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Research Article

Multiplicity Results for Positive Solutions to Differential Systems of Singular Coupled Integral Boundary Value Problems

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By constructing a special cone and using a fixed-point theorem in cone, this paper investigates the existence of multiple solutions of coupled integral boundary value problems for a nonlinear singular differential system.

1. Introduction

In recent years, singular uncoupled boundary value problems to differential systems have been studied widely and there are many excellent results (see [1–18] and references therein). Naturally we hope there are the same excellent results on singular uncoupled boundary value problems to differential systems with coupled boundary conditions. Many researchers put their efforts to study the existence of solutions for differential systems with coupled boundary conditions (see [19–30] and references therein).

In [19], Asif and Khan addressed the question of the existence of coupled four-point boundary value conditions

$$\begin{aligned} -x''(t) &= f_1(t, x(t), y(t)), \quad t \in (0,1), \\ -y''(t) &= f_2(t, x(t), y(t)), \quad t \in (0,1), \\ x(0) &= y(0) = 0, \\ x(1) &= \alpha y(\xi), \\ y(1) &= \beta x(\eta), \end{aligned} \tag{1}$$

where the parameters α, β, ξ, and η satisfy $\xi, \eta \in (0,1)$, $0 < \alpha\beta\xi\eta < 1$. The main tool in [19] is the Guo-Krasnosel'skiĭ fixed-point theorem.

In [29], the authors studied the following nonlinear semipositone fractional differential equation with four-point coupled boundary value problem:

$$\begin{aligned} &D_{0+}^{\alpha} u + \lambda f(t, u, v) = 0, \quad t \in (0,1), \ \lambda > 0, \\ &D_{0+}^{\alpha} v + \lambda g(t, u, v) = 0, \\ &u^{(i)}(0) = v^{(i)}(0) = 0, \quad 0 \le i \le n-2, \\ &u(1) = a v(\xi), \\ &v(1) = b u(\eta), \end{aligned} \tag{2}$$

where λ is a parameter, a, b, ξ, and η satisfy $\xi, \eta \in (0,1)$, $0 < ab\xi\eta < 1$, $\alpha \in (n-1, n]$ is a real number and $n \ge 3$, and $D_{0+}^{\alpha} u$ is Riemann-Liouville's fractional derivative. The existence of positive solutions is established by using a nonlinear alternative of Leray-Schauder type and Guo-Krasnosel'skiĭ fixed-point theorem in a cone.

In [20], Cui and Sun, using fixed-point index theory, studied the existence of positive solutions for superlinear differential system

$$\begin{aligned} -x''(t) &= f_1(t, x(t), y(t)), \quad t \in (0,1), \\ -y''(t) &= f_2(t, x(t), y(t)), \quad t \in (0,1), \end{aligned}$$

$$\begin{aligned} &x(0) = y(0) = 0, \\ &x(1) = \alpha[y], \\ &y(1) = \beta[x], \end{aligned} \tag{3}$$

where $\alpha[x]$ and $\beta[x]$ are bounded linear functionals on $C[0,1]$ given by

$$\begin{aligned} \alpha[x] &= \int_0^1 x(t)\,dA(t), \\ \beta[x] &= \int_0^1 x(t)\,dB(t), \end{aligned} \tag{4}$$

involving Stieltjes integrals; in particular, A and B are functions of bounded variation with positive measures.

We should note that the nonlinear terms in two equations for the above problems have the same features. For instance, the nonlinear terms in two equations are both superlinear [20, 29] or both sublinear [19]. However, to the best of our knowledge, only a few papers discuss differential system under the case that the nonlinear terms of the system have different behaviors. Motivated by [19, 20, 29], the purpose of this paper is to establish the existence of multiple positive solutions for differential system with coupled integral boundary value problems (3) when f_1 is superlinear in x and y and f_2 is sublinear in x and y. Also suppose that $f_1(t,x,y)$ may be singular at $t=0$, $t=1$ and $f_2(t,x,y)$ may be singular at $t=0$, $t=1$, $x=0$, and $y=0$. Our main features are threefold. Firstly, our study is on singular nonlinear differential systems with general boundary value conditions. Secondly, f_2 is allowed to be not only singular at $t=0$ and 1 but also singular at $x=0$ and $y=0$. Finally, a special cone is constructed to overcome difficulties due to singularities of nonlinear term.

In the rest of this section, let us list the following assumptions:

(H_1) $\kappa_1 > 0, \kappa_2 > 0$, and $\kappa > 0$, where

$$\begin{aligned} \kappa_1 &= \alpha[t] = \int_0^1 t\,dA(t), \\ \kappa_2 &= \beta[t] = \int_0^1 t\,dB(t), \\ \kappa &= 1 - \kappa_1\kappa_2. \end{aligned} \tag{5}$$

(H_2) $f_1 \in C((0,1)\times[0,\infty)^2, [0,\infty))$ satisfy

$$0 < \int_0^1 s(1-s)\,f_1(s,1,1)\,ds < +\infty, \tag{6}$$

and there exist constants λ_{1j}, μ_{1j} $(0 < \lambda_{1j} \le \mu_{1j} < 1$, $j = 1,2$, $\lambda_{11}+\lambda_{12} > 1)$ such that, for $t \in (0,1)$, $x, y \in (0,\infty)$,

$$\begin{aligned} &c^{\mu_{11}} f_1(t,x,y) \le f_1(t,cx,y) \le c^{\lambda_{11}} f_1(t,x,y), \\ &c^{\mu_{12}} f_1(t,x,y) \le f_1(t,x,cy) \le c^{\lambda_{12}} f_1(t,x,y), \\ &\qquad \text{if } 0 < c \le 1. \end{aligned} \tag{7}$$

(H_3) $f_2 \in C((0,1)\times(0,\infty)^2, [0,\infty))$ and there exist constants λ_{2j}, μ_{2j} $(-\infty < \lambda_{2j} \le \mu_{2j} < 1$, $j=1,2$, $\mu_{21}+\mu_{22} < 1)$ such that, for $t\in(0,1)$, $x,y\in(0,\infty)$,

$$\begin{aligned} &c^{\mu_{21}} f_2(t,x,y) \le f_2(t,cx,y) \le c^{\lambda_{21}} f_2(t,x,y), \\ &c^{\mu_{22}} f_2(t,x,y) \le f_2(t,x,cy) \le c^{\lambda_{22}} f_2(t,x,y), \\ &\qquad \text{if } 0 < c \le 1. \end{aligned} \tag{8}$$

And they satisfy one of the following conditions:

(H_{31}) $\lambda_{21} > 0$, $\lambda_{22} > 0$, $0 < \int_0^1 s(1-s)f_2(s,1,1)ds < +\infty$, $\rho\int_0^1 s(1-s)f_1(s,1,1)ds + \rho\int_0^1 s(1-s)f_2(s,1,1)ds < 1$.

(H_{32}) $\mu_{21} < 0$, $\mu_{22} < 0$, $0 < \int_0^1 s(1-s)f_2(s,s,s)ds < +\infty$, $\rho\int_0^1 s(1-s)f_1(s,1,1)ds + \rho\gamma^{\mu_{21}+\mu_{22}}\int_0^1 s(1-s)f_2(s,s,s)ds < 1$.

(H_{33}) $\lambda_{21} > 0$, $\mu_{22} < 0$, $0 < \int_0^1 s(1-s)f_2(s,1,s)ds < +\infty$, $\rho\int_0^1 s(1-s)f_1(s,1,1)ds + \rho\gamma^{\mu_{22}}\int_0^1 s(1-s)f_2(s,1,s)ds < 1$.

(H_{34}) $\mu_{21} < 0$, $\lambda_{22} > 0$, $0 < \int_0^1 s(1-s)f_2(s,s,1)ds < +\infty$. $\rho\int_0^1 s(1-s)f_1(s,1,1)ds + \rho\gamma^{\mu_{21}}\int_0^1 s(1-s)f_2(s,s,1)ds < 1$, where

$$0 < \gamma = \frac{\nu}{\rho} < 1, \tag{9}$$

$$\begin{aligned} \rho &= \max\left\{\frac{\alpha[t]}{\kappa}\beta[1]+1, \frac{\beta[t]}{\kappa}\alpha[1]+1, \frac{1}{\kappa}\beta[1], \frac{1}{\kappa}\cdot\alpha[1]\right\}, \\ \nu &= \min\left\{\frac{\alpha[t]}{\kappa}\beta[t(1-t)], \frac{\beta[t]}{\kappa}\alpha[t(1-t)], \frac{1}{\kappa}\cdot\beta[t(1-t)], \frac{1}{\kappa}\alpha[t(1-t)]\right\}. \end{aligned} \tag{10}$$

Remark 1. Equations (7)-(8) imply

$$\begin{aligned} &c^{\lambda_{i1}} f_i(t,x,y) \le f_i(t,cx,y) \le c^{\mu_{i1}} f_i(t,x,y), \\ &c^{\lambda_{i2}} f_i(t,x,y) \le f_i(t,x,cy) \le c^{\mu_{i2}} f_i(t,x,y), \\ &\qquad \text{if } c \ge 1,\ i = 1,2. \end{aligned} \tag{11}$$

Conversely, (11) implies (7)-(8).

Remark 2. Equation (7) implies

$$f_1(t,x_1,y_1) \le f_1(t,x_2,y_2), \quad \text{if } 0 < x_1 \le x_2,\ 0 < y_1 \le y_2. \tag{12}$$

Remark 3. (H_{31}) implies that $f_2(t,x,y)$ is nondecreasing in x and y.

(H_{32}) implies that $f_2(t,x,y)$ is nonincreasing in x and y.

(H_{33}) implies that $f_2(t,x,y)$ is nondecreasing in x and nonincreasing in y.

(H_{34}) implies that $f_2(t,x,y)$ is nonincreasing in x and nondecreasing in y.

2. Main Results

For each $u \in E := C[0,1]$, we write $\|u\| = \max\{|u(t)| : t \in [0,1]\}$. Clearly, $(E, \|\cdot\|)$ is a Banach space. Similarly, for each $(x, y) \in E\times E$, we write $\|(x,y)\|_1 = \max\{\|x\|, \|y\|\}$. For any real constant $r > 0$, define $\Omega_r = \{(x, y) \in E \times E : \|(x, y)\|_1 < r\}$. Define

$$P = \{(x, y) \in E \times E : x(t) \geq \gamma t \|(x, y)\|_1,\ y(t) \geq \gamma t \|(x, y)\|_1,\ t \in [0,1]\}, \tag{13}$$

where γ is given by (9). Clearly, $(E\times E, \|\cdot\|_1)$ is a Banach space and P is a cone of $E \times E$.

Remark 4. The cone P defined by (13) is completely different from the cone used in the uncoupled boundary value problems. This means that the cone P has the following property:

$$\begin{aligned} &\text{if} \quad (x, y) \in P \setminus \{\theta\}, \\ &\text{then} \quad \gamma t \|(x, y)\|_1 \leq x(t), \\ &\qquad\quad y(t) \leq \|(x, y)\|_1, \\ &\qquad\qquad\qquad\qquad \text{for } t \in [0,1], \end{aligned} \tag{14}$$

which is crucial in the definition of T and in the proof of Lemma 9.

Our main result is the following theorems.

Theorem 5. *Assume that (H_1), (H_2), and (H_3) are satisfied. Then differential system (3) has at least two positive solutions $(x_1, y_1), (x_2, y_2) \in C[0,1]\times C[0,1]$ such that $0 < \|(x_1, y_1)\|_1 < 1 < \|(x_2, y_2)\|_1$.*

Note. We need only to prove this theorem under condition (H_{33}), since the proof is similar when (H_{31}) or (H_{32}) or (H_{34}) is satisfied.

The proof of Theorem 5 is based on the following theorem in [31].

Lemma 6. *Let E be a Banach space and P a cone in E. Suppose that Ω_1 and Ω_2 are two bounded open subsets of E with $\theta \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$. Let operator $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be completely continuous. Suppose that one of the two conditions*

$$\begin{aligned} \text{(i)} \quad & \|Tx\| \leq \|x\|, \quad \forall x \in P \cap \partial\Omega_1, \\ & \|Tx\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_2, \\ \text{(ii)} \quad & \|Tx\| \geq \|x\|, \quad \forall x \in P \cap \partial\Omega_1, \\ & \|Tx\| \leq \|x\|, \quad \forall x \in P \cap \partial\Omega_2 \end{aligned} \tag{15}$$

is satisfied. Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Lemma 7 (see [20]). *Assume that (H_1) holds. Let $u, v \in E$; then the system of BVPs*

$$\begin{aligned} &-x''(t) = u(t), \\ &-y''(t) = v(t), \\ &\qquad\qquad t \in [0,1], \\ &x(0) = y(0) = 0, \\ &x(1) = \alpha[y], \\ &y(1) = \beta[x] \end{aligned} \tag{16}$$

has integral representation

$$\begin{aligned} x(t) &= \int_0^1 G_1(t,s)\, u(s)\, ds + \int_0^1 H_1(t,s)\, v(s)\, ds, \\ y(t) &= \int_0^1 G_2(t,s)\, v(s)\, ds + \int_0^1 H_2(t,s)\, u(s)\, ds, \end{aligned} \tag{17}$$

where

$$\begin{aligned} k(t,s) &= \begin{cases} t(1-s), & 0 \leq t \leq s \leq 1, \\ s(1-t), & 0 \leq s \leq t \leq 1, \end{cases} \\ G_1(t,s) &= \frac{k_1 t}{\kappa} \int_0^1 k(s,\tau)\, dB(\tau) + k(t,s), \\ H_1(t,s) &= \frac{t}{\kappa} \int_0^1 k(s,\tau)\, dA(\tau), \\ G_2(t,s) &= \frac{k_2 t}{\kappa} \int_0^1 k(s,\tau)\, dA(\tau) + k(t,s), \\ H_2(t,s) &= \frac{t}{\kappa} \int_0^1 k(s,\tau)\, dB(\tau). \end{aligned} \tag{18}$$

Remark 8 (see [20]). From (13) and (H_1), for $t \in [0,1]$, we have

$$\begin{aligned} &G_i(t,s) \leq \rho s(1-s), \\ &H_i(t,s) \leq \rho s(1-s), \\ &G_i(t,s) \geq \nu t s(1-s), \\ &H_i(t,s) \geq \nu t s(1-s), \\ &\qquad\qquad i = 1, 2. \end{aligned} \tag{19}$$

Define an operator $T : P \setminus \{\theta\} \to P$ by

$$T(x,y) = (T_1(x,y), T_2(x,y)), \tag{20}$$

where operators $T_1, T_2 : P \setminus \{\theta\} \to Q = \{u \in E \mid u(t) \geq 0,\ t \in [0,1]\}$ are defined by

$$\begin{aligned} T_1(x,y)(t) = &\int_0^1 G_1(t,s)\, f_1(s, x(s), y(s))\, ds \\ &+ \int_0^1 H_1(t,s)\, f_2(s, x(s), y(s))\, ds, \end{aligned}$$

$$T_2(x,y)(t) = \int_0^1 G_2(t,s) f_2(s, x(s), y(s))\, ds + \int_0^1 H_2(t,s) f_1(s, x(s), y(s))\, ds, \quad t \in [0,1]. \tag{21}$$

Lemma 9. *Assume that* (H_1), (H_2), *and* (H_3) *hold. Then, for any* $0 < a < b < +\infty$, $T : P \cap (\overline{\Omega_b} \setminus \Omega_a) \to P$ *is a completely continuous operator.*

Proof. For $(x,y) \in P \cap (\overline{\Omega_b} \setminus \Omega_a)$, let c be a positive number such that $\|(x,y)\|_1/c < 1$ and $c > 1$. From (H_2), (H_3), and Remark 2, we have

$$\begin{aligned} f_1(t, x(t), y(t)) &\le f_1(t,c,c) \le c^{\mu_{11}+\mu_{12}} f_1(t,1,1), \\ f_2(t, x(t), y(t)) &\le f_2(t, c, \gamma \|(x,y)\|_1 t) \\ &\le c^{\mu_{21}} f_2(t, 1, \min\{\gamma \|(x,y)\|_1, 1\} t) \\ &\le c^{\mu_{21}} \min\{\gamma a, 1\}^{\lambda_{22}} f_2(t,1,t). \end{aligned} \tag{22}$$

Hence for any $t \in [0,1]$, by Remark 8, we get

$$\begin{aligned} T_i(x,y)(t) &\le \rho \int_0^1 s(1-s) f_1(s, x(s), y(s))\, ds + \rho \int_0^1 s(1-s) f_2(s, x(s), y(s))\, ds \\ &\le \rho \int_0^1 s(1-s) f_1(s,c,c)\, ds + \rho \int_0^1 s(1-s) f_2(s, c, \gamma s \|(x,y)\|_1)\, ds \\ &\le \rho c^{\mu_{11}+\mu_{12}} \int_0^1 s(1-s) f_1(s,1,1)\, ds \\ &\quad + \rho c^{\mu_{21}} \min\{\gamma a, 1\}^{\lambda_{22}} \int_0^1 s(1-s) f_2(s,1,s)\, ds, \quad i = 1,2. \end{aligned} \tag{23}$$

Thus, T is well defined on $P \cap (\overline{\Omega_b} \setminus \Omega_a)$.

Next we show that $T(P \cap (\overline{\Omega_b} \setminus \Omega_a)) \subset P$. By Remark 8, for $\tau, t, s \in [0,1]$, we obtain

$$\begin{aligned} G_i(t,s) &\ge \gamma t G_i(\tau,s), \\ H_i(t,s) &\ge \gamma t H_i(\tau,s), \quad i = 1,2, \\ H_1(t,s) &\ge \gamma t G_2(\tau,s), \\ G_1(t,s) &\ge \gamma t H_2(\tau,s), \\ H_2(t,s) &\ge \gamma t G_1(\tau,s), \\ G_2(t,s) &\ge \gamma t H_1(\tau,s). \end{aligned} \tag{24}$$

Hence, for $(x,y) \in P \cap (\overline{\Omega_b} \setminus \Omega_a)$, $t \in [0,1]$, we have

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 G_1(t,s) f_1(s,x(s),y(s))\, ds + \int_0^1 H_1(t,s) f_2(s,x(s),y(s))\, ds \\ &\ge \gamma t \int_0^1 G_1(\tau,s) f_1(s,x(s),y(s))\, ds + \gamma t \int_0^1 H_1(\tau,s) f_2(s,x(s),y(s))\, ds \\ &= \gamma t T_1(x,y)(\tau), \\ T_1(x,y)(t) &= \int_0^1 G_1(t,s) f_1(s,x(s),y(s))\, ds + \int_0^1 H_1(t,s) f_2(s,x(s),y(s))\, ds \\ &\ge \gamma t \int_0^1 H_2(\tau,s) f_1(s,x(s),y(s))\, ds + \gamma t \int_0^1 G_2(\tau,s) f_2(s,x(s),y(s))\, ds \\ &= \gamma t T_2(x,y)(\tau). \end{aligned} \tag{25}$$

Then $T_1(x,y)(t) \ge \gamma t \|T_1(x,y)\|$ and $T_1(x,y)(t) \ge \gamma t \|T_2(x,y)\|$, that is, $T_1(x,y)(t) \ge \gamma t \|(T_1(x,y), T_2(x,y))\|_1$. In the same way, we can prove that $T_2(x,y)(t) \ge \gamma t \|(T_1(x,y), T_2(x,y))\|_1$. Therefore, $T(P \cap (\overline{\Omega_b} \setminus \Omega_a)) \subset P$.

Moreover, $T : P \cap (\overline{\Omega_b} \setminus \Omega_a) \to P$ is a completely continuous operator. This is a standard textbook result using Ascoli-Arzela theorem (see, e.g., [31]) and is omitted. □

Proof of Theorem 5. By (H_2) and (H_3), we can get

$$\begin{aligned} t^{\mu_{11}+\mu_{12}} f_1(t,1,1) &\le f_1(t,t,t) \le f_1(t,1,1), \\ t^{\mu_{21}-\lambda_{22}} f_2(t,1,t) &\le f_2(t,t,1) \le f_2(t,1,t), \quad t \in (0,1). \end{aligned} \tag{26}$$

This implies that

$$\begin{aligned} 0 &< \int_0^1 s(1-s) f_1(s,s,s)\, ds < +\infty, \\ 0 &< \int_0^1 s(1-s) f_2(s,s,1)\, ds < +\infty. \end{aligned} \tag{27}$$

Choose constants r and R such that

$$R > \max\left\{\gamma^{-1}, 1, \left(\frac{\nu}{4} \cdot \gamma^{\lambda_{11}+\lambda_{12}} \int_0^1 s(1-s) f_1(s,s,s)\, ds\right)^{-1/(\lambda_{11}+\lambda_{12}-1)}\right\},$$
$$0 < r < \min\left\{\frac{1}{2}, \left(\frac{\nu}{4} \cdot \gamma^{\mu_{21}} \int_0^1 s(1-s) f_2(s,s,1)\, ds\right)^{1/(1-\mu_{21}-\mu_{22})}\right\}. \tag{28}$$

It follows from Lemma 9 that $T : P \cap (\overline{\Omega_R} \setminus \Omega_r) \to P$ is a completely continuous operator. Moreover, by Lemma 7, if $(x, y) \in P \cap (\overline{\Omega_R} \setminus \Omega_r)$ is a fixed point of T, then (x, y) is a solution of differential system (3).

For any $t \in [0, 1]$, $(x, y) \in P \cap \partial\Omega_r$, it follows from the definition of cone P that

$$\gamma r t = \gamma t \|(x,y)\|_1 \le x(t), \quad y(t) \le r < 1. \tag{29}$$

Thus for any $(x, y) \in P \cap \partial\Omega_r$, by (H_{33}) and Remarks 2 and 8, we have

$$\begin{aligned} T_i(x,y)(t) &\ge \frac{\nu}{4} \int_0^1 s(1-s) f_2(s, \gamma r s, r)\, ds \\ &\ge \frac{\nu}{4} \gamma^{\mu_{21}} r^{\mu_{21}+\mu_{22}} \int_0^1 s(1-s) f_2(s,s,1)\, ds \\ &\ge r = \|(x,y)\|_1, \quad i = 1,2,\ t \in \left[\frac{1}{4}, 1\right]. \end{aligned} \tag{30}$$

Consequently,

$$\|T(x,y)\|_1 \ge \|(x,y)\|_1, \quad \forall (x,y) \in P \cap \partial\Omega_r. \tag{31}$$

Again, for any $t \in [0, 1]$, $(x, y) \in P \cap \partial\Omega_R$, we have

$$\begin{aligned} \gamma R t = \gamma t \|(x,y)\|_1 \le x(t), \\ y(t) \le R. \end{aligned} \tag{32}$$

Thus for any $(x, y) \in P \cap \partial\Omega_R$, noting that $\gamma R > 1$, we have

$$\begin{aligned} T_i(x,y)(t) &\ge \frac{\nu}{4} \int_0^1 s(1-s) f_1(s, \gamma R s, \gamma R s)\, ds \\ &\ge \frac{\nu}{4} (\gamma R)^{\lambda_{11}+\lambda_{12}} \int_0^1 s(1-s) f_1(s,s,s)\, ds \\ &\ge R = \|(x,y)\|_1, \quad i = 1,2,\ t \in \left[\frac{1}{4}, 1\right]. \end{aligned} \tag{33}$$

This guarantees

$$\|T(x,y)\|_1 \ge \|(x,y)\|_1, \quad \forall (x,y) \in P \cap \partial\Omega_R. \tag{34}$$

On the other hand, for any $t \in [0, 1]$, $(x, y) \in P \cap \partial\Omega_1$, it follows from the definition of cone P that

$$\begin{aligned} \gamma t \le x(t), \\ y(t) \le 1. \end{aligned} \tag{35}$$

Thus for any $(x, y) \in P \cap \partial\Omega_1$, by (H_{33}) and Remarks 2 and 8, we have

$$\begin{aligned} T_i(x,y)(t) &\le \rho \int_0^1 s(1-s) f_1(s,1,1)\, ds \\ &\quad + \rho \int_0^1 s(1-s) f_2(s,1,\gamma s)\, ds \\ &\le \rho \int_0^1 s(1-s) f_1(s,1,1)\, ds \\ &\quad + \rho \gamma^{\mu_{22}} \int_0^1 s(1-s) f_2(s,1,s)\, ds < 1 \\ &= \|(x,y)\|_1, \quad i = 1,2. \end{aligned} \tag{36}$$

That is,

$$\|T(x,y)\|_1 < \|(x,y)\|_1, \quad \forall (x,y) \in P \cap \partial\Omega_1. \tag{37}$$

Therefore, from (31), (37), and Lemma 6, it follows that differential system (3) has one positive solution $(x_1, y_1) \in P$ with $r \le \|(x_1, y_1)\|_1 < 1$. In the same way, from (34), (37), and Lemma 6, it follows that differential system (3) has one positive solution $(x_2, y_2) \in P$ with $1 < \|(x_1, y_1)\|_1 \le R$. □

3. An Example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the singular differential system with coupled boundary value problem

$$\begin{aligned} -x'' &= \frac{x^{p_1} y^{q_1}}{\pi \sqrt{t(1-t)}}, \\ -y'' &= \frac{x^{p_2}}{4\sqrt{y}} t, \\ x(0) &= y(0) = 0, \\ x(1) &= y\left(\frac{1}{4}\right) + y\left(\frac{1}{2}\right), \\ y(1) &= \int_0^1 x(t)\, dt, \end{aligned} \tag{38}$$

where $0 \le p_1, q_1, p_2 < +\infty$, $p_1 + q_1 > 1$, $p_2 < 1$.

Let

$$A(t)=\begin{cases}0, & t\in\left[0,\dfrac{1}{4}\right),\\ 1, & t\in\left[\dfrac{1}{4},\dfrac{1}{2}\right),\\ 2, & t\in\left[\dfrac{1}{2},1\right],\end{cases}$$
$$B(t)=t,$$
$$\lambda_{11}=\mu_{11}=p_1,$$
$$\lambda_{12}=\mu_{12}=q_1,$$
$$\lambda_{21}=\mu_{21}=p_2,$$
$$\lambda_{22}=\mu_{22}=-\frac{1}{2},$$
$$f_1(t,x,y)=\frac{x^{p_1}y^{q_1}}{\pi\sqrt{t(1-t)}},$$
$$f_2(t,x,y)=\frac{x^{p_2}}{4\sqrt{y}}t; \tag{39}$$

then

$$\kappa_1=\frac{3}{4},$$
$$\kappa_2=\frac{1}{2},$$
$$\kappa=1-\kappa_1\kappa_2=\frac{5}{8},$$
$$\rho=\frac{16}{5},$$
$$\nu=\frac{1}{5},$$
$$\gamma=\frac{1}{16},$$
$$\rho\int_0^1 s(1-s)f_1(s,1,1)\,ds+\rho\gamma^{\mu_{22}}\int_0^1 s(1-s)f_2(s,1,s)\,ds=\frac{16}{5}\left(\frac{1}{8}+\frac{4}{35}\right)=\frac{134}{175}<1. \tag{40}$$

So all conditions of Theorem 5 are satisfied for (38), and our conclusion follows from Theorem 5.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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