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# Explicit iterations for Lipschitzian semigroups with the Meir-Keeler type contraction in Banach spaces

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**Abstract**

In this paper, we introduce and prove strong convergence theorems for a new viscosity iteration scheme for approximating common fixed points of a Lipschitzian semigroup on a compact and convex subset of a smooth Banach space. Our results extend and improve recent results.
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**Keywords:** Meir-Keeler type mapping; Lipschitzian mapping; variational inequality; viscosity approximation method; fixed point

## 1 Introduction

Let $C$ be a nonempty, closed and convex subset of a Banach space $E$. Recall that a self mapping $f : E \to E$ is an *$\alpha$-contraction* on $C$ if there exists a constant $\alpha \in (0, 1)$ such that for any $x, y \in E$, we have

$$\|f(x) - f(y)\| \leq \alpha \|x - y\|.$$

A function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is said to be an *L-function* if $\psi(0) = 0$, $\psi(t) > 0$ for any $t > 0$, and for every $t > 0$ and $s > 0$, there exists $u > s$ such that $\psi(t) \leq s$ for all $t \in [s, u]$. This implies that $\psi(t) < t$ for all $t > 0$.

A mapping $f : E \to E$ is said to be a *$(\psi, L)$-contraction* if there exists an *L*-function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|f(x) - f(y)\| \leq \psi \|x - y\|, \quad \forall x, y \in E \text{ with } x \neq y.$$

If $\psi(t) = kt$ for all $t > 0$, where $k \in (0, 1)$, then $f$ is a *contraction*.

A mapping $f$ is called a *Meir-Keeler type* mapping if for each $\epsilon > 0$, there exists $\delta(\epsilon) > 0$ such that for all $x, y \in E$, if $\epsilon < \|x - y\| < \epsilon + \delta$, then $\|f(x) - f(y)\| < \epsilon$.

A mapping $T : C \to C$ is said to be

(i) *nonexpansive* if

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C;$$

(ii) *Lipschitzian* with a Lipschitz constant $l > 0$ if

$$\|Tx - Ty\| \le l\|x - y\|, \quad \forall x, y \in C;$$

(iii) *asymptotically nonexpansive* if there exists a sequence $\{k_n\}$ of positive numbers satisfying the property $\lim_{n \to \infty} k_n = 1$ and

$$\left\|T^n x - T^n y\right\| \le k_n \|x - y\|, \quad \forall x, y \in C.$$

Every nonexpansive mappings are asymptotically nonexpansive with respect to the sequence $k_n = 1$, $n \in \mathbb{N}$. Also, every asymptotically nonexpansive mappings are uniformly $l$-Lipschitzian with $l = \sup_{n \in \mathbb{N}} k_n$.

Let $S$ be a semigroup. Then a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of $C$ into itself is called Lipschitzian mappings on $C$ if for each $s \in S$, the mapping $T_s$ is a Lipschitzian mapping on $C$ with a Lipschitz constant $k_s$, and $T_{st} = T_s T_t$ for all $s, t \in S$. A family $\mathcal{S}$ is called a *Lipschitzian semigroup* on $C$ if it satisfies the following:

1. $T_{st} x = T_s T_t x$ for all $s, t \in S$ and $x \in C$;
2. for each $s \in S$, $T_s$ is a Lipschitzian mapping of $C$ into itself, *i.e.*, there is $k_s \ge 0$ such that

$$\|T_s x - T_s y\| \le k_s \|x - y\| \quad \text{for all } x, y \in C.$$

A Lipschitzian semigroup $\mathcal{S}$ is called *uniformly l-Lipschitzian* if $k_s = l$ for all $s \in S$.

Let $F(\mathcal{S})$ denote the common fixed point set $\{x \in C : T_s x = x, \forall s \in S\}$ of the mappings $T_s$.

For a semigroup $S$, we define a partial preordering $\prec$ on $S$ by $a \prec b$ if and only if $aS \supset bS$. If $S$ is a *left reversible semigroup* (*i.e.*, $aS \cap bS \ne \emptyset$ for $a, b \in S$), then it is a directed set. (Indeed, for every $a, b \in S$, applying $aS \cap bS \ne \emptyset$, there exist $a', b' \in S$ with $aa' = bb'$; by taking $c = aa' = bb'$, we have $cS \subseteq aS \cap bS$, and then $a \prec c$ and $b \prec c$.)

Let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of a left reversible semigroup $S$ as Lipschitzian mappings on $C$ with Lipschitz constants $\{k_s : s \in S\}$. We shall say that $\mathcal{S}$ is an *asymptotically nonexpansive semigroup* on $C$, if there holds the uniform Lipschitzian condition $\lim_s k_s \le 1$ on the Lipschitz constants.

In 1953, Mann [1] introduced an iterative method as follows: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{1.1}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a sequence of real numbers in $[0, 1]$. The Mann iteration can guarantee in general only weak convergence. The Mann iteration has been extensively investigated for nonexpansive mappings and modified for strong convergence. Later, in 2000, Reich and Zaslavski [2] introduced the Krasnoselskii-Mann iterations of a generic nonexpansive operator on a closed and convex, but not necessarily bounded, subset of a hyperbolic space with a unique fixed point.

In 1967, Halpern [3] considered the following algorithm:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) T x_n, \tag{1.2}$$

where $T$ is nonexpansive and the initial guess element $x \in C$ is arbitrary.

In 2004, Xu [4] introduced and proved the following viscosity approximation methods for nonexpansive mappings in a uniformly smooth Banach space:

$$x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \tag{1.3}$$

where $f : C \to C$ is a contraction mapping.

In 2007, Lau, Miyake and Takahashi [5] introduced the following Mann's implicit iteration process:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T(\mu_n)x_n, \tag{1.4}$$

for a semigroup $\mathcal{S} = \{T_s : s \in S\}$ of nonexpansive mappings on a compact and convex subset $C$ of a smooth and strictly convex Banach space.

In the same year, Zhang *et al.* [6] introduced the following composite iteration scheme:

$$\begin{cases} y_n = \beta_n x_n + (1 - \beta_n)T(t_n)x_n, \\ x_{n+1} = \alpha_n x + (1 - \alpha_n)y_n, \end{cases} \tag{1.5}$$

where $\{T(t) : t \geq 0\}$ is a nonexpansive semigroup, $x$ is an arbitrary point in $C$. Under a suitable condition, they proved strong convergence theorems of an explicit composite iteration scheme for nonexpansive semigroups in a reflexive Banach space with a uniformly Gâteaux differentiable norm, uniformly smooth Banach space and uniformly convex Banach space with a weakly continuous normalized duality mapping.

In 2008, Shahram Saeidi [7] introduced the following viscosity iterative scheme:

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n)x_n, \tag{1.6}$$

for a representation of $S$ as Lipschitzian mappings on a compact and convex subset $C$ of a smooth Banach space $E$ with respect to a left regular sequence $\{\mu_n\}$ of means defined on an appropriate invariant subspace of $l^\infty(S)$; for some related results, we refer the readers to [8, 9].

Motivated and inspired by the idea of Zhang *et al.* [6] and Saeidi [7], we introduce the explicit viscosity iterative process by Meir-Keeler type contraction in a smooth Banach space. Then we prove that the sequence $\{x_n\}$ converges strongly to a common fixed point of $\mathcal{S} = \{T_s : s \in S\}$, where $S$ is a left reversible semigroup, which is the unique solution of the variational inequality

$$\langle (f - I)q, J(p - q) \rangle \leq 0, \quad \forall p \in F(\mathcal{S}).$$

## 2 Preliminaries

Let $E$ be a Banach space and let $E^*$ be the topological dual of $E$. The value of $x^* \in E^*$ at $x \in E$ will be denoted by $\langle x, x^* \rangle$ or $x^*(x)$. With each $x \in E$, we associate the set

$$J(x) = \left\{ x^* \in E^* : \langle x, x^* \rangle = \|x^*\|^2 = \|x\|^2 \right\}.$$

Using the Hahn-Banach theorem, it immediately follows that $J(x) \neq \emptyset$ for each $x \in E$. A Banach space $E$ is said to be smooth if the duality mapping $J$ of $E$ is single-valued. We

know that if $E$ is smooth, then $J$ is norm-to-weak$^*$ continuous; see [8, 9]. Let $E$ be a Banach space and let $C$ be a closed and convex subset of $E$. Then

$$\|x - y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x - y, y \rangle, \tag{2.1}$$

and

$$\left\| \lambda x + (1 - \lambda)y \right\|^2 = \lambda \|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \tag{2.2}$$

for all $x, y \in E$ and $\lambda \in [0, 1]$.

Let $S$ be a semigroup. We denote by $l^\infty(S)$ the Banach space of all bounded real valued functions on $S$ with a supremum norm. For each $s \in S$, we define $l_s$ and $r_s$ on $l^\infty(S)$ by $(l_s g)(t) = g(st)$ and $(r_s g)(t) = g(ts)$ for each $t \in S$ and $g \in l^\infty(S)$. Let $X$ be a subspace of $l^\infty(S)$ containing 1 and let $X^*$ be its topological dual. An element $\mu$ of $X^*$ is said to be a *mean* on $X$ if $\|\mu\| = \mu(1) = 1$. We often write $\mu_t(g(t))$ instead of $\mu(g)$ for $\mu \in X^*$ and $g \in X$. Let $X$ be *left invariant* (resp. right invariant), *i.e.*, $l_s(X) \subset X$ (resp. $r_s(X) \subset X$) for each $s \in S$. A mean $\mu$ on $X$ is said to be *left invariant* (resp. right invariant) if $\mu(l_s g) = \mu(g)$ (resp. $\mu(r_s g) = \mu(g)$) for each $s \in S$ and $g \in X$. A subspace $X$ is said to be *left* (resp. right) *amenable* if $X$ has a left (resp. right) invariant mean. A semigroup $X$ is *amenable* if $X$ is both left and right amenable. If a semigroup $S$ is left amenable, then $S$ is left reversible [10, 11].

A net $\{\mu_\alpha\}$ of means on $X$ is said to be *strongly left regular* if

$$\lim_\alpha \left\| l_s^* \mu_\alpha - \mu_\alpha \right\| = 0$$

for each $s \in S$, where $l_s^*$ is the adjoint operator of $l_s$. Let $C$ be a nonempty, closed and convex subset of $E$. Throughout this paper, $S$ will always denote a semigroup with an identity $e$. $S$ is called *left reversible* if any two right ideals in $S$ have nonvoid intersection, *i.e.*, $aS \cap bS \neq \emptyset$ for any $a, b \in S$. In this case, we can define a partial ordering $\prec$ on $S$ by $a \prec b$ if and only if $aS \supset bS$. It is easy to see $t \prec ts$ (for all $t, s \in S$). Further, if $t \prec s$, then $pt \prec ps$ for all $p \in S$. If a semigroup $S$ is left amenable, then $S$ is left reversible. But the converse is false. Denote by $C_a$ the set of almost periodic elements in $C$, *i.e.*, all $x \in C$ such that $\mathcal{S} = \{T_s x : s \in S\}$ is relatively compact in the norm topology of $E$. We will call a subspace $X$ of $l^\infty(S)$, $\mathcal{S}$-*stable* if the functions $s \mapsto \langle T_s x, x^* \rangle$ and $s \mapsto \|T_s x - y\|$ on $S$ are in $X$ for all $x, y \in C$ and $x^* \in E^*$. We know that if $\mu$ is a mean on $X$ and if for each $x^* \in E^*$, the function $s \mapsto \langle T_s x, x^* \rangle$ is contained in $X$ and $C$ is weakly compact, then there exists a unique point $x_0$ of $E$ such that

$$\mu_s \langle T_s x, x^* \rangle = \langle x_0, x^* \rangle$$

for each $x^* \in E^*$. We denote such a point $x_0$ by $T(\mu)x$. Note that $T(\mu)z = z$ for each $z \in F(\mathcal{S})$; see [12–14]. Let $D$ be a subset of $B$ where $B$ is a subset of a Banach space $E$ and let $P$ be a retraction of $B$ onto $D$. Then $P$ is said to be sunny [15] if for each $x \in B$ and $t \geq 0$ with $Px + t(x - Px) \in B$,

$$P(Px + t(x - Px)) = Px.$$

A subset $D$ is said to be a sunny nonexpansive retract of $B$ if there exists a sunny nonexpansive retraction $P$ of $B$ onto $D$. If $E$ is smooth and $P$ is a retraction of $B$ onto $D$, then $P$

is sunny and nonexpansive if and only if for each $x \in B$ and $z \in D$,

$$\langle x - Px, J(z - Px) \rangle \leq 0. \tag{2.3}$$

For more details, see [8, 9].

**Lemma 2.1** ([16]) *Let S be a left reversible semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty, weakly compact and convex subset C of a Banach space E into C, with the uniform Lipschitzian condition $\lim_s k_s \leq 1$ on the Lipschitz constants of the mappings. Let X be a left invariant $\mathcal{S}$-stable subspace of $l^\infty(S)$ containing 1, and $\mu$ be a left invariant mean on X. Then $F(\mathcal{S}) = F(T(\mu)) \cap C_a$.*

**Corollary 2.2** ([7]) *Let $\{\mu_n\}$ be an asymptotically left invariant sequence of means on X. If $z \in C_a$ and $\liminf_{n \to \infty} \|T(\mu_n)z - z\| = 0$, then z is a common fixed point of $\mathcal{S}$.*

**Lemma 2.3** ([7]) *Let S be a left reversible semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty weakly compact and convex subset C of a Banach space E into C, with the uniform Lipschitzian condition $\lim_s k_s \leq 1$ on the Lipschitz constants of the mappings. Let X be a left invariant subspace of $l^\infty(S)$ containing 1 such that the mappings $s \mapsto \langle T_s x, x^* \rangle$ be in X for all $x \in X$ and $x^* \in E^*$, and $\{\mu_n\}$ be a strongly left regular sequence of means on X. Then*

$$\limsup_{n \to \infty} \sup_{x,y \in C} \left( \|T(\mu_n)x - T(\mu_n)y\| - \|x - y\| \right) \leq 0.$$

**Remark 2.4** From Lemma 2.3, taking

$$c_n = \sup_{x,y \in C} \left( \|T(\mu_n)x - T(\mu_n)y\| - \|x - y\| \right), \quad \forall n, \tag{2.4}$$

we obtain that $\limsup_{n \to \infty} c_n \leq 0$. Moreover,

$$\|T(\mu_n)x - T(\mu_n)y\| \leq \|x - y\| + c_n, \quad \forall x, y \in C. \tag{2.5}$$

**Corollary 2.5** ([7]) *Let S be a left reversible semigroup and let $\mathcal{S} = \{T_s : s \in S\}$ be a representation of S as Lipschitzian mappings from a nonempty, compact and convex subset C of a Banach space E into C, with the uniform Lipschitzian condition $\lim_s k_s \leq 1$. Let X be a left invariant $\mathcal{S}$-stable subspace of $l^\infty(S)$ containing 1, and $\mu$ be a left invariant mean on X. Then $T(\mu)$ is nonexpansive and $F(\mathcal{S}) \neq \emptyset$. Moreover, if E is smooth, then $F(\mathcal{S})$ is a sunny nonexpansive retract of C and the sunny nonexpansive retraction of C onto $F(\mathcal{S})$ is unique.*

**Lemma 2.6** ([8, 9]) *Let E be a real Banach space. Then, for any given $x, y \in E$ and $j(x + y) \in J(x + y)$, the following inequality holds:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle.$$

**Lemma 2.7** ([17]) *Let $\{x_n\}$ and $\{y_n\}$ be two bounded sequences in a Banach space E and let $\{\beta_n\}$ be a sequence in $[0, 1]$ with $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$. Suppose that*

$x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ *for all integers* $n \geq 0$ *and* $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$.
*Then* $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

**Lemma 2.8** ([4]) *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1 - \sigma_n)a_n + \rho_n,$$

*where* $\{\alpha_n\}$ *is a sequence in* $(0, 1)$ *and* $\{\delta_n\}$ *is a sequence in* $\mathbb{R}$ *such that*
(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(2) $\limsup_{n\to\infty} \frac{\delta_n}{\alpha_n} \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.
*Then* $\lim_{n\to\infty} a_n = 0$.

**Lemma 2.9** ([18]) *Let* $(Y, d)$ *be a metric space and let* $f : Y \to Y$ *be a mapping. The following assertions are equivalent*:
1. *f is a Meir-Keeler type mapping*;
2. *there exists a L-function* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *such that f is a* $(\psi, L)$-*contraction*.

**Lemma 2.10** ([19]) *Let E be a Banach space and let C be a convex subset of E. Let* $T : C \to C$ *be a nonexpansive mapping and f be a* $(\psi, L)$-*contraction. Then the following assertions hold*:
1. $T \circ f$ *is a* $(\psi, L)$-*contraction on C and has a unique fixed point in C*;
2. *for each* $\alpha \in (0, 1)$, *the mapping* $x \to \alpha f(x) + (1 - \alpha)Tx$ *is of Meir-Keeler-type and it has a unique fixed point in C*.

**Lemma 2.11** ([19]) *Let E be a Banach space and let C be a convex subset of E. Let* $f : C \to C$ *be a Meir-Keeler-type contraction. Then for each* $\epsilon > 0$, *there exists* $r \in (0, 1)$ *such that, for each* $x, y \in C$ *with* $\|x - y\| \geq \epsilon$, *we have* $\|f(x) - f(y)\| \leq r\|x - y\|$.

## 3 Main results

In this paper, we suppose that $\psi$ from the definition of $(\psi, L)$-contraction is continuous, strictly increasing. Let $\eta(t) = t - \psi(t)$ for all $t \in \mathbb{R}^+$, we have that $\lim_{t\to\infty} \eta(t) = \infty$ and $\eta(t)$ is strictly increasing and onto. Consequently, we have that and $\eta(t)$ is a bijection on $\mathbb{R}^+$.

**Theorem 3.1** *Let C be a nonempty, compact and convex subset of a smooth Banach space E. Let S be a left reversible semigroup and* $\mathcal{S} = \{T_s : s \in S\}$ *be a representation of S as Lipschitzian mappings from C into itself, with the uniform Lipschitzian condition* $\lim_s k_s \leq 1$, *and f be a Meir-Keeler contraction of C into itself. Let X be a left invariant* $\mathcal{S}$-*stable subspace of* $l^\infty(S)$ *containing 1,* $\{\mu_n\}$ *be a strongly left regular sequence of means on X such that* $\lim_{n\to\infty} \|\mu_{n+1} - \mu_n\| = 0$ *and* $\{c_n\}$ *be the sequence defined in* (2.4) *with* $\limsup_{n\to\infty} c_n \leq 0$. *Suppose the sequences* $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ *and* $\{\delta_n\}$ *in* $(0, 1)$ *satisfy* $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. *The following conditions are satisfied*:
(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
(ii) $\lim_{n\to\infty} \delta_n = 0$;
(iii) $\limsup_{n\to\infty} \frac{c_n}{\alpha_n} \leq 0$;
(iv) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.

*For arbitrary $x_1 \in C$, generate a sequence $\{x_n\}$ by*

$$
\begin{cases}
y_n = \delta_n x_n + (1 - \delta_n) T(\mu_n) x_n, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n.
\end{cases}
\tag{3.1}
$$

*Then $\{x_n\}$ converges strongly to $q \in F(\mathcal{S})$, which is the unique solution of the variational inequality*

$$
\langle (f - I)q, J(p - q) \rangle \leq 0, \quad \forall p \in F(\mathcal{S}).
$$

*Equivalently, we have $q = Pfq$, where $P$ is the unique sunny nonexpansive retraction of $C$ onto $F(\mathcal{S})$.*

*Proof* First, we prove that $\{x_n\}$ is bounded. That is, if we take a point $p \in F(\mathcal{S})$, we will show that $\|x_n - p\| \leq M$ for all $n \in \mathbb{N}$. Let $p \in F(\mathcal{S})$. It is obvious that $M = \max\{\|x_1 - p\|, \eta^{-1}\|f(p) - p\|\}$. By induction we suppose that $M = \max\{\|x_k - p\|, \eta^{-1}\|f(p) - p\|\}$, where $k \in \mathbb{N}$. Since $T(\mu_k)$ is nonexpansive, we have

$$
\begin{aligned}
\|y_k - p\| &= \left\| \delta_n x_k + (1 - \delta_k) T(\mu_k) x_k - p \right\| \\
&\leq \delta_k \|x_k - p\| + (1 - \delta_k) \left\| T(\mu_k) x_k - p \right\| \\
&\leq \delta_k \|x_k - p\| + (1 - \delta_k) \left( \|x_k - p\| \right) \\
&\leq \|x_k - p\|.
\end{aligned}
\tag{3.2}
$$

Since $f$ is a Meir-Keeler contraction, we have that

$$
\begin{aligned}
\|x_{k+1} - p\| &= \left\| \alpha_k f(x_k) + \beta_k x_k + \gamma_k y_k - p \right\| \\
&= \left\| \alpha_k \left( f(x_k) - p \right) + \beta_k (x_k - p) + \gamma_n (y_k - p) \right\| \\
&\leq \alpha_k \left\| f(x_k) - p \right\| + \beta_k \|x_k - p\| + \gamma_k \|y_k - p\| \\
&\leq \alpha_k \left\| f(x_k) - f(p) \right\| + \alpha_k \left\| f(p) - p \right\| + \beta_k \|x_k - p\| + \gamma_k \|y_k - p\| \\
&\leq \alpha_k \psi \|x_k - p\| + \alpha_k \left\| f(p) - p \right\| + \beta_k \|x_k - p\| + \gamma_k \|y_k - p\| \\
&= \alpha_k \psi \|x_k - p\| + (\beta_k + \gamma_k) \|x_k - p\| + \alpha_k \eta \left( \eta^{-1} \right) \left\| f(p) - p \right\| \\
&= \alpha_k \psi \|x_k - p\| + (1 - \alpha_k) \|x_k - p\| + \alpha_k \eta \left( \eta^{-1} \right) \left\| f(p) - p \right\| \\
&\leq \alpha_k \psi (M) + (1 - \alpha_k)(M) + \alpha_k \eta (M) \\
&\leq \alpha_k \psi (M) + (1 - \alpha_k)(M) + \alpha_k \left( M - \psi (M) \right) \\
&= M.
\end{aligned}
$$

That is, $\{x_{k+1}\}$ is bounded. By induction we have that $\{x_n\}$ is bounded, and so are the sequences $\{f(x_n)\}$, $\{T(\mu_n)x_n\}$ and $\{y_n\}$. As $T(\mu_n)$ is bounded, we have $\{T_s x_n\}$ is also bounded. Denote $D = \sup_{s \in S} \|T_s x_n\|$, then it follows that

$$
\begin{aligned}
\left\| T(\mu_{n+1}) x_n - T(\mu_n) x_n \right\| &= \sup\left\{ \left| \langle T(\mu_{n+1}) x_n - T(\mu_n) x_n, z^* \rangle \right| : z^* \in E^*, \left\| z^* \right\| = 1 \right\} \\
&= \sup\left\{ \left| (\mu_{n+1})_s \langle T_s x_n, z^* \rangle - (\mu_n)_s \langle T_s x_n, z^* \rangle \right| : z^* \in E^*, \left\| z^* \right\| = 1 \right\}
\end{aligned}
$$

$$\leq \|\mu_{n+1} - \mu_n\| \sup_{s \in S} \|T_s x_n\| \|z^*\|$$

$$\leq \|\mu_{n+1} - \mu_n\| \sup_{s \in S} \|T_s x_n\|$$

$$= \|\mu_{n+1} - \mu_n\| D.$$

Since $\lim_{n \to \infty} \|\mu_{n+1} - \mu_n\| = 0$, we obtain that

$$\lim_{n \to \infty} \left\| T(\mu_{n+1})x_n - T(\mu_n)x_n \right\| = 0. \tag{3.3}$$

Next, we will show that $\lim_{n \to \infty} \|x_{n+1} - x_n\| = 0$ and by Lemma 2.3, we observe that

$$
\begin{aligned}
\|y_{n+1} - y_n\| &= \left\| \delta_{n+1}x_{n+1} + (1 - \delta_{n+1})T(\mu_{n+1})x_{n+1} - \left(\delta_n x_n + (1 - \delta_n)T(\mu_n)x_n\right) \right\| \\
&= \left\| \delta_{n+1}x_{n+1} - \delta_{n+1}x_n + \delta_{n+1}x_n + (1 - \delta_{n+1})T(\mu_{n+1})x_{n+1} \right. \\
&\qquad \left. - (1 - \delta_{n+1})T(\mu_n)x_n + (1 - \delta_{n+1})T(\mu_n)x_n - \delta_n x_n - (1 - \delta_n)T(\mu_n)x_n \right\| \\
&= \left\| \delta_{n+1}(x_{n+1} - x_n) + (\delta_{n+1} - \delta_n)x_n + (1 - \delta_{n+1})\left(T(\mu_{n+1})x_{n+1}\right.\right. \\
&\qquad \left.\left. - T(\mu_n)x_n\right) + (\delta_n - \delta_{n+1})T(\mu_n)x_n \right\| \\
&\leq \delta_{n+1}\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\left(\|x_n\| + \left\|T(\mu_n)x_n\right\|\right) \\
&\qquad + \left\| T(\mu_{n+1})x_{n+1} - T(\mu_n)x_n \right\| \\
&\leq \delta_{n+1}\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\left(\|x_n\| + \left\|T(\mu_n)x_n\right\|\right) \\
&\qquad + \left\| T(\mu_{n+1})x_{n+1} - T(\mu_n)x_{n+1} \right\| + \left\| T(\mu_n)x_{n+1} - T(\mu_n)x_n \right\| \\
&\leq \delta_{n+1}\|x_{n+1} - x_n\| + |\delta_{n+1} - \delta_n|\left(\|x_n\| + \left\|T(\mu_n)x_n\right\|\right) \\
&\qquad + \left\| T(\mu_{n+1})x_{n+1} - T(\mu_n)x_{n+1} \right\| + \|x_{n+1} - x_n\| + c_n.
\end{aligned}
$$

Setting $x_{n+1} = (1 - \beta_n)z_n + \beta_n x_n$, we see that

$$z_n = \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n}.$$

Then we compute

$$
\begin{aligned}
&\|z_{n+1} - z_n\| \\
&= \left\| \frac{x_{n+2} - \beta_{n+1}x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n y_n}{1 - \beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}f(x_{n+1}) + \gamma_{n+1}y_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_{n+1}f(x_n)}{1 - \beta_{n+1}} + \frac{\alpha_{n+1}f(x_n)}{1 - \beta_{n+1}} \right. \\
&\qquad \left. - \frac{\gamma_{n+1}y_n}{1 - \beta_{n+1}} + \frac{\gamma_{n+1}y_n}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + \gamma_n y_n}{1 - \beta_n} \right\| \\
&= \left\| \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\left(f(x_{n+1}) - f(x_n)\right) + \frac{\gamma_{n+1}}{1 - \beta_{n+1}}(y_{n+1} - y_n) \right.
\end{aligned}
$$

$$+ \left( \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right) f(x_n) + \left( \frac{\gamma_{n+1}}{1-\beta_{n+1}} - \frac{\gamma_n}{1-\beta_n} \right) y_n \Big\|$$

$$= \Big\| \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left( f(x_{n+1}) - f(x_n) \right) + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (y_{n+1} - y_n)$$

$$+ \left( \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right) f(x_n) + \left( \frac{1-\beta_{n+1}-\alpha_{n+1}}{1-\beta_{n+1}} - \frac{1-\beta_n-\alpha_n}{1-\beta_n} \right) y_n \Big\|$$

$$= \Big\| \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left( f(x_{n+1}) - f(x_n) \right) + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (y_{n+1} - y_n)$$

$$+ \left( \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right) f(x_n) + \left( \frac{-\alpha_{n+1}}{1-\beta_{n+1}} + \frac{\alpha_n}{1-\beta_n} \right) y_n \Big\|$$

$$= \Big\| \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left( f(x_{n+1}) - f(x_n) \right) + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (y_{n+1} - y_n)$$

$$+ \left( \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right) f(x_n) - \left( \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right) y_n \Big\|$$

$$= \Big\| \frac{\alpha_{n+1}}{1-\beta_{n+1}} \left( f(x_{n+1}) - f(x_n) \right) + \frac{\gamma_{n+1}}{1-\beta_{n+1}} (y_{n+1} - y_n)$$

$$+ \left( \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right) \left( f(x_n) - y_n \right) \Big\|$$

$$\leq \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} \right| \big\| f(x_{n+1}) - f(x_n) \big\| + \left| \frac{\gamma_{n+1}}{1-\beta_{n+1}} \right| \| y_{n+1} - y_n \|$$

$$+ \left| \left( \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right) \right| \big\| f(x_n) - y_n \big\|$$

$$= \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} \right| \big\| f(x_{n+1}) - f(x_n) \big\| + \left| 1 - \frac{\alpha_{n+1}}{1-\beta_{n+1}} \right| \| y_{n+1} - y_n \|$$

$$+ \left| \left( \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right) \right| \big\| f(x_n) - y_n \big\|$$

$$\leq \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} \right| \big\| f(x_{n+1}) - f(x_n) \big\| + \| y_{n+1} - y_n \|$$

$$+ \left| \left( \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right) \right| \big\| f(x_n) - y_n \big\|$$

$$\leq \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} \right| \big\| f(x_{n+1}) - f(x_n) \big\| + |\delta_{n+1}| \| x_{n+1} - x_n \| + |\delta_{n+1} - \delta_n| \big( \| x_n \| + \big\| T(\mu_n) x_n \big\| \big)$$

$$+ \big\| T(\mu_{n+1}) x_{n+1} - T(\mu_n) x_{n+1} \big\| + \| x_{n+1} - x_n \| + c_n$$

$$+ \left| \left( \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right) \right| \big\| f(x_n) - y_n \big\|.$$

It follows that

$$\| z_{n+1} - z_n \| - \| x_{n+1} - x_n \| \leq \left( \frac{\alpha_{n+1}}{1-\beta_{n+1}} + \delta_{n+1} \right) \| x_{n+1} - x_n \|$$

$$+ \left| \frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n} \right| \big( \big\| f(x_n) \big\| + \| y_n \| \big)$$

$$+ |\delta_{n+1} - \delta_n| \big( \| x_n \| + \big\| T(\mu_n) x_n \big\| \big)$$

$$+ \big\| T(\mu_{n+1}) x_{n+1} - T(\mu_n) x_{n+1} \big\| + c_n.$$

From (i), (ii), (iv), (3.3) and Lemma 2.3, we have

$$\limsup_{n\to\infty}\big(\|z_{n+1}-z_n\|-\|x_{n+1}-x_n\|\big)\le 0.$$

Applying Lemma 2.7, we obtain $\lim_{n\to\infty}\|z_n-x_n\|=0$ and also

$$\|x_{n+1}-x_n\|=(1-\beta_n)\|z_n-x_n\|\to 0,\quad\text{as }n\to\infty.$$

That is,

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.4}$$

Next, we will show that the set of all limit points of $\{x_n\}$ is a subset of $F(\mathcal{S})$. Note that

$$
\begin{aligned}
\|x_{n+1}-x_n\| &= \big\|\alpha_n f(x_n)+\beta_n x_n+\gamma_n y_n\big\| \\
&= \big\|\alpha_n f(x_n)+\beta_n x_n+\gamma_n\big(\delta_n x_n+(1-\delta_n)T(\mu_n)x_n\big)-x_n\big\| \\
&= \big\|\alpha_n f(x_n)-(1-\beta_n)x_n+\gamma_n\big(\delta_n x_n+(1-\delta_n)T(\mu_n)x_n\big)\big\| \\
&= \big\|\alpha_n f(x_n)-(1-\beta_n)x_n+\gamma_n\delta_n x_n+\gamma_n T(\mu_n)x_n-\gamma_n\delta_n T(\mu_n)x_n\big\| \\
&= \big\|\alpha_n f(x_n)-(1-\beta_n)x_n+\gamma_n\delta_n x_n+(1-\alpha_n-\beta_n)T(\mu_n)x_n-\gamma_n\delta_n T(\mu_n)x_n\big\| \\
&= \big\|\alpha_n\big(f(x_n)-T(\mu_n)x_n\big)+(1-\beta_n)\big(T(\mu_n)x_n-x_n\big)+\gamma_n\delta_n\big(x_n-T(\mu_n)x_n\big)\big\| \\
&= \big\|\alpha_n\big(f(x_n)-T(\mu_n)x_n\big)+(-1+\beta_n+\gamma_n\delta_n)\big(x_n-T(\mu_n)x_n\big)\big\| \\
&\le \alpha_n\big\|f(x_n)-T(\mu_n)x_n\big\|+(-1+\beta_n+\gamma_n\delta_n)\big\|x_n-T(\mu_n)x_n\big\|.
\end{aligned}
$$

It follows that

$$\big\|x_n-T(\mu_n)x_n\big\|\le\frac{1}{1-\beta_n-\gamma_n\delta_n}\big(\alpha_n\big\|f(x_n)-T(\mu_n)x_n\big\|-\|x_{n+1}-x_n\|\big).$$

From (i), (ii), (iv) and (3.4), we have

$$\lim_{n\to\infty}\big\|x_n-T(\mu_n)x_n\big\|=0. \tag{3.5}$$

Let $p$ be a limit point of $\{x_n\}$ and $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging strongly to $p$. From Lemma 2.3, we obtain that

$$
\begin{aligned}
\limsup_{k\to\infty}\big\|p-T(\mu_{n_k})p\big\| &\le \limsup_{k\to\infty}\big(\|p-x_{n_k}\|+\big\|x_{n_k}-T(\mu_{n_k})x_{n_k}\big\| \\
&\qquad +\big\|T(\mu_{n_k})x_{n_k}-T(\mu_{n_k})p\big\|\big) \\
&\le \limsup_{k\to\infty}\big(2\|p-x_{n_k}\|+\big\|x_{n_k}-T(\mu_{n_k})x_{n_k}\big\|+c_{n_k}\big) \\
&\le 0.
\end{aligned}
$$

From (3.5), (2.4) and Corollary 2.2, we get $p\in F(\mathcal{S})$.

We know that there exists a unique sunny nonexpansive retraction $P$ of $C$ onto $F(\mathcal{S})$, and from the Banach contraction mapping principle, we known that $Pf$ has a unique fixed point $q$ which by (2.3) is the unique solution of

$$\langle (f - I)q, J(p - q) \rangle \leq 0, \quad \forall p \in F(\mathcal{S}). \tag{3.6}$$

Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ converging to $p \in C$. From the smoothness of $E$ and (3.6), we have that

$$\limsup_{n \to \infty} \langle (f - I)q, J(x_n - q) \rangle = \limsup_{k \to \infty} \langle (f - I)q, J(x_{n_k} - q) \rangle$$
$$= \langle (f - I)q, J(p - q) \rangle \leq 0. \tag{3.7}$$

Finally, we show that the sequence $\{x_n\}$ converges strongly to $q = Pfq$. Now, we have

$$\begin{aligned}
\|y_n - q\| &= \left\| \delta_n x_n + (1 - \delta_n)T(\mu_n)x_n - q \right\| \\
&= \left\| (1 - \delta_n)\big(T(\mu_n)x_n - q\big) + \delta_n(x_n - q) \right\| \\
&\leq (1 - \delta_n)\left\| T(\mu_n)x_n - q \right\| + \delta_n \|x_n - q\| \\
&\leq (1 - \delta_n)\|x_n - q\| + c_n + \delta_n \|x_n - q\| \\
&= \|x_n - q\| + c_n. \tag{3.8}
\end{aligned}$$

From Lemma 2.6, (3.8) and (2.2), we have

$$\begin{aligned}
\|x_{n+1} - q\|^2 &= \left\| \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n - q \right\|^2 \\
&= \left\| \big(\gamma_n(y_n - q) + \beta_n(x_n - q)\big) + \alpha_n\big(f(x_n) - q\big) \right\|^2 \\
&\leq \left\| \gamma_n(y_n - q) + \beta_n(x_n - q) \right\|^2 + 2\alpha_n\langle f(x_n) - q, J(x_{n+1} - q) \rangle \\
&= \left\| (1 - \beta_n)\frac{\gamma_n}{1 - \beta_n}(y_n - q) + \beta_n\left(\frac{1 - \beta_n}{1 - \beta_n}\right)(x_n - q) \right\|^2 \\
&\quad + 2\alpha_n\langle f(x_n) - f(q), J(x_{n+1} - q) \rangle + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q) \rangle \\
&\leq (1 - \beta_n)\left\| \frac{\gamma_n}{1 - \beta_n}(y_n - q) \right\|^2 + \beta_n \|x_n - q\|^2 \\
&\quad + 2r\alpha_n\|x_n - q\|\|x_{n+1} - q\| + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q) \rangle \\
&\leq \frac{\gamma_n^2}{1 - \beta_n}\|y_n - q\|^2 + \beta_n\|x_n - q\|^2 \\
&\quad + r\alpha_n\big(\|x_n - q\|^2 + \|x_{n+1} - q\|^2\big) + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q) \rangle \\
&\leq \frac{\gamma_n^2}{1 - \beta_n}\|x_n - q\|^2 + \frac{\gamma_n^2 c_n}{1 - \beta_n} + \beta_n\|x_n - q\|^2 \\
&\quad + r\alpha_n\|x_n - q\|^2 + r\alpha_n\|x_{n+1} - q\|^2 + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q) \rangle \\
&= \left( \frac{\gamma_n^2}{1 - \beta_n} + \beta_n + r\alpha_n \right)\|x_n - q\|^2 + \frac{\gamma_n^2 c_n}{1 - \beta_n} \\
&\quad + r\alpha_n\|x_{n+1} - q\|^2 + 2\alpha_n\langle f(q) - q, J(x_{n+1} - q) \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left( \frac{((1 - \beta_n) - \alpha_n)^2}{1 - \beta_n} + \beta_n + r\alpha_n \right) \|x_n - q\|^2 + \frac{\gamma_n^2 c_n}{1 - \beta_n} \\
&\quad + r\alpha_n \|x_{n+1} - q\|^2 + 2\alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle \\
&= \left( \frac{(1 - \beta_n)^2 - 2(1 - \beta_n)\alpha_n + \alpha_n^2}{1 - \beta_n} + \beta_n + r\alpha_n \right) \|x_n - q\|^2 \\
&\quad + \frac{\gamma_n^2 c_n}{1 - \beta_n} + r\alpha_n \|x_{n+1} - q\|^2 + 2\alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle \\
&= \left( 1 - \beta_n - 2\alpha_n + \frac{\alpha_n^2}{1 - \beta_n} + \beta_n + r\alpha_n \right) \|x_n - q\|^2 + \frac{\gamma_n^2 c_n}{1 - \beta_n} \\
&\quad + r\alpha_n \|x_{n+1} - q\|^2 + 2\alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle \\
&= \left( (1 - r\alpha_n) + (2r\alpha_n - 2\alpha_n) + \frac{\alpha_n^2}{1 - \beta_n} \right) \|x_n - q\|^2 + \frac{\gamma_n^2 c_n}{1 - \beta_n} \\
&\quad + r\alpha_n \|x_{n+1} - q\|^2 + 2\alpha_n \langle f(q) - q, J(x_{n+1} - q) \rangle.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\|x_{n+1} - q\|^2 &\leq \left( 1 - \frac{2\alpha_n(1 - r)}{1 - r\alpha_n} + \frac{\alpha_n^2}{(1 - r\alpha_n)(1 - \beta_n)} \right) \|x_n - q\|^2 + \frac{\gamma_n^2 c_n}{(1 - r\alpha_n)(1 - \beta_n)} \\
&\quad + \frac{2\alpha_n}{1 - r\alpha_n} \langle f(q) - q, J(x_{n+1} - q) \rangle \\
&\leq \left( 1 - \frac{2\alpha_n(1 - r)}{1 - r\alpha_n} \right) \|x_n - q\|^2 + \frac{\alpha_n}{1 - r\alpha_n} \left( \frac{\alpha_n}{1 - \beta_n} \|x_n - q\|^2 \right. \\
&\quad \left. + \frac{\gamma_n^2 c_n}{\alpha_n(1 - \beta_n)} + 2 \langle f(q) - q, J(x_{n+1} - q) \rangle \right) \\
&:= (1 - \sigma_n) \|x_n - q\|^2 + \rho_n,
\end{aligned}
$$

where $\sigma_n := \frac{2\alpha_n(1 - r)}{1 - r\alpha_n}$ and $\rho_n := \frac{\alpha_n}{1 - r\alpha_n} \left( \frac{\alpha_n}{1 - \beta_n} \|x_n - q\|^2 + \frac{\gamma_n^2 c_n}{\alpha_n(1 - \beta_n)} + 2 \langle f(q) - q, J(x_{n+1} - q) \rangle \right)$. Now, from (i), (iii), (iv), (3.7) and Lemma 2.8, $\|x_n - q\| \to 0$ as $n \to \infty$. This proof is completed. $\square$

**Corollary 3.2** *Let $C$ be a nonempty, compact and convex subset of a smooth Banach space $E$. Let $S$ be a left reversible semigroup and $\mathcal{S} = \{T_s : s \in S\}$ be a representation of $S$ as Lipschitzian mappings from $C$ into itself, with the uniform Lipschitzian condition $\lim_s k_s \leq 1$, and $f$ be an $\alpha$-contraction of $C$ into itself. Let $X$ be a left invariant $\mathcal{S}$-stable subspace of $l^\infty(S)$ containing $1$, $\{\mu_n\}$ be a strongly left regular sequence of means on $X$ such that $\lim_{n \to \infty} \|\mu_{n+1} - \mu_n\| = 0$ and $\{c_n\}$ be the sequence defined by (2.4) with $\limsup_{n \to \infty} c_n \leq 0$. Suppose the sequences $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ and $\{\delta_n\}$ in $(0, 1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. The following conditions are satisfied:*

(i) $\lim_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=0}^\infty \alpha_n = \infty$;

(ii) $\lim_{n \to \infty} \delta_n = 0$;

(iii) $\limsup_{n \to \infty} \frac{c_n}{\alpha_n} \leq 0$;

(iv) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$.

*For arbitrary given $x_1 \in C$, generate a sequence $\{x_n\}$ by*

$$
\begin{cases}
y_n = \delta_n x_n + (1 - \delta_n) T(\mu_n) x_n, \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n.
\end{cases}
\tag{3.9}
$$

*Then $\{x_n\}$ converges strongly to $q \in F(\mathcal{S})$, which is the unique solution of the variational inequality*

$$\big\langle (f - I)q, J(p - q) \big\rangle \leq 0, \quad \forall p \in F(\mathcal{S}).$$

*Equivalently, we have $q = Pfq$, where $P$ is the unique sunny nonexpansive retraction of $C$ onto $F(\mathcal{S})$.*

**Corollary 3.3** *Let $C$ be a nonempty, compact and convex subset of a smooth Banach space $E$. Let $S$ be a left reversible semigroup and $\mathcal{S} = \{T_s : s \in S\}$ be a representation of $S$ as Lipschitzian mappings from $C$ into itself, with the uniform Lipschitzian condition $\lim_s k_s \leq 1$, and $f$ be a Meir-Keeler contraction of $C$ into itself. Let $X$ be a left invariant $\mathcal{S}$-stable subspace of $l^\infty(S)$ containing $1$, $\{\mu_n\}$ be a strongly left regular sequence of means on $X$ such that $\lim_{n \to \infty} \|\mu_{n+1} - \mu_n\| = 0$ and $\{c_n\}$ be the sequence defined by (2.4) with $\limsup_{n \to \infty} c_n \leq 0$. Suppose the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. The following conditions are satisfied:*

(i) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=0}^\infty \alpha_n = \infty$;
(ii) $\limsup_{n \to \infty} \frac{c_n}{\alpha_n} \leq 0$;
(iii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$.

*For arbitrary $x_1 \in C$, generate a sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n) x_n. \tag{3.10}$$

*Then $\{x_n\}$ converges strongly to $q \in F(\mathcal{S})$, which is the unique solution of the variational inequality*

$$\big\langle (f - I)q, J(p - q) \big\rangle \leq 0, \quad \forall p \in F(\mathcal{S}).$$

*Equivalently, we have $q = Pfq$, where $P$ is the unique sunny nonexpansive retraction of $C$ onto $F(\mathcal{S})$.*

**Corollary 3.4** *Let $C$ be a nonempty, compact and convex subset of a smooth Banach space $E$. Let $S$ be a left reversible semigroup and $\mathcal{S} = \{T_s : s \in S\}$ be a representation of $S$ as Lipschitzian mappings from $C$ into itself, with the uniform Lipschitzian condition $\lim_s k_s \leq 1$, and $f$ be an $\alpha$-contraction of $C$ into itself. Let $X$ be a left invariant $\mathcal{S}$-stable subspace of $l^\infty(S)$ containing $1$, $\{\mu_n\}$ be a strongly left regular sequence of means on $X$ such that $\lim_{n \to \infty} \|\mu_{n+1} - \mu_n\| = 0$ and $\{c_n\}$ be the sequence defined by (2.4) with $\limsup_{n \to \infty} c_n \leq 0$. Suppose the sequences $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \geq 1$. The following conditions are satisfied:*

(i) $\lim_{n \to \infty} \alpha_n = 0$ and $\sum_{n=0}^\infty \alpha_n = \infty$;
(ii) $\limsup_{n \to \infty} \frac{c_n}{\alpha_n} \leq 0$;
(iii) $0 < \liminf_{n \to \infty} \beta_n \leq \limsup_{n \to \infty} \beta_n < 1$.

*For arbitrary $x_1 \in C$, generate a sequence $\{x_n\}$ by*

$$x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n T(\mu_n) x_n. \tag{3.11}$$

Then $\{x_n\}$ converges strongly to $q \in F(\mathcal{S})$, which is the unique solution of the variational inequality

$$\langle (f - I)q, J(p - q) \rangle \le 0, \quad \forall p \in F(\mathcal{S}).$$

Equivalently, we have $q = Pfq$, where $P$ is the unique sunny nonexpansive retraction of $C$ onto $F(\mathcal{S})$.

**Corollary 3.5** *Let $C$ be a nonempty, compact and convex subset of a smooth Banach space $E$. Let $S$ be a left reversible semigroup and $\mathcal{S} = \{T_s : s \in S\}$ be a representation of $S$ as nonexpansive mappings from $C$ into itself and $f$ be an $\alpha$-contraction of $C$ into itself. Let $X$ be a left invariant $\mathcal{S}$-stable subspace of $l^\infty(S)$ containing $1$, $\{\mu_n\}$ be a strongly left regular sequence of means on $X$ such that $\lim_{n \to \infty} \|\mu_{n+1} - \mu_n\| = 0$ and $\{c_n\}$ be the sequence defined by (2.4) with $\limsup_{n \to \infty} c_n \le 0$. Suppose the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\delta_n\}$ in $(0,1)$ satisfy $\alpha_n + \beta_n + \gamma_n = 1$, $n \ge 1$. The following conditions are satisfied:*

(i) $\lim_{n \to \infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(ii) $\lim_{n \to \infty} \delta_n = 0$;

(iii) $\limsup_{n \to \infty} \frac{c_n}{\alpha_n} \le 0$;

(iv) $0 < \liminf_{n \to \infty} \beta_n \le \limsup_{n \to \infty} \beta_n < 1$.

*For arbitrary $x_1 \in C$, generate a sequences $\{x_n\}$ by*

$$\begin{cases} y_n = \delta_n x_n + (1 - \delta_n) T(\mu_n) x_n, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n y_n. \end{cases} \tag{3.12}$$

Then $\{x_n\}$ converges strongly to $q \in F(\mathcal{S})$, which is the unique solution of the variational inequality

$$\langle (f - I)q, J(p - q) \rangle \le 0, \quad \forall p \in F(\mathcal{S}).$$

Equivalently, we have $q = Pfq$, where $P$ is the unique sunny nonexpansive retraction of $C$ onto $F(\mathcal{S})$.

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