + x_2^{\frac{1}{p}}\right)^p = \inf_{r \in \mathcal{Q}\cap(0,1)} \left\{r^{-p}x_1 + (1 - r)^{-p}x_2\right\}, \tag{2.13}$$

where $\mathcal{Q}$ is the set of all rational numbers in $\mathcal{R}$. Let $x_1 := |X|^p$ and $x_2 := |Y|^p$. From (2.13), we have

$$\left(|X| + |Y|\right)^p \le r^{-p}|X|^p + (1 - r)^{-p}|Y|^p \quad \text{a.s.}$$

for all $r \in \mathcal{Q} \cap (0,1)$. It follows from (d) and (e) that

$$\mathcal{E}_{s,t}\left[\left(|X| + |Y|\right)^p\right] \le r^{-p}\mathcal{E}_{s,t}\left[|X|^p\right] + (1 - r)^{-p}\mathcal{E}_{s,t}\left[|Y|^p\right] \quad \text{a.s.}$$

for all $r \in \mathcal{Q} \cap (0,1)$. Taking the infimum with respect to r in $\mathcal{Q} \cap (0,1)$, we have

$$\mathcal{E}_{s,t}\left[\left(|X| + |Y|\right)^p\right] \le \left\{\left(\mathcal{E}_{s,t}\left[|X|^p\right]\right)^{\frac{1}{p}} + \left(\mathcal{E}_{s,t}\left[|Y|^p\right]\right)^{\frac{1}{p}}\right\}^p \quad \text{a.s.}$$

Thus, (2.12) holds true. □

3 Jensen's inequality for *g*-evaluations

In this section, first, we present some notations, notions, and propositions which are useful in this paper.

Let

$$\begin{aligned}
\mathcal{S}^p(0,t;P;\mathcal{R}) &:= \Big\{V : V_s \text{ is } \mathcal{R}\text{-valued } \mathcal{F}_s\text{-adapted continuous process with} \\
&\qquad E_P\Big[\sup_{0\le s\le t}|V_s|^p\Big] < \infty\Big\}, \\
\mathcal{S}(0,t;P;\mathcal{R}) &:= \bigcup_{p>1}\mathcal{S}^p(0,t;P;\mathcal{R}), \\
L^p\left(0,t;P;\mathcal{R}^d\right) &:= \Bigg\{V : V_s \text{ is } \mathcal{R}^d\text{-valued and } \mathcal{F}_s\text{-adapted process with} \\
&\qquad E_P\Bigg[\left(\int_0^t |V_s|^2\,\mathrm{d}s\right)^{\frac{p}{2}}\Bigg] < \infty\Bigg\}, \\
\mathcal{L}\left(0,t;P;\mathcal{R}^d\right) &:= \bigcup_{p>1} L^p\left(0,t;P;\mathcal{R}^d\right), \\
\mathcal{M}^p(0,t;P;\mathcal{R}) &:= \Bigg\{V : V_s \text{ is } \mathcal{R}\text{-valued } \mathcal{F}_s\text{-adapted process with}
\end{aligned}$$

$$E_P\left[\left(\int_0^t |V_s|\,\mathrm{d}s\right)^p\right] < \infty\bigg\},$$

$$\mathcal{M}(0,t;P;\mathcal{R}) := \bigcup_{p>1} \mathcal{M}^p(0,t;P;\mathcal{R})$$

and

$$\mathcal{L}(\mathcal{F}_t) := \bigcup_{p>1} L^p(\mathcal{F}_t).$$

For each $t \in [0,T]$, we consider the following BSDE with terminal time t:

$$y_s = X + \int_s^t g(r,y_r,z_r)\,\mathrm{d}r - \int_s^t z_r \cdot \mathrm{d}B_r, \quad s \in [0,t]. \tag{3.1}$$

Here the function g:

$$g(\omega,t,y,z) : \Omega \times [0,T] \times \mathcal{R} \times \mathcal{R}^d \mapsto \mathcal{R}$$

satisfies the following assumptions:

(B.1) there exist two non-negative deterministic functions $\alpha(t)$ and $\beta(t)$ such that for all $y_1, y_2 \in \mathcal{R}$, $z_1, z_2 \in \mathcal{R}^d$,

$$\big|g(t,y_1,z_1) - g(t,y_2,z_2)\big| \le \alpha(t)|y_1 - y_2| + \beta(t)|z_1 - z_2|, \quad \forall t \in [0,T],$$

where $\alpha(t)$ and $\beta(t)$ satisfy $\int_0^T \alpha^2(t)\,\mathrm{d}t < \infty$, $\int_0^T \beta^2(t)\,\mathrm{d}t < \infty$;

(B.2) $g(t,0,0) \in \mathcal{M}(0,t;P;\mathcal{R})$;

(B.3) $g(t,y,0) = 0$, $\mathrm{d}P \times \mathrm{d}t$-a.s., $\forall y \in \mathcal{R}$.

It is well known that (see Zong [11]) if we suppose that the function g satisfies (B.1) and (B.2), then for each given $X \in \mathcal{L}(\mathcal{F}_t)$, there exists a unique solution $(Y^X, Z^X) \in \mathcal{S}(0,t;P;\mathcal{R}) \times \mathcal{L}(0,t;P;\mathcal{R}^d)$ of BSDE (3.1).

Example 3.1 For each given $\xi \in \mathcal{L}(\mathcal{F}_T)$, the BSDE

$$y_t = \xi + \int_t^T \left(\frac{1}{\sqrt[5]{s}} y_s + \frac{1}{\sqrt[8]{T-s}}|z_s|\right)\mathrm{d}s - \int_t^T z_s \cdot \mathrm{d}B_s, \quad t \in [0,T],$$

has a unique solution in $\mathcal{S}(0,T;P;\mathcal{R}) \times \mathcal{L}(0,T;P;\mathcal{R}^d)$.

We denote $\mathcal{E}^g_{s,t}[X] := Y^X_s$. We thus define a system of operators:

$$\mathcal{E}^g_{s,t}[X] : X \in \mathcal{L}(\mathcal{F}_t) \mapsto \mathcal{L}(\mathcal{F}_s), \quad 0 \le s \le t \le T.$$

This system is completely determined by the above given function g. We have the following.

Proposition 3.1 *We assume that the function g satisfies* (B.1) *and* (B.2). *Then the system of operators $\mathcal{E}^g_{s,t}[\cdot]$, $0 \le s \le t \le T$ is an $\mathcal{F}_t$-consistent nonlinear evaluation defined in $\mathcal{L}(\mathcal{F}_T)$.*

The proof of Proposition 3.1 is very similar to that of Corollary 2.9 in [13], so we omit it.

Remark 3.1 From Proposition 3.1, we know that the dynamically consistent nonlinear evaluation $\mathcal{E}^g_{s,t}[\cdot]$, $0 \leq s \leq t \leq T$ is completely determined by the given function g. Thus, we call $\mathcal{E}^g_{s,t}[\cdot]$, $0 \leq s \leq t \leq T$ a g-evaluation.

Definition 3.1 (g-Expectation) (see Zong [11]) Suppose that the function g satisfies (B.1) and (B.3). The g-expectation $\mathcal{E}_g[\cdot] : \mathcal{L}(\mathcal{F}_T) \mapsto \mathcal{R}$ is defined by $\mathcal{E}_g[\xi] = Y_0^\xi$.

Definition 3.2 (Conditional g-expectation) (see Zong [11]) Suppose that the function g satisfies (B.1) and (B.3). The conditional g-expectation of ξ with respect to $\mathcal{F}_t$ is defined by $\mathcal{E}_g[\xi|\mathcal{F}_t] = Y_t^\xi$.

Proposition 3.2 (see Zong [11]) *$\mathcal{E}_g[\xi|\mathcal{F}_t]$ is the unique random variable η in $\mathcal{L}(\mathcal{F}_t)$ such that*

$$\mathcal{E}_g[1_A\xi] = \mathcal{E}_g[1_A\eta], \quad \forall A \in \mathcal{F}_t.$$

Proposition 3.3 *For any $\xi_n \in \mathcal{L}(\mathcal{F}_t)$, if $\lim_{n\to\infty} \xi_n = \xi$ a.s. and $|\xi_n| \leq \eta$ a.s. with $\eta \in \mathcal{L}(\mathcal{F}_t)$, then for $0 \leq s \leq t \leq T$,*

$$\lim_{n\to\infty} \mathcal{E}^g_{s,t}[\xi_n] = \mathcal{E}^g_{s,t}[\xi] \quad a.s.$$

The proof of Proposition 3.3 is very similar to that of Theorem 3.1 in Hu and Chen [24], so we omit it.

In the following, we study Jensen's inequality for g-evaluations. First, we introduce some notions on g.

Definition 3.3 Let $g : \Omega \times [0,T] \times \mathcal{R} \times \mathcal{R}^d \mapsto \mathcal{R}$. The function g is said to be super-homogeneous if for each $(y,z) \in \mathcal{R} \times \mathcal{R}^d$ and $\lambda \in \mathcal{R}$, then $g(t,\lambda y,\lambda z) \geq \lambda g(t,y,z)$, $\mathrm{d}P \times \mathrm{d}t$-a.s. The function g is said to be positively homogeneous if for each $(y,z) \in \mathcal{R} \times \mathcal{R}^d$ and $\lambda \geq 0$, then $g(t,\lambda y,\lambda z) = \lambda g(t,y,z)$, $\mathrm{d}P \times \mathrm{d}t$-a.s. The function g is said to be sub-additive if, for any $(y,z),(\overline{y},\overline{z}) \in \mathcal{R} \times \mathcal{R}^d$, $g(t,y+\overline{y},z+\overline{z}) \leq g(t,y,z)+g(t,\overline{y},\overline{z})$, $\mathrm{d}P \times \mathrm{d}t$-a.s. The function g is said to be super-additive if, for any $(y,z),(\overline{y},\overline{z}) \in \mathcal{R} \times \mathcal{R}^d$, $g(t,y+\overline{y},z+\overline{z}) \geq g(t,y,z)+g(t,\overline{y},\overline{z})$, $\mathrm{d}P \times \mathrm{d}t$-a.s.

Theorem 3.1 *Suppose that $\mathcal{E}^g_{s,t}[\cdot]$, $0 \leq s \leq t \leq T$ is a g-evaluation, then the following three statements are equivalent:*

(i) *Jensen's inequality for g-evaluation $\mathcal{E}^g_{s,t}[\cdot]$ holds in general, i.e., for each convex function $\varphi(x) : \mathcal{R} \mapsto \mathcal{R}$ and each $\xi \in \mathcal{L}(\mathcal{F}_t)$, if $\varphi(\xi) \in \mathcal{L}(\mathcal{F}_t)$, then we have*

$$\mathcal{E}^g_{s,t}\big[\varphi(\xi)\big] \geq \varphi\big(\mathcal{E}^g_{s,t}[\xi]\big) \quad a.s.;$$

(ii) *$\forall(\xi,a,b) \in \mathcal{L}(\mathcal{F}_t) \times \mathcal{R} \times \mathcal{R}$, $\mathcal{E}^g_{s,t}[a\xi + b] \geq a\mathcal{E}^g_{s,t}[\xi] + b$ a.s.;*

(iii) *g is independent of y and super-homogeneous with respect to z.*

Theorem 3.2 *Suppose that $\mathcal{E}^g_{s,t}[\cdot]$, $0 \leq s \leq t \leq T$ is a g-evaluation, then the following three statements are equivalent:*

(i) *the n-dimensional* ($n > 1$) *Jensen inequality for the g-evaluation* $\mathcal{E}^g_{s,t}[\cdot]$ *holds in general, i.e., for each convex function* $\varphi : R^n \mapsto R$ *and* $\xi_i \in \mathcal{L}(\mathcal{F}_t)$ ($i = 1, 2, \ldots, n$), *if* $\varphi(\xi_1, \xi_2, \ldots, \xi_n) \in \mathcal{L}(\mathcal{F}_t)$, *then we have*

$$\mathcal{E}^g_{s,t}\big[\varphi(\xi_1, \xi_2, \ldots, \xi_n)\big] \geq \varphi\big(\mathcal{E}^g_{s,t}[\xi_1], \mathcal{E}^g_{s,t}[\xi_2], \ldots, \mathcal{E}^g_{s,t}[\xi_n]\big) \quad a.s.;$$

(ii) $\mathcal{E}^g_{s,t}$ *is linear in* $\mathcal{L}(\mathcal{F}_t)$;

(iii) *g is independent of y and linear with respect to z, i.e., g is of the form* $g(t, y, z) = g(t, z) = \alpha_t \cdot z$, $\mathrm{d}P \times \mathrm{d}t$-*a.s.*, $\forall (y, z) \in \mathcal{R} \times \mathcal{R}^d$, *where* α *is a* R^d-*valued progressively measurable process*.

In order to prove Theorems 3.1 and 3.2, we need the following lemmas. These lemmas can be found in Zong and Hu [33].

Lemma 3.1 *Suppose that the function g satisfies* (B.1) *and* (B.2). *Then the following three conditions are equivalent*:

(i) *The function g is independent of y*.

(ii) *The corresponding dynamically consistent nonlinear evaluation* $\mathcal{E}^g[\cdot]$ *satisfies*: *for each* $0 \leq s \leq t \leq T$, $\mathcal{F}_t$ *measurable simple function X and* $y \in \mathcal{R}$,

$$\mathcal{E}^g_{s,t}[X + y] = \mathcal{E}^g_{s,t}[X] + y \quad a.s.$$

(iii) *The corresponding dynamically consistent nonlinear evaluation* $\mathcal{E}^g[\cdot]$ *satisfies*: *for each* $0 \leq s \leq t \leq T$, $X \in \mathcal{L}(\mathcal{F}_t)$, *and* $\eta \in \mathcal{L}(\mathcal{F}_s)$,

$$\mathcal{E}^g_{s,t}[X + \eta] = \mathcal{E}^g_{s,t}[X] + \eta \quad a.s.$$

Lemma 3.2 *Suppose that the function g satisfies* (B.1) *and* (B.2). *Then the following three conditions are equivalent*:

(i) *The function g is positively homogeneous*.

(ii) *The corresponding dynamically consistent nonlinear evaluation* $\mathcal{E}^g[\cdot]$ *satisfies*: *for each* $0 \leq s \leq t \leq T$, $\lambda \geq 0$, *and* $\mathcal{F}_t$ *measurable simple function X*,

$$\mathcal{E}^g_{s,t}[\lambda X] = \lambda \mathcal{E}^g_{s,t}[X] \quad a.s.$$

(iii) *The corresponding dynamically consistent nonlinear evaluation* $\mathcal{E}^g[\cdot]$ *is positively homogeneous*: *for each* $0 \leq s \leq t \leq T$, $\lambda \geq 0$, *and* $X \in \mathcal{L}(\mathcal{F}_t)$,

$$\mathcal{E}^g_{s,t}[\lambda X] = \lambda \mathcal{E}^g_{s,t}[X] \quad a.s.$$

Lemma 3.3 *Suppose that the function g satisfies* (B.1) *and* (B.2). *Then the following three conditions are equivalent*:

(i) *The function g is sub-additive* (*super-additive*).

(ii) *The corresponding dynamically consistent nonlinear evaluation* $\mathcal{E}^g[\cdot]$ *satisfies*: *for each* $0 \leq s \leq t \leq T$ *and* $\mathcal{F}_t$ *measurable simple functions X and* $\overline{X}$,

$$\mathcal{E}^g_{s,t}[X + \overline{X}] \leq (\geq)\, \mathcal{E}^g_{s,t}[X] + \mathcal{E}^g_{s,t}[\overline{X}] \quad a.s.$$

(iii) *The corresponding dynamically consistent nonlinear evaluation* $\mathcal{E}^g[\cdot]$ *is sub-additive (super-additive): for each* $0 \le s \le t \le T$ *and* $X, \overline{X} \in \mathcal{L}(\mathcal{F}_t)$,

$$\mathcal{E}^g_{s,t}[X + \overline{X}] \le (\ge)\, \mathcal{E}^g_{s,t}[X] + \mathcal{E}^g_{s,t}[\overline{X}] \quad \textit{a.s.}$$

Lemma 3.4 *Suppose that the functions* g *and* $\overline{g}$ *satisfy* (B.1) *and* (B.2). *Then the following three conditions are equivalent*:

(i) $g(t,y,z) \ge \overline{g}(t,y,z)$, $\mathrm{d}P \times \mathrm{d}t$*-a.s.*, $\forall (y,z) \in \mathcal{R} \times \mathcal{R}^d$.

(ii) *The corresponding dynamically consistent nonlinear evaluations* $\mathcal{E}^g[\cdot]$ *and* $\mathcal{E}^{\overline{g}}[\cdot]$ *satisfy, for each* $0 \le s \le t \le T$ *and* $\mathcal{F}_t$ *measurable simple function* X,

$$\mathcal{E}^g_{s,t}[X] \ge \mathcal{E}^{\overline{g}}_{s,t}[X] \quad \textit{a.s.}$$

(iii) *The corresponding dynamically consistent nonlinear evaluations* $\mathcal{E}^g[\cdot]$ *and* $\mathcal{E}^{\overline{g}}[\cdot]$ *satisfy, for each* $0 \le s \le t \le T$ *and* $X \in \mathcal{L}(\mathcal{F}_t)$,

$$\mathcal{E}^g_{s,t}[X] \ge \mathcal{E}^{\overline{g}}_{s,t}[X] \quad \textit{a.s.}$$

In particular, $\mathcal{E}^g[\cdot] \equiv \mathcal{E}^{\overline{g}}[\cdot]$ *if and only if* $g \equiv \overline{g}$.

Proof of Theorem 3.1 From Theorem 2.1, we only need to prove (ii) $\Leftrightarrow$ (iii). (iii) $\Rightarrow$ (ii) is obvious.

In the following, we prove (ii) $\Rightarrow$ (iii). First, we prove that g is independent of y. Suppose (ii) holds, then we have, for any $(\xi, y) \in \mathcal{L}(\mathcal{F}_t) \times \mathcal{R}$,

$$\mathcal{E}^g_{s,t}[\xi + y] = \mathcal{E}^g_{s,t}[\xi] + y \quad \text{a.s.} \tag{3.2}$$

By Lemma 3.1, we can deduce that g is independent of y.

Next we prove that g is super-homogeneous with respect to z. By (ii), we have, for any $(\xi, \lambda) \in \mathcal{L}(\mathcal{F}_t) \times R$,

$$\lambda \mathcal{E}^g_{s,t}[\xi] \le \mathcal{E}^g_{s,t}[\lambda \xi] \quad \text{a.s.} \tag{3.3}$$

For each $(s,z) \in [0,t] \times \mathcal{R}^d$, let $Y^{s,z}_\cdot$ be the solution of the following stochastic differential equation (SDE for short) defined on $[s,t]$:

$$Y^{s,z}_t = -\int_s^t g(r,z)\,\mathrm{d}r + z \cdot (B_t - B_s). \tag{3.4}$$

From (3.3), we have

$$\mathcal{E}^g_{r,t}\big[\lambda Y^{s,z}_t\big] \ge \lambda \mathcal{E}^g_{r,t}\big[Y^{s,z}_t\big] = \lambda Y^{s,z}_r, \quad 0 \le s \le r \le t \le T.$$

Thus, $(\lambda Y^{s,z}_r)_{r\in[s,t]}$ is an $\mathcal{E}_g$-submartingale. From the decomposition theorem of an $\mathcal{E}_g$-supermartingale (see Zong and Hu [33]), it follows that there exists an increasing process $(A_r)_{r\in[s,t]}$ such that

$$\lambda Y^{s,z}_t = -\int_s^t g(r,Z_r)\,\mathrm{d}r + A_t - A_s + \int_s^t Z_r \cdot \mathrm{d}B_r, \quad t \in [s,T].$$

This with $\lambda Y_t^{s,z} = -\int_s^t \lambda g(r,z)\,\mathrm{d}r + \int_s^t \lambda z \cdot \mathrm{d}B_r$ yields $Z_r \equiv \lambda z$ and

$$\lambda g(t,z) \le g(t,\lambda z), \quad \mathrm{d}P \times \mathrm{d}t\text{-a.s.} \tag{3.5}$$

The proof of Theorem 3.1 is complete. □

Remark 3.2 The condition that g is super-homogeneous with respect to z implies that g is positively homogeneous with respect to z. Indeed, for each fixed $\lambda > 0$, by (3.5), we have $\frac{1}{\lambda} g(t,\lambda z) \le g(t,z)$, $\mathrm{d}P \times \mathrm{d}t$-a.s., *i.e.*,

$$g(t,\lambda z) \le \lambda g(t,z), \quad \mathrm{d}P \times \mathrm{d}t\text{-a.s.} \tag{3.6}$$

Thus by (3.5) and (3.6), for any $\lambda > 0$,

$$g(t,\lambda z) = \lambda g(t,z), \quad \mathrm{d}P \times \mathrm{d}t\text{-a.s.} \tag{3.7}$$

In particular, choosing $\lambda = 2$, we have $2g(t,0) = g(t,0)$, $\mathrm{d}P \times \mathrm{d}t$-a.s. Hence $g(t,0) = 0$, $\mathrm{d}P \times \mathrm{d}t$-a.s. Thus, for $\lambda = 0$ (3.7) still holds.

Proof of Theorem 3.2 From Theorem 2.2, we only need to prove (ii) ⇔ (iii). (iii) ⇒ (ii) is obvious.

In the following, we prove (ii) ⇒ (iii). From the proof of Theorem 3.1, we can obtain, for any $\lambda \in \mathcal{R}$ and $(y,z) \in \mathcal{R} \times \mathcal{R}^d$, $g(t,y,\lambda z) = g(t,\lambda z) \ge \lambda g(t,z)$, $\mathrm{d}P \times \mathrm{d}t$-a.s. Using the same method, we have $g(t,y,\lambda z) = g(t,\lambda z) \le \lambda g(t,z)$, $\mathrm{d}P \times \mathrm{d}t$-a.s., $\forall \lambda \in \mathcal{R}$, $(y,z) \in \mathcal{R} \times \mathcal{R}^d$. The above arguments imply that, for any $\lambda \in \mathcal{R}$ and $(y,z) \in \mathcal{R} \times \mathcal{R}^d$,

$$g(t,y,\lambda z) = g(t,\lambda z) = \lambda g(t,z), \quad \mathrm{d}P \times \mathrm{d}t\text{-a.s.} \tag{3.8}$$

On the other hand, by Lemma 3.3, we have, for any $(y,z), (\overline{y},\overline{z}) \in \mathcal{R} \times \mathcal{R}^d$,

$$g(t,y+\overline{y},z+\overline{z}) = g(t,y,z) + g(t,\overline{y},\overline{z}), \quad \mathrm{d}P \times \mathrm{d}t\text{-a.s.} \tag{3.9}$$

It follows from (3.8) and (3.9) that (iii) holds true. The proof of Theorem 3.2 is complete. □

From Theorem 3.1(iii), we know that, for any $y \in \mathcal{R}$, $g(t,y,0) = g(t,0) = 0$, $\mathrm{d}P \times \mathrm{d}t$-a.s. Hence, $\mathcal{E}_{s,t}^g[\cdot] = \mathcal{E}_g[\cdot|\mathcal{F}_s]$. Thus, Theorem 3.1 can be rewritten as follows.

Corollary 3.1 *Suppose that* $\mathcal{E}_{s,t}^g[\cdot]$, $0 \le s \le t \le T$ *is a g-evaluation, then the following four statements are equivalent*:

(i) *Jensen's inequality for the g-evaluation* $\mathcal{E}_{s,t}^g[\cdot]$ *holds in general, i.e., for each convex function* $\varphi(x): \mathcal{R} \mapsto \mathcal{R}$ *and each* $\xi \in \mathcal{L}(\mathcal{F}_t)$, *if* $\varphi(\xi) \in \mathcal{L}(\mathcal{F}_t)$, *then we have*

$$\mathcal{E}_{s,t}^g\big[\varphi(\xi)\big] \ge \varphi\big(\mathcal{E}_{s,t}^g[\xi]\big) \quad a.s.;$$

(ii) $\forall (\xi,a,b) \in L^2(\mathcal{F}_T) \times \mathcal{R} \times \mathcal{R}$, $\mathcal{E}_{0,T}^g[a\xi + b] \ge a\mathcal{E}_{0,T}^g[\xi] + b$, *and, for any* $y \in \mathcal{R}$, $g(t,y,0) = 0$, $\mathrm{d}P \times \mathrm{d}t$-*a.s.*;

(iii) $\forall(\xi, a, b) \in L^2(\mathcal{F}_t) \times \mathcal{R} \times \mathcal{R}$, $\mathcal{E}^g_{s,t}[a\xi + b] \geq a\mathcal{E}^g_{s,t}[\xi] + b$ *a.s.*;
(iv) *g is independent of y and super-homogeneous with respect to z.*

Similarly, Theorem 3.2 can be rewritten as follows.

Corollary 3.2 *Suppose that* $\mathcal{E}^g_{s,t}[\cdot]$, $0 \leq s \leq t \leq T$ *is a g-evaluation, then the following four statements are equivalent*:
(i) *the n-dimensional* ($n > 1$) *Jensen inequality for g-evaluation* $\mathcal{E}^g_{s,t}[\cdot]$ *holds in general, i.e., for each convex function* $\varphi : \mathcal{R}^n \mapsto \mathcal{R}$ *and* $\xi_i \in \mathcal{L}(\mathcal{F}_t)$ ($i = 1, 2, \ldots, n$), *if* $\varphi(\xi_1, \xi_2, \ldots, \xi_n) \in \mathcal{L}(\mathcal{F}_t)$, *then we have*

$$\mathcal{E}^g_{s,t}\big[\varphi(\xi_1, \xi_2, \ldots, \xi_n)\big] \geq \varphi\big(\mathcal{E}^g_{s,t}[\xi_1], \mathcal{E}^g_{s,t}[\xi_2], \ldots, \mathcal{E}^g_{s,t}[\xi_n]\big) \quad a.s.;$$

(ii) $\mathcal{E}^g_{0,T}$ *is linear in* $L^2(\mathcal{F}_T)$ *and, for any* $y \in \mathcal{R}$, $g(t, y, 0) = 0$, $\mathrm{d}P \times \mathrm{d}t$*-a.s.*;
(iii) $\mathcal{E}^g_{s,t}$ *is linear in* $L^2(\mathcal{F}_t)$;
(iv) *for each* $(y, z) \in \mathcal{R} \times \mathcal{R}^d$, $g(t, y, z) = g(t, z) = \alpha_t \cdot z$, $\mathrm{d}P \times \mathrm{d}t$*-a.s., where* α *is a* $\mathcal{R}^d$*-valued progressively measurable process.*

Proof of Corollary 3.1 From Proposition 3.3 and Theorem 3.1, we only need to prove (ii) $\Leftrightarrow$ (iii). It is obvious that (iii) implies (ii).

In the following, we prove that (ii) implies (iii). Suppose (ii) holds. For each $(X, t, k) \in L^2(\mathcal{F}_T) \times [0, T] \times \mathcal{R}$, by (ii), we know that for each $A \in \mathcal{F}_t$,

$$\begin{aligned}
\mathcal{E}^g_{0,T}\big[1_A(X + k)\big] &= \mathcal{E}^g_{0,T}[1_A X + 1_A k - k] + k \\
&= \mathcal{E}^g_{0,T}\big[1_A X + 1_{A^C}(-k)\big] + k \\
&= \mathcal{E}^g_{0,t}\big[\mathcal{E}^g_{t,T}\big[1_A X + 1_{A^C}(-k)\big]\big] + k \\
&= \mathcal{E}^g_{0,t}\big[1_A \mathcal{E}^g_{t,T}[X] + 1_{A^C}(-k)\big] + k \\
&= \mathcal{E}^g_{0,t}\big[1_A \mathcal{E}^g_{t,T}[X] + 1_{A^C}(-k) + k\big] \\
&= \mathcal{E}^g_{0,t}\big[1_A\big(\mathcal{E}^g_{t,T}[X] + k\big)\big].
\end{aligned}$$

Thus

$$\mathcal{E}^g_{t,T}[X + k] = \mathcal{E}^g_{t,T}[X] + k \quad \text{a.s.} \tag{3.10}$$

For each $\lambda \neq 0$, define $\mathcal{E}^\lambda_{t,T}[\cdot] := \frac{\mathcal{E}^g_{t,T}[\lambda \cdot]}{\lambda}$, $\forall t \in [0, T]$. It is easy to check that $\mathcal{E}^g_{t,T}[\cdot]$ and $\mathcal{E}^\lambda_{t,T}[\cdot]$ are two $\mathcal{F}$-expectations in $L^2(\mathcal{F}_T)$ (the notion of $\mathcal{F}$-expectation can be seen in Coquet *et al.* [20]). If $\lambda > 0$, for each $\xi \in L^2(\mathcal{F}_T)$, $\mathcal{E}^\lambda_{0,T}[\xi] \geq \mathcal{E}^g_{0,T}[\xi]$. In a similar manner to Lemma 4.5 in Coquet *et al.* [20], we can obtain

$$\mathcal{E}^\lambda_{t,T}[\xi] \geq \mathcal{E}^g_{t,T}[\xi] \quad \text{a.s.}, \forall t \in [0, T]. \tag{3.11}$$

If $\lambda < 0$, for each $\xi \in L^2(\mathcal{F}_T)$, $\mathcal{E}^\lambda_{0,T}[\xi] \leq \mathcal{E}^g_{0,T}[\xi]$. In a similar manner to Lemma 4.5 in Coquet *et al.* [20] again, we have

$$\mathcal{E}^\lambda_{t,T}[\xi] \leq \mathcal{E}^g_{t,T}[\xi] \quad \text{a.s.}, \forall t \in [0, T]. \tag{3.12}$$

From (3.11) and (3.12), we have, for any $(\xi, \lambda) \in L^2(\mathcal{F}_T) \times \mathcal{R}$,

$$\mathcal{E}^g_{t,T}[\lambda\xi] \geq \lambda\mathcal{E}^g_{t,T}[\xi] \quad \text{a.s.}, \forall t \in [0, T]. \tag{3.13}$$

From (3.10) and (3.13), we have, for any $(\xi, a, b) \in L^2(\mathcal{F}_T) \times \mathcal{R} \times \mathcal{R}$,

$$\mathcal{E}^g_{t,T}[a\xi + b] \geq a\mathcal{E}^g_{t,T}[\xi] + b \quad \text{a.s.}, \forall t \in [0, T].$$

Since, for any $y \in \mathcal{R}$, $g(t, y, 0) = 0$, $\mathrm{d}P \times \mathrm{d}t$-a.s., we have

$$\mathcal{E}^g_{s,t}[a\xi + b] = \mathcal{E}^g_{s,T}[a\xi + b] \geq a\mathcal{E}^g_{s,T}[\xi] + b = a\mathcal{E}^g_{s,t}[\xi] + b \quad \text{a.s.}, \forall(\xi, a, b) \in L^2(\mathcal{F}_t) \times \mathcal{R} \times \mathcal{R}.$$

Therefore, (iii) holds true. The proof of Corollary 3.1 is complete. □

Proof of Corollary 3.2 From Proposition 3.3 and Theorem 3.2, we only need to prove (ii) ⇔ (iii). It is obvious that (iii) implies (ii).

In the following, we prove that (ii) implies (iii). Suppose (ii) holds. By Proposition 3.3, we know that for each sequence $\{X_n\}_{n=1}^\infty \subset L^2(\mathcal{F}_T)$ such that $X_n(\omega) \downarrow 0$ for all ω, $\mathcal{E}^g_{0,T}[X_n] \downarrow 0$. By the well-known Daniell-Stone theorem (*cf.*, *e.g.*, Yan [34], Theorem 3.6.8, p.83), there exists a unique probability measure P_α defined on $(\Omega, \mathcal{F}_T)$ such that

$$\mathcal{E}^g_{0,T}[\xi] = E_{P_\alpha}[\xi], \quad \forall\xi \in L^2(\mathcal{F}_T) \tag{3.14}$$

holds. Indeed, from (iv), we know that $\frac{\mathrm{d}P_\alpha}{\mathrm{d}P} = \exp(\int_0^T \alpha_t \cdot \mathrm{d}B_t - \frac{1}{2}\int_0^T |\alpha_t|^2\,\mathrm{d}t)$.

On the other hand, since, for any $y \in \mathcal{R}$, $g(t, y, 0) = 0$, $\mathrm{d}P \times \mathrm{d}t$-a.s., we can obtain

$$\mathcal{E}^g_{s,t}[\xi] = \mathcal{E}^g_{s,T}[\xi] \quad \text{a.s.}, \forall\xi \in L^2(\mathcal{F}_t). \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\mathcal{E}^g_{s,t}[\xi] = E_{P_\alpha}[\xi|\mathcal{F}_s] \quad \text{a.s.}, \forall\xi \in L^2(\mathcal{F}_t).$$

Therefore, $\mathcal{E}^g_{s,t}$ is linear in $L^2(\mathcal{F}_t)$. The proof of Corollary 3.2 is complete. □

From Corollary 3.2, we can immediately obtain the following.

Theorem 3.3 *Suppose that* $\mathcal{E}^g_{s,t}[\cdot]$, $0 \leq s \leq t \leq T$ *is a g-evaluation, then the following two statements are equivalent*:

(i) $\mathcal{E}^g_{s,t}$ *is linear in* $\mathcal{L}(\mathcal{F}_t)$;

(ii) *there exists a unique probability measure* P_α *defined on* $(\Omega, \mathcal{F}_T)$ *such that, for any* $\xi \in \mathcal{L}(\mathcal{F}_t)$,

$$\mathcal{E}^g_{s,t}[\xi] = E_{P_\alpha}[\xi|\mathcal{F}_s] \quad a.s.$$

The following result can be seen as an extension of Theorem 3.3.

Theorem 3.4 *Suppose that* $\mathcal{E}^g_{s,t}[\cdot]$, $0 \leq s \leq t \leq T$ *is a g-evaluation, then the following two statements are equivalent*:

(i) $\mathcal{E}^g_{s,t}$ *is sublinear in* $\mathcal{L}(\mathcal{F}_t)$, *i.e.*,

(f) $\mathcal{E}^g_{s,t}[\lambda X] = \lambda \mathcal{E}^g_{s,t}[X]$ *a.s.*, *for any* $X \in \mathcal{L}(\mathcal{F}_t)$ *and* $\lambda \geq 0$;

(g) $\mathcal{E}_{s,t}[X+Y] \leq \mathcal{E}^g_{s,t}[X] + \mathcal{E}^g_{s,t}[Y]$ *a.s.*, *for any* $(X,Y) \in \mathcal{L}(\mathcal{F}_t) \times \mathcal{L}(\mathcal{F}_t)$;

(h) $\mathcal{E}^g_{s,t}[\mu] = \mu$ *a.s.*, *for any* $\mu \in \mathcal{R}$;

(ii) *for any* $\xi \in \mathcal{L}(\mathcal{F}_t)$,

$$\mathcal{E}^g_{s,t}[\xi] = \sup_{Q_\theta \in \Lambda} E_{Q_\theta}[\xi | \mathcal{F}_s] \quad a.s.,$$

where Λ *is a set of probability measures on* $(\Omega, \mathcal{F}_T)$ *and defined by*

$$\Lambda := \{ Q_\theta : E_{Q_\theta}[\xi] \leq \mathcal{E}^g_{0,T}[\xi], \forall \xi \in \mathcal{L}(\mathcal{F}_T) \}.$$

Proof It is obvious that (ii) implies (i).

In the following, we prove that (i) implies (ii). Suppose (i) holds. Since $\mathcal{E}_{0,T}[\cdot]$ is a sublinear expectation in $\mathcal{L}(\mathcal{F}_T)$, by Lemma 2.4 in Peng [35], we know that there exists a family of linear expectations $\{E_\theta : \theta \in \Theta\}$ on $(\Omega, \mathcal{F}_T)$ such that, for any $\xi \in \mathcal{L}(\mathcal{F}_T)$,

$$\mathcal{E}^g_{0,T}[\xi] = \sup_{\theta \in \Theta} E_\theta[\xi]. \tag{3.16}$$

On the other hand, by Proposition 3.3, we know that for each sequence $\{X_n\}_{n=1}^\infty \subset \mathcal{L}(\mathcal{F}_T)$ such that $X_n(\omega) \downarrow 0$ for all ω, $\mathcal{E}^g_{0,T}[X_n] \downarrow 0$. By the well-known Daniell-Stone theorem, we can deduce that for each $\theta \in \Theta$ and $\xi \in \mathcal{L}(\mathcal{F}_T)$, there exists a unique probability measure Q_θ defined on $(\Omega, \mathcal{F}_T)$ such that

$$E_\theta[\xi] = E_{Q_\theta}[\xi]. \tag{3.17}$$

It follows from (3.16) and (3.17) that, for any $\xi \in \mathcal{L}(\mathcal{F}_T)$,

$$\mathcal{E}^g_{0,T}[\xi] = \sup_{Q_\theta \in \Lambda} E_{Q_\theta}[\xi]. \tag{3.18}$$

Let Π be a set of probability measures on $(\Omega, \mathcal{F}_T)$ defined by

$$\Pi := \left\{ P_\alpha : \alpha \in \Theta^g, \frac{\mathrm{d}P_\alpha}{\mathrm{d}P} = \exp\left(\int_0^T \alpha_t \cdot \mathrm{d}B_t - \frac{1}{2} \int_0^T |\alpha_t|^2 \,\mathrm{d}t \right) \right\},$$

where $\Theta^g := \{(\alpha_t)_{t \in [0,T]} : \alpha$ is $\mathcal{R}^d$-valued, progressively measurable and, for any $(y,z) \in \mathcal{R} \times \mathcal{R}^d$, $\alpha_t \cdot z \leq g(t,y,z)$, $\mathrm{d}P \times \mathrm{d}t$-a.s.$\}$. In order to prove (ii), now we prove that $\Pi = \Lambda$.

For any $\alpha \in \Theta^g$, we define $g^\alpha(t,y,z) := \alpha_t \cdot z$, $\forall t \in [0,T]$, $(y,z) \in \mathcal{R} \times \mathcal{R}^d$. Then, for any $\xi \in \mathcal{L}(\mathcal{F}_T)$, by the well-known Girsanov theorem, we can deduce that

$$\mathcal{E}^{g^\alpha}_{0,T}[\xi] = E_{P_\alpha}[\xi].$$

Since, for any $(y,z) \in \mathcal{R} \times \mathcal{R}^d$, $\alpha_t \cdot z = g^\alpha(t,y,z) \leq g(t,y,z)$, $\mathrm{d}P \times \mathrm{d}t$-a.s., it follows from the well-known comparison theorem for BSDEs that $E_{P_\alpha}[\xi] = \mathcal{E}^{g^\alpha}_{0,T}[\xi] \leq \mathcal{E}^g_{0,T}[\xi]$. Hence $\Pi \subseteq \Lambda$.

Next let us prove that $\Lambda \subseteq \Pi$. For each $Q_\theta \in \Lambda$, since $E_{Q_\theta}[\cdot] \leq \mathcal{E}^g_{0,T}[\cdot]$, $\forall \xi, \eta \in L^2(\mathcal{F}_T)$, we have

$$E_{Q_\theta}[\xi + \eta] - E_{Q_\theta}[\eta] = E_{Q_\theta}[\xi] \leq \mathcal{E}^g_{0,T}[\xi]. \tag{3.19}$$

Denote $g^{\beta}(t,y,z) := \beta(t)|z|$, $\forall t \in [0,T]$, $(y,z) \in \mathcal{R} \times \mathcal{R}^d$. From Lemmas 3.1 and 3.2 and applying the well-known comparison theorem for BSDEs again, we have

$$\mathcal{E}^g_{0,T}[\xi] = \mathcal{E}_g[\xi] \leq \mathcal{E}_{g^{\beta}}[\xi]. \tag{3.20}$$

From (3.19) and (3.20), we can deduce that $E_{Q_\theta}[\xi + \eta] - E_{Q_\theta}[\eta] \leq \mathcal{E}_{g^{\beta}}[\xi]$. Then, in a similar manner to Theorem 7.1 in Coquet *et al.* [20], we know that there exists a unique function g^{θ} defined on $\Omega \times [0,T] \times \mathcal{R} \times \mathcal{R}^d$ satisfying the following three conditions:

(H.1) $g^{\theta}(t,y,0) = 0$, $\mathrm{d}P \times \mathrm{d}t$-a.s., $\forall y \in \mathcal{R}$;

(H.2) $|g^{\theta}(t,y_1,z_1) - g^{\theta}(t,y_2,z_2)| \leq \beta(t)|z_1 - z_2|$, $\forall (y_1,z_1),(y_2,z_2) \in \mathcal{R} \times \mathcal{R}^d$, where $\beta(t)$ is a non-negative deterministic function satisfying that $\int_0^T \beta^2(t)\,\mathrm{d}t < \infty$;

(H.3) $\mathcal{E}_{g^{\theta}}[\xi|\mathcal{F}_t] = E_{Q_\theta}[\xi|\mathcal{F}_t]$ a.s., $\forall \xi \in L^2(\mathcal{F}_T)$.

It follows from the linearity of $(\mathcal{E}_{g^{\theta}}[\cdot|\mathcal{F}_t])_{t\in[0,T]}$ and Theorem 3.2 that g^{θ} is linear with respect to z. Therefore, there exists a $\mathcal{R}^d$-valued progressively measurable process $(\theta_t)_{t\in[0,T]}$ such that $g^{\theta}(t,y,z) = \theta_t \cdot z$, $\mathrm{d}P \times \mathrm{d}t$-a.s., $\forall (y,z) \in \mathcal{R} \times \mathcal{R}^d$. In view of $Q_\theta \in \Lambda$ and (H.3), we have for each $\xi \in L^2(\mathcal{F}_T)$, $\mathcal{E}_{g^{\theta}}[\xi] = E_{Q_\theta}[\xi] \leq \mathcal{E}^g_{0,T}[\xi]$. Then in a similar manner to Lemma 4.5 in Coquet *et al.* [20] and by Lemma 3.4, we can obtain $g^{\theta}(t,y,z) = \theta_t \cdot z \leq g(t,y,z)$, $\mathrm{d}P \times \mathrm{d}t$-a.s., $\forall (y,z) \in \mathcal{R} \times \mathcal{R}^d$. For θ, we define the probability measure P_θ satisfying $\frac{\mathrm{d}P_\theta}{\mathrm{d}P} = \exp(\int_0^T \theta_t \cdot \mathrm{d}B_t - \frac{1}{2}\int_0^T |\theta_t|^2\,\mathrm{d}t)$, then $P_\theta \in \Pi$ and $E_{P_\theta}[\xi] = \mathcal{E}_{g^{\theta}}[\xi] = E_{Q_\theta}[\xi]$, $\forall \xi \in L^2(\mathcal{F}_T)$. Hence, $Q_\theta = P_\theta \in \Pi$. Thus, $\Lambda \subseteq \Pi$. Therefore, we have $\Pi = \Lambda$.

Finally, we prove that, for any $s,t \in [0,T]$ satisfying $s \leq t$ and $\xi \in \mathcal{L}(\mathcal{F}_t)$, $\mathcal{E}^g_{s,t}[\xi] = \sup_{Q_\theta \in \Lambda} E_{Q_\theta}[\xi|\mathcal{F}_s]$ a.s. It follows from (H.3), the well-known comparison theorem for BSDEs, and Proposition 3.3 that

$$\mathcal{E}^g_{s,t}[\xi] \geq \mathcal{E}_{g^{\theta}}[\xi|\mathcal{F}_s] = E_{Q_\theta}[\xi|\mathcal{F}_s] \quad \text{a.s.}, \forall \xi \in \mathcal{L}(\mathcal{F}_t).$$

Hence, for any $s,t \in [0,T]$ satisfying $s \leq t$ and $\xi \in \mathcal{L}(\mathcal{F}_t)$,

$$\mathcal{E}^g_{s,t}[\xi] \geq \sup_{Q_\theta \in \Lambda} E_{Q_\theta}[\xi|\mathcal{F}_s] \quad \text{a.s.} \tag{3.21}$$

On the other hand, by Lemmas 3.1, 3.2, and 3.3, we can deduce that g is independent of y and positively homogeneous, sub-additive with respect to z. For any $\xi \in \mathcal{L}(\mathcal{F}_T)$, let $(Y^{\xi}_t, Z^{\xi}_t)_{t\in[0,T]}$ denote the solution of the following BSDE:

$$y_t = \xi + \int_t^T g(s,z_s)\,\mathrm{d}s - \int_t^T z_s \cdot \mathrm{d}B_s, \quad \forall t \in [0,T].$$

By a measurable selection theorem (*cf.*, *e.g.*, El Karoui and Quenez [21], p.215), we can deduce that there exists a progressively measurable process $\alpha^{\xi} \in \Theta^g$ such that

$$g(t, Z^{\xi}_t) = \alpha^{\xi}_t \cdot Z^{\xi}_t, \quad \mathrm{d}P \times \mathrm{d}t\text{-a.s.} \tag{3.22}$$

From (3.22) and applying the well-known Girsanov theorem, we have $\mathcal{E}^g_{s,t}[\xi] = \mathcal{E}^g_{s,T}[\xi] = E_{P_{\alpha^{\xi}}}[\xi|\mathcal{F}_s]$ a.s. Hence, for any $\xi \in \mathcal{L}(\mathcal{F}_t)$,

$$\mathcal{E}^g_{s,t}[\xi] \leq \sup_{P_\alpha \in \Pi} E_{P_\alpha}[\xi|\mathcal{F}_s] = \sup_{Q_\theta \in \Lambda} E_{Q_\theta}[\xi|\mathcal{F}_s] \quad \text{a.s.} \tag{3.23}$$

It follows from (3.21) and (3.23) that

$$\mathcal{E}^g_{s,t}[\xi] = \sup_{Q_\theta \in \Lambda} E_{Q_\theta}[\xi|\mathcal{F}_s] \quad \text{a.s.}, \forall \xi \in \mathcal{L}(\mathcal{F}_t).$$

The proof of Theorem 3.4 is complete. □

4 Hölder's inequality and Minkowski's inequality for g-evaluations

In this section, we give a sufficient condition on g under which Hölder's inequality and Minkowski's inequality for g-evaluations hold true.

First, we give the following lemma.

Lemma 4.1 *Suppose that the function g satisfies* (B.1) *and* (B.2). *Let g satisfy the following conditions*:

(i) *for any $y_1 \geq 0$, $y_2 \geq 0$, and $(z_1, z_2) \in \mathcal{R}^d \times \mathcal{R}^d$,*

$$g(t, y_1 + y_2, z_1 + z_2) \leq g(t, y_1, z_1) + g(t, y_2, z_2), \quad \mathrm{d}P \times \mathrm{d}t\text{-a.s.};$$

(ii) *for any $\lambda \geq 0$, $y \geq 0$, and $z \in \mathcal{R}^d$,*

$$g(t, \lambda y, \lambda z) \leq \lambda g(t, y, z), \quad \mathrm{d}P \times \mathrm{d}t\text{-a.s.},$$

then $\mathcal{E}^g_{s,t}[\cdot]$ satisfies the following conditions:

(j) $\mathcal{E}^g_{s,t}[\xi + \eta] \leq \mathcal{E}^g_{s,t}[\xi] + \mathcal{E}^g_{s,t}[\eta]$ *a.s., for any $(\xi, \eta) \in \mathcal{L}_+(\mathcal{F}_t) \times \mathcal{L}_+(\mathcal{F}_t)$;*

(k) $\mathcal{E}^g_{s,t}[\lambda\xi] = \lambda \mathcal{E}^g_{s,t}[\xi]$ *a.s., for any $\xi \in \mathcal{L}_+(\mathcal{F}_t)$ and $\lambda \geq 0$.*

The key idea of the proof of Lemma 4.1 is the well-known comparison theorem for BSDEs. The proof is very similar to that of Proposition 4.2 in Jia [26]. So we omit it.

Applying Lemma 4.1 and Theorems 2.3 and 2.4, we immediately have the following Hölder inequality and Minkowski inequality for g-evaluations.

Theorem 4.1 *Let g satisfy the conditions of Lemma* 4.1, *then, for any $X, Y \in \mathcal{L}(\mathcal{F}_t)$ and $|X|^p, |Y|^q \in \mathcal{L}(\mathcal{F}_t)$ ($p, q > 1$ and $1/p + 1/q = 1$), we have*

$$\mathcal{E}^g_{s,t}\big[|XY|\big] \leq \big(\mathcal{E}^g_{s,t}\big[|X|^p\big]\big)^{\frac{1}{p}} \big(\mathcal{E}^g_{s,t}\big[|Y|^q\big]\big)^{\frac{1}{q}} \quad \textit{a.s.}$$

Theorem 4.2 *Let g satisfy the conditions of Lemma* 4.1, *then, for any $X, Y \in \mathcal{L}(\mathcal{F}_t)$, and $|X|^p, |Y|^p \in \mathcal{L}(\mathcal{F}_t)$ ($p > 1$), we have*

$$\big(\mathcal{E}^g_{s,t}\big[|X + Y|^p\big]\big)^{\frac{1}{p}} \leq \big(\mathcal{E}^g_{s,t}\big[|X|^p\big]\big)^{\frac{1}{p}} + \big(\mathcal{E}^g_{s,t}\big[|Y|^p\big]\big)^{\frac{1}{p}} \quad \textit{a.s.}$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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