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An identity involving the mean value of two-term character sums

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Northwest University, Xi'an, Shaanxi,
P.R. China**Abstract**

The main purpose of this paper is, using the properties of Gauss sums and the estimate for character sums, to study the mean value problem of the two-term character sums and give an interesting identity for it.

MSC: 11M20**Keywords:** two-term character sums; mean value; identity; analytic method; Gauss sums**1 Introduction**

Let $q \geq 3$ be an integer, χ be a non-principal character mod q . For any integer n with $(n, q) = 1$, the two-term character sums $N(n, r, s, \chi; q)$ are defined as

$$N(n, r, s, \chi; q) = \sum_{a=1}^q \chi(a^r + na^s),$$

where $r > s$ are two fixed positive integers. These sums play a very important role in the study of analytic number theory, so they caused many number theorists' interest and favor. Some works related to $N(n, r, s, \chi; q)$ can be found in [1–4]. In fact, the sums $N(n, r, s, \chi; q)$ is a special case of the general character sums of the polynomials

$$\sum_{a=N+1}^{N+M} \chi(f(a)), \quad (1)$$

where M and N are any positive integers, and $f(x)$ is a polynomial. If $q = p$ is an odd prime, then Weil (see [1]) obtained the following important conclusion: Let χ be a q th-order character mod p . If $f(x)$ is not a perfect q th power mod p , then we have the estimate

$$\sum_{x=N+1}^{N+M} \chi(f(x)) \ll p^{\frac{1}{2}} \ln p, \quad (2)$$

where ' \ll ' constant depends only on the degree of $f(x)$.

Now we are concerned about whether there exists a computational formula for the mean value

$$\sum_{\chi \bmod q}^* \left| \sum_{a=1}^q \chi(a^r + na^s) \right|^{2k}, \quad (3)$$

where $s > r \geq 1$ are two integers. In this paper, we shall use the analytic method and the properties of Gauss sums to study this problem, and give an exact computational formula for (3). That is, we shall prove the following theorem.

Theorem 1 *Let $q > 1$ be an odd integer. Then, for any real number k and integer n with $(n, q) = 1$, we have the identity*

$$\sum_{\chi \bmod q}^* \left| \sum_{a=1}^q \chi(a^2 + na) \right|^{2k} = J(q) \cdot q^k \cdot \prod_{p \parallel q} \left(1 - \frac{1}{p-2} + \frac{1}{p^k(p-2)} \right),$$

where $\sum_{\chi \bmod q}^*$ denotes the summation over all primitive characters mod q , $p^\alpha \parallel q$ denotes that $p^\alpha | q$ and $p^{\alpha+1} \nmid q$, and $J(q) = \sum_{d|q} \mu(d) \phi\left(\frac{q}{d}\right)$ denotes the number of all primitive characters mod q .

Theorem 2 *Let $q > 1$ be an odd integer. Then, for any real number k and integer n with $(n, q) = 1$, we have the identity*

$$\begin{aligned} \sum_{\chi \bmod q}^* \left| \sum_{a=1}^q \chi(a^3 + na^2) \right|^{2k} &= J(q) \cdot q^k \cdot \prod_{p \parallel q, 3 \nmid p-1} \left(1 - \frac{1}{p-2} + \frac{1}{p^k(p-2)} \right) \\ &\quad \times \prod_{p \parallel q, 3 \mid p-1} \left(1 - \frac{3}{p-2} + \frac{3}{p^k(p-2)} \right). \end{aligned}$$

From these two theorems we may immediately deduce the following corollaries.

Corollary 1 *Let q be an odd square-full number (that is, for any prime p , if $p|q$, then $p^2|q$). Then, for any real number k and integer n with $(n, q) = 1$, we have the identity*

$$\sum_{\chi \bmod q}^* \left| \sum_{a=1}^q \chi(a^2 + na) \right|^{2k} = \sum_{\chi \bmod q}^* \left| \sum_{a=1}^q \chi(a^3 + na^2) \right|^{2k} = J(q) \cdot q^k.$$

Corollary 2 *Let q be an odd square-free number (that is, for any prime p , $p^2 \nmid q$). Then, for any real number k and integer n with $(n, q) = 1$, we have the identity*

$$\sum_{\chi \bmod q}^* \left| \sum_{a=1}^q \chi(a^2 + na) \right|^{2k} = q^k \cdot \prod_{p|q} \left(p - 3 + \frac{1}{p^k} \right).$$

Corollary 3 *Let q be an odd square-full number. Then, for any integer n with $(n, q) = 1$, we have the identity*

$$\sum_{\chi \bmod q}^* \left| \sum_{a=1}^q \chi(a^2 + na) \right|^{-2} = \sum_{\chi \bmod q}^* \left| \sum_{a=1}^q \chi(a^3 + na^2) \right|^{-2} = \frac{J(q)}{q}.$$

For general characters $\chi \pmod q$, whether there exists an identity for

$$\sum_{\chi \pmod q} \left| \sum_{a=1}^q \chi(a^r + na^s) \right|^{2k}$$

is an interesting open problem, where $r > s$ are two positive integers.

2 Several lemmas

In this section, we shall give several lemmas, which are necessary in the proof of our theorems. Hereinafter, we shall use many properties of character sums and Gauss sums, all of which can be found in references [5] and [6], so they will not be repeated here. First we have the following lemmas.

Lemma 1 *Let $q > 1$ be an integer, $J(q)$ denotes the number of all primitive characters $\pmod q$. Then we have the identity*

$$J(q) = \sum_{d|q} \mu(d) \cdot \phi\left(\frac{q}{d}\right),$$

where $\mu(n)$ is the Möbius function, $\phi(n)$ is the Euler function.

Proof This is a well-known result, here we give a simple proof. It is clear that for any non-principal character $\chi \pmod q$, there exists one and only one $d|q$ and a primitive character $\chi^* \pmod d$ such that $\chi = \chi^* \cdot \chi_0$, where χ_0 denotes the principal character $\pmod q$. So, from these properties, we have

$$\phi(q) = \sum_{\chi \pmod q} 1 = \sum_{d|q} J(d).$$

From this identity and the Möbius inversion formula, we may immediately deduce

$$J(q) = \sum_{d|q} \mu(d) \cdot \phi\left(\frac{q}{d}\right).$$

This proves Lemma 1. □

Lemma 2 *Let p be an odd prime, $i \geq 2$ be an integer and $q = p^i$. Then, for any real number k and integer n with $(n, q) = 1$, we have the identities*

$$\sum_{\chi \pmod q}^* \left| \sum_{a=1}^q \chi(a^2 + na) \right|^{2k} = \sum_{\chi \pmod q}^* q^k = \phi^2(q) \cdot q^{k-1}$$

and

$$\sum_{\chi \pmod p}^* \left| \sum_{a=1}^p \chi(a^2 + na) \right|^{2k} = 1 + (p-3) \cdot p^k.$$

Proof For any primitive character $\chi \pmod q$, from the properties of Gauss sums, we know that

$$\begin{aligned} \sum_{a=1}^q \chi(a^2 + na) &= \frac{1}{\tau(\bar{\chi})} \sum_{a=1}^q \sum_{b=1}^q \bar{\chi}(b) e\left(\frac{b(a^2 + na)}{q}\right) \\ &= \frac{1}{\tau(\bar{\chi})} \sum_{a=1}^q \sum_{b=1}^q \bar{\chi}(b\bar{a}) e\left(\frac{b\bar{a}(a^2 + na)}{q}\right) \\ &= \frac{1}{\tau(\bar{\chi})} \sum_{b=1}^q \bar{\chi}(b) e\left(\frac{nb}{q}\right) \sum_{a=1}^q \chi(a) e\left(\frac{ba}{q}\right) \\ &= \frac{\tau(\chi)}{\tau(\bar{\chi})} \sum_{b=1}^q \bar{\chi}^2(b) e\left(\frac{nb}{q}\right) \\ &= \chi^2(n) \frac{\tau(\chi) \cdot \tau(\bar{\chi}^2)}{\tau(\bar{\chi})}. \end{aligned} \tag{4}$$

Note that $q = p^i$ and $i \geq 2$, so if χ is a primitive character mod q , then $\bar{\chi}^2$ is also a primitive character mod q , so that $|\tau(\chi)| = |\tau(\bar{\chi})| = |\tau(\bar{\chi}^2)| = \sqrt{q}$. From these identities, Lemma 1 and formula (4), we may immediately deduce that

$$\sum_{\chi \pmod q}^* \left| \sum_{a=1}^q \chi(a^2 + na) \right|^{2k} = \sum_{\chi \pmod q}^* q^k = \phi^2(q) \cdot q^{k-1}.$$

This proves the first formula of Lemma 2.

Now we prove the second formula. If χ is the Legendre symbol mod p , then we have $|\tau(\bar{\chi}^2)| = 1$. So, from Lemma 1 and the method of proving (4), we have

$$\sum_{\chi \pmod p}^* \left| \sum_{a=1}^p \chi(a^2 + na) \right|^{2k} = 1 + (p-3)p^k.$$

This completes the proof of Lemma 2. □

Lemma 3 *Let $p > 3$ be an odd prime, $\alpha \geq 2$ be an integer and $q = p^\alpha$. Then, for any real number k and n with $(n, q) = 1$, we have the identities*

$$\sum_{\chi \pmod q}^* \left| \sum_{a=1}^q \chi(a^3 + na^2) \right|^{2k} = \phi^2(q) \cdot q^{k-1}$$

and

$$\sum_{\chi \pmod p}^* \left| \sum_{a=1}^p \chi(a^3 + na^2) \right|^{2k} = \begin{cases} 3 + (p-5) \cdot p^k & \text{if } p \equiv 1 \pmod 3; \\ 1 + (p-3) \cdot p^k & \text{if } (3, p-1) = 1. \end{cases}$$

Proof From the properties of primitive characters mod q and the method of proving Lemma 2, we have

$$\begin{aligned} \sum_{a=1}^q \chi(a^3 + na^2) &= \frac{1}{\tau(\bar{\chi})} \sum_{a=1}^q \sum_{b=1}^q \bar{\chi}(b) e\left(\frac{b(a^3 + na^2)}{q}\right) \\ &= \frac{1}{\tau(\bar{\chi})} \sum_{b=1}^q \bar{\chi}(b) e\left(\frac{nb}{q}\right) \sum_{a=1}^q \chi^2(a) e\left(\frac{ba}{q}\right) \\ &= \chi^3(n) \frac{\tau(\chi^2) \cdot \tau(\bar{\chi}^3)}{\tau(\bar{\chi})}. \end{aligned} \tag{5}$$

Since χ is a primitive character mod q and $(3, q) = 1$, so χ^2 and $\bar{\chi}^3$ are also two primitive characters mod q . So, from Lemma 1, (5) and the properties of Gauss sums, we can deduce the first identity of Lemma 3.

Note that if $(3, p - 1) = 1$, χ_1 is the Legendre symbol mod p , then for any non-principal character χ mod p , we have $|\tau(\bar{\chi}^3)| = \sqrt{p}$ and $|\tau(\chi_1^2)| = 1$. So, from Lemma 1 and the method of proving (5), we have

$$\sum_{\chi \bmod p}^* \left| \sum_{a=1}^p \chi(a^3 + na^2) \right|^{2k} = 1 + (p - 3)p^k. \tag{6}$$

If $3|p - 1$, then there exist two 3-order characters mod p , so from the method of proving (6), we have the identity

$$\sum_{\chi \bmod p}^* \left| \sum_{a=1}^p \chi(a^3 + na^2) \right|^{2k} = 3 + (p - 5)p^k. \tag{7}$$

Combining (6) and (7), we can deduce the second identity of Lemma 3. □

Lemma 4 Let $q_1 > 1$ and $q_2 > 1$ be two integers with $(q_1, q_2) = 1$, $\chi_1 \bmod q_1$ and $\chi_2 \bmod q_2$. Then, for any integer n with $(n, q_1 q_2) = 1$, we have

$$\left| \sum_{a=1}^{q_1 q_2} \chi_1 \chi_2(a^2 + na) \right| = \left| \sum_{a=1}^{q_1} \chi_1(a^2 + na) \right| \cdot \left| \sum_{b=1}^{q_2} \chi_1(b^2 + nb) \right|.$$

Proof From the properties of complete residue system $q_1 q_2$, we have

$$\begin{aligned} &\left| \sum_{a=1}^{q_1 q_2} \chi_1 \chi_2(a^2 + na) \right| \\ &= \left| \sum_{a=1}^{q_1} \sum_{b=1}^{q_2} \chi_1 \chi_2((aq_2 + bq_1)^2 + n(aq_2 + bq_1)) \right| \\ &= \left| \sum_{a=1}^{q_1} \chi_1((aq_2 + bq_1)^2 + n(aq_2 + bq_1)) \sum_{b=1}^{q_2} \chi_2((aq_2 + bq_1)^2 + n(aq_2 + bq_1)) \right| \\ &= \left| \sum_{a=1}^{q_1} \chi_1((aq_2)^2 + n(aq_2)) \sum_{b=1}^{q_2} \chi_2((bq_1)^2 + n(bq_1)) \right| \end{aligned}$$

$$\begin{aligned}
 &= \left| \sum_{a=1}^{q_1} \chi_1(a^2 + na) \sum_{b=1}^{q_2} \chi_2(b^2 + nb) \right| \\
 &= \left| \sum_{a=1}^{q_1} \chi_1(a^2 + na) \right| \cdot \left| \sum_{b=1}^{q_2} \chi_1(b^2 + nb) \right|.
 \end{aligned}$$

This proves Lemma 4. □

3 Proof of the theorems

In this section, we shall complete the proof of our theorems. First we prove Theorem 1. For any odd number $q > 1$, it is clear that there exist two integers M and N such that $q = M \cdot N$, where M is a square-free number, and N is a square-full number. Now, for any primitive character $\chi \pmod q$, there exist two primitive characters $\chi_1 \pmod M$ and $\chi_2 \pmod N$ such that $\chi = \chi_1 \chi_2$. Note that $J(N) = \frac{\phi^2(N)}{N}$, so from these properties, Lemma 1, Lemma 2 and Lemma 4, we have

$$\begin{aligned}
 &\sum_{\chi \pmod q}^* \left| \sum_{a=1}^q \chi(a^2 + na) \right|^{2k} \\
 &= \left(\sum_{\chi \pmod M}^* \left| \sum_{a=1}^M \chi(a^2 + na) \right|^{2k} \right) \cdot \left(\sum_{\chi \pmod N}^* \left| \sum_{a=1}^N \chi(a^2 + na) \right|^{2k} \right) \\
 &= \prod_{p|M} (1 + (p-3) \cdot p^k) \prod_{p^\alpha \parallel N} (\phi^2(p^\alpha) \cdot p^{\alpha(k-1)}) \\
 &= J(q) \cdot q^k \cdot \prod_{p \parallel q} \left(1 - \frac{1}{p-2} + \frac{1}{p^k(p-2)} \right),
 \end{aligned}$$

where $p^\alpha \parallel q$ denotes that $p^\alpha | q$ and $p^{\alpha+1} \nmid q$. This proves Theorem 1.

Now we prove Theorem 2. From Lemma 3, Lemma 4 and the method of proving Theorem 1, we have

$$\begin{aligned}
 &\sum_{\chi \pmod q}^* \left| \sum_{a=1}^q \chi(a^3 + na^2) \right|^{2k} \\
 &= \left(\sum_{\chi \pmod M}^* \left| \sum_{a=1}^M \chi(a^3 + na^2) \right|^{2k} \right) \cdot \left(\sum_{\chi \pmod N}^* \left| \sum_{a=1}^N \chi(a^3 + na^2) \right|^{2k} \right) \\
 &= \prod_{p|M} (1 + (p-3) \cdot p^k) \prod_{p^\alpha \parallel N} (\phi^2(p^\alpha) \cdot p^{\alpha(k-1)}) \\
 &= J(q) \cdot q^k \cdot \prod_{p \parallel q, 3 \nmid p-1} \left(1 - \frac{1}{p-2} + \frac{1}{p^k(p-2)} \right) \\
 &\quad \times \prod_{p \parallel q, 3 \mid p-1} \left(1 - \frac{3}{p-2} + \frac{3}{p^k(p-2)} \right).
 \end{aligned}$$

This completes the proof of our theorems.

Competing interests

The authors did not provide this information.

Authors' contributions

The authors did not provide this information.

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