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Research Article

Discussion on the Time-Harmonic Elastodynamic Half-Space Green's Function Obtained by Superposition

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The time-harmonic elastodynamic half-space Green's function derived by Banerjee and Mamoon by way of superposition is discussed and examined against another semianalytical solution and a numerical solution. It is shown that Banerjee and Mamoon's solution gives infinite z-displacement response when the depth of the source goes to infinity, which is unreasonable and does not agree with other solutions. A possible problem in the derivation is that it is inappropriate to directly extend the results of Mindlin's superposition method for the elastostatic half-space problem to the dynamic case. The superposition of the six full-space elastodynamic solutions does not satisfy the required boundary conditions of the half-space elastodynamic problem as in the static case and thus does not solve the dynamic half-space problem.

1. Introduction

The elastodynamic Green's function for the half-space is fundamental to the application of boundary element method (BEM) to situations involving semi-infinite media. Various derivations of the elastic displacement due to a subsurface transient or time-harmonic point force can be found in the literature [1–4]. Often expressed in Fourier-Bessel integral forms, numerical evaluation and application of these solutions are usually complex and time-consuming [5–7]. The time-harmonic elastodynamic half-space Green's function under discussion, proposed by Banerjee and Mamoon [4], is derived by extending the superposition technique devised by Mindlin [8] for the elastostatic half-space problem to the dynamic case of a periodic point force in a semi-infinite solid.

2. Banerjee and Mamoon's Green's Function

Consider the situation where the periodic force is normal to the free boundary of the half-space, as depicted in Figure 1. The semi-infinite solid is bounded by the plane $z = 0$, with the positive z-axis pointing to the interior of the body. A periodic force $Pe^{i\omega t}$ is applied at point $(0, 0, +c)$ in the positive z-direction, where ω is the circular frequency and i is the imaginary unit. The displacements at point $\boldsymbol{\xi}(x, y, z)$ are to be found. The distance between $\boldsymbol{\xi}$ and the real source point is given by $R_1 = \{r^2 + (z - c)^2\}^{1/2}$, whereas the distance between $\boldsymbol{\xi}$ and the image point $(0, 0, -c)$ is denoted by $R_2 = \{r^2 + (z - c)^2\}^{1/2}$.

According to Banerjee and Mamoon (B&M for short) [4], the solution to this problem is composed of six individual components, corresponding to solutions to six problems in an infinite solid. The first two problems represent single periodic forces at $(0, 0, +c)$ and $(0, 0, -c)$, respectively; the third, fourth, and sixth problems represent a dynamic double force, a dynamic center of compression, and a dynamic doublet [9] at $(0, 0, -c)$, respectively; and the fifth problem represents a line of dynamic centers of compression extending from $z = -c$ to $z = -\infty$. To facilitate the ensuing discussion and save space, here only the z-displacement component of the sixth problem is given:

$$u_z^6 = K_6 \left[A_6 \left(\frac{\partial R_2}{\partial z} \right)^2 + B_6 \right], \tag{1a}$$

$$K_6 = -\frac{c^2}{2\pi\mu (3 - 4\nu)}, \tag{1b}$$

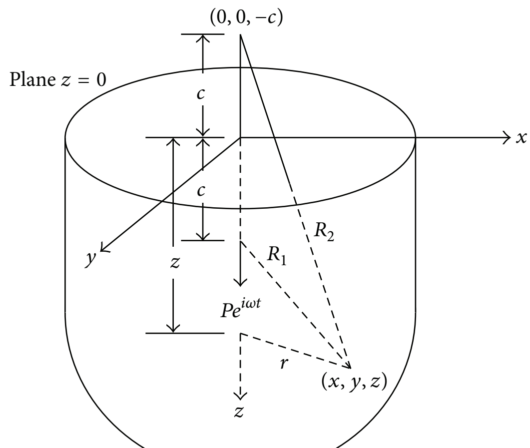


Figure 1: Periodic force normal to the boundary in the interior of a semi-infinite solid (after Banerjee and Mamoon [4]).

$$A_6 = -\left(\frac{15c_2^2}{s^2R_2^2}+\frac{15c_2}{sR_2}+\frac{sR_2}{c_2}+6\right)\frac{e^{-sR_2/c_2}}{R_2^3} + \frac{c_2^2}{c_1^2}\left(\frac{15c_1^2}{s^2R_2^2}+\frac{15c_1}{sR_2}+\frac{s^2R_2^2}{c_1^2}+\frac{4sR_2}{c_1}+9\right)\frac{e^{-sR_2/c_1}}{R_2^3}, \tag{1c}$$

$$B_6 = \left(\frac{3c_2^2}{s^2R_2^2}+\frac{3c_2}{sR_2}+1\right)\frac{e^{-sR_2/c_2}}{R_2^3} - \frac{c_2^2}{c_1^2}\left(\frac{3c_1^2}{s^2R_2^2}+\frac{3c_1}{sR_2}+\frac{sR_2}{c_1}+2\right)\frac{e^{-sR_2/c_1}}{R_2^3}, \tag{1d}$$

where μ is the shear modulus, ν is the Poisson's ratio, c_1 and c_2 are the pressure and shear wave velocities, respectively, and $s = i\omega$ is the Laplace transformed parameter (note that, in the digital copy of the original paper, the square in (1a) is missing).

3. A Possible Problem and Comparison with Other Solutions

A close inspection on (1a)–(1d) reveals that, after substituting A_6 and K_6 into (1a), the third term in the second component of A_6 yields

$$-\frac{c_2^2}{2\pi\mu c_1^4(3-4\nu)}\cdot\frac{c^2(z+c)^2}{R_2^3}\cdot e^{-sR_2/c_1}. \tag{2}$$

When the depth of the periodic point force goes to infinity, that is, $c \to \infty$, this term also goes to infinity. Further examination shows that this term cannot be cancelled out when summing up all the six solutions, which means that the z-displacement at $\boldsymbol{\xi}(x, y, z)$ goes to infinity as $c \to \infty$. However, the displacements at a given point $\boldsymbol{\xi}(x, y, z)$ with finite z are expected to vanish when $c \to \infty$, since the distance between source and receiver goes to infinity.

To demonstrate the discrepancy between B&M's solution and other solutions, a test problem is used. The semi-infinite solid is characterized by mass density $\rho = 2000\,\mathrm{kg/m^3}$, shear modulus $\mu = 180$ MPa, and Poisson's ratio $\nu = 0.2$. The excitation frequency $f = 10$ Hz, and thus the shear wavelength is $\lambda_2 = 30$ m. The amplitude of the periodic force is $P = 1$ N. The free surface z-displacement responses at different radii as the depth of the source changes from $c = 0$ to $c = 10\lambda_2$ are calculated using B&M's solution and a semi-analytical solution presented by Maurel et al. in [6]. Both solutions are coded in MATLAB. The results are compared in Figure 2, where the z-displacement calculated using the former solution oscillates and tends to increase when c increases, while that obtained using the latter solution decays quickly with increasing c.

For further comparison, a numerical solution is computed using the commercial FEM software ABAQUS. As shown in Figure 3, a 2D axisymmetric model of size 2000 m × 2000 m is built. The top surface is free; the bottom and side surfaces are fixed. A periodic force $P = 1$ N of frequency $f = 10$ Hz is applied along the axis of symmetry in the positive z-direction. The P wave speed is approximately 490 m/s. To avoid reflections from the fixed boundaries, the simulation time is set to be 4 s. The time-dependent free surface z-displacement responses at radii $r = \lambda_2$ and $r = 10\lambda_2$ are plotted in Figure 4, from which it can be seen that the surface points begin to oscillate periodically and stably in a short time after the initial arrival of the wave. Therefore, the discrete inverse Fourier transform is applied to the time series between 2 s and 4 s to obtain the displacement response at 10 Hz.

Figure 5 compares the frequency-domain z-displacement responses along the free surface when the source is buried at different depths computed using these three methods. In general, the numerical solution agrees with Maurel's solution, while B&M's solution is different. And the difference increases with source depth. In [4], Banerjee and Mamoon compared their solution with those obtained by Whittaker and Christiano [10] and Kobayashi and Nishimura [11] for the time-harmonic Boussinesq problem; that is, $c = 0$. The results for $a_0 = \omega r/c_2 = 2\pi r/\lambda_2$ between 0 and 2.0 are shown in Figures 7–10 in [4] and good agreement is displayed. The position of $a_0 = 2.0$ is denoted by a vertical line in Figure 5(a), from which it can be seen that when $c = 0$, B&M's solution agrees reasonably well with other solutions as $a_0 < 2.0$ and deviates from other solutions as $a_0 > 2.0$.

4. Discussion

The above comparison indicates that Banerjee and Mamoon's solution for the elastodynamic half-space problem is incorrect. The reason might be that the results of Mindlin's superposition method for the elastostatic problem cannot be simply extended to the dynamic case.

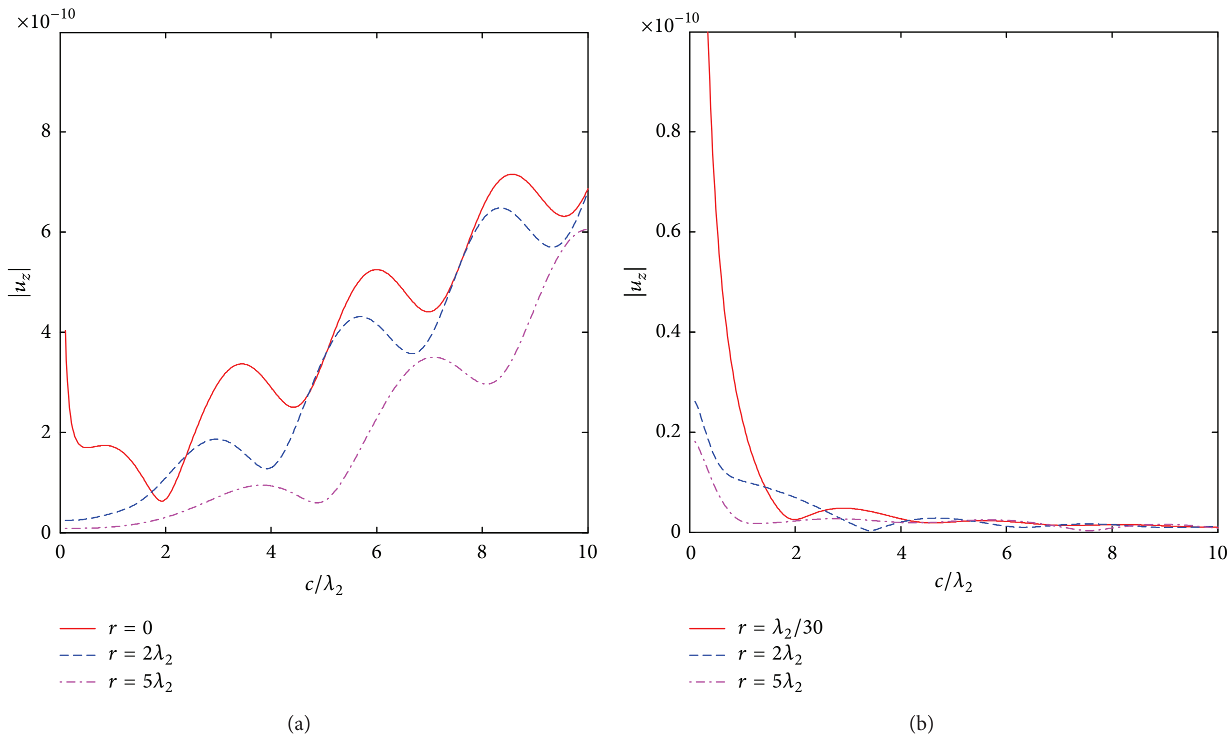


Figure 2: Comparison of free surface responses at different radii for $c = 0$ to $c = 10\lambda_2$: (a) B&M's solution and (b) solution of Maurel et al. [6].

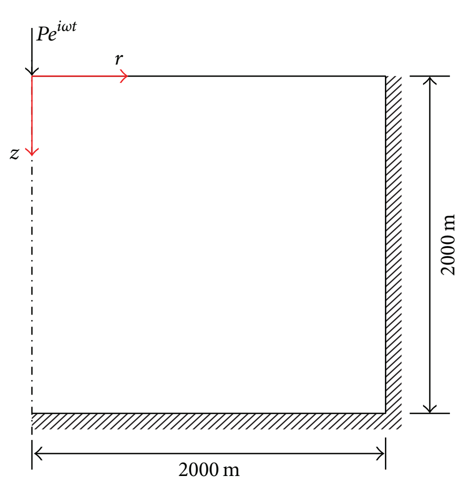


Figure 3: The 2D axisymmetric model for numerical simulation.

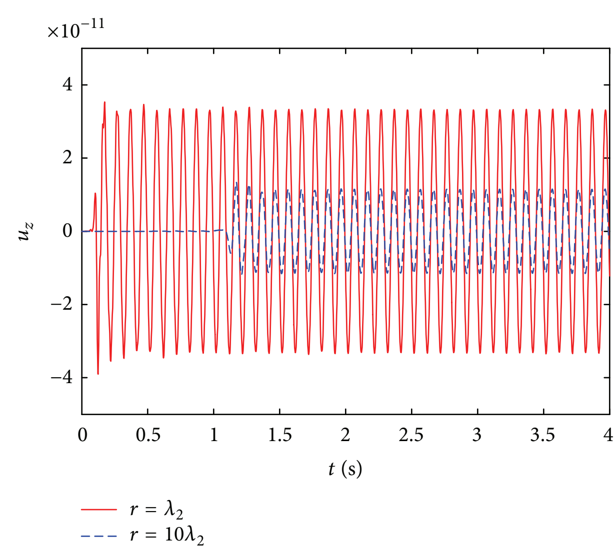


Figure 4: The time-dependent free surface z-displacement responses at different radii.

For the static problem, the coefficients of the six solution components are determined by imposing the boundary conditions and equilibrium condition [8]. The boundary conditions for the free surface $z = 0$ are

$$[\sigma_{zz}]_{z=0} = [\sigma_{zr}]_{z=0} = 0, \tag{3}$$

where σ_{zz} and σ_{zr} are the normal and shear stresses. The equilibrium condition is given by

$$P = -\int_0^{\infty} 2\pi r \sigma_{zz} \mathrm{d}r, \quad (z > c), \tag{4}$$

where P is the static point force applied at $(0, 0, +c)$.

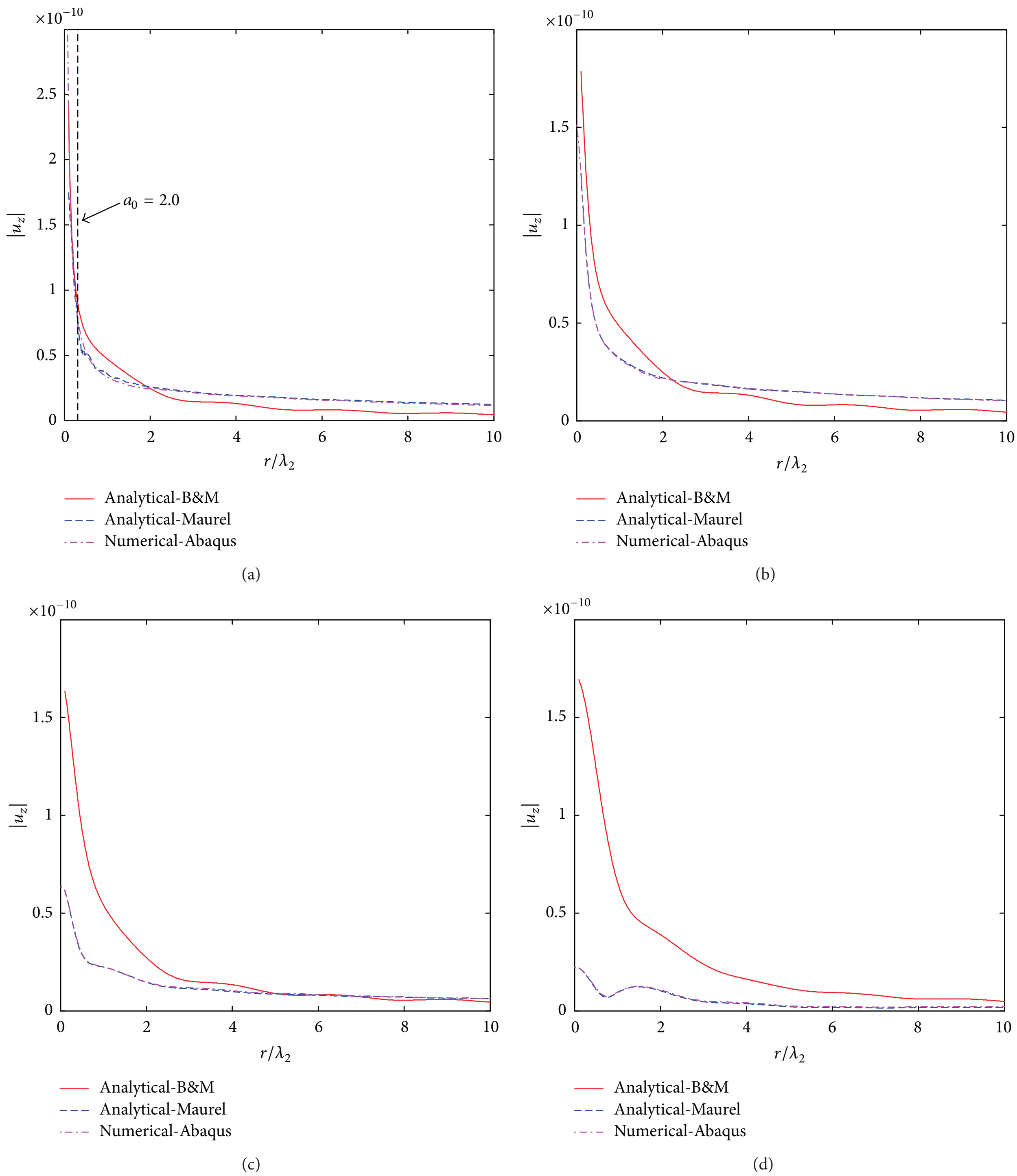


FIGURE 5: Comparison of free surface responses computed using different methods and for different source depths: (a) $c = 0$, (b) $c = \lambda_2/4$, (c) $c = \lambda_2/2$, and (d) $c = \lambda_2$.

For the dynamic case, the boundary conditions still hold. However, whether these boundary conditions can be satisfied is not explicitly stated in [4]. Furthermore, the equilibrium condition given by (4) will be inapplicable to the dynamic case due to the presence of the inertia force that is associated with acceleration. To consider the inertia force caused by time-harmonic excitation, the equilibrium condition can be rewritten as

$$P = -\int_0^{\infty} 2\pi r \sigma_{zz} dr - \int_0^{z} \int_0^{\infty} 2\pi r \rho \omega^2 u_z dr\, dz, \quad (z > c), \tag{5}$$

where u_z is the displacement in z-direction.

The boundary conditions are examined first. The expressions of the stresses are derived. Since these expressions are lengthy, only the stress components σ_{zz} and σ_{zr} on the free surface, that is, $z = 0$, for the first and fifth problems are presented here:

$$\sigma_{zz}^{1} = (\lambda + 2\mu) K_1 \left(A_1 - B_1 \frac{c^2}{R_1^2} \right) + \lambda K_1 \left(C_1 - B_1 \frac{r^2}{R_1^2} \right), \tag{6a}$$

$$K_1 = \frac{1}{4\pi\mu}, \tag{6b}$$

$$A_1 = \left(\frac{9c_2^2}{s^2R_1^2} + \frac{9c_2}{sR_1} + \frac{sR_1}{c_2} + 4 \right) \frac{ce^{-sR_1/c_2}}{R_1^3} - \frac{c_2^2}{c_1^2} \left(\frac{9c_1^2}{s^2R_1^2} + \frac{9c_1}{sR_1} + 3 \right) \frac{ce^{-sR_1/c_1}}{R_1^3}, \tag{6c}$$

$$B_1 = \left(\frac{15c_2^2}{s^2R_1^2} + \frac{15c_2}{sR_1} + \frac{sR_1}{c_2} + 6 \right) \frac{ce^{-sR_1/c_2}}{R_1^3} - \frac{c_2^2}{c_1^2} \left(\frac{15c_1^2}{s^2R_1^2} + \frac{15c_1}{sR_1} + \frac{sR_1}{c_1} + 6 \right) \frac{ce^{-sR_1/c_1}}{R_1^3}, \tag{6d}$$

$$C_1 = \left(\frac{6c_2^2}{s^2R_1^2} + \frac{6c_2}{sR_1} + 2 \right) \frac{ce^{-sR_1/c_2}}{R_1^3} - \frac{c_2^2}{c_1^2} \left(\frac{6c_1^2}{s^2R_1^2} + \frac{6c_1}{sR_1} + 2 \right) \frac{ce^{-sR_1/c_1}}{R_1^3}, \tag{6e}$$

where $\lambda = 2\mu\nu/(1-2\nu)$ is the Lame constant:

$$\sigma_{zz}^{5} = \lambda K_5 \int_c^\infty f_1 \mathrm{d}\zeta + (\lambda + 2\mu) K_5 \int_c^\infty f_2 \mathrm{d}\zeta, \tag{7a}$$

$$K_5 = \frac{(1-\nu)(1-2\nu)}{\pi\mu(3-4\nu)}, \tag{7b}$$

$$f_1 = -A_5 \frac{r^2}{R^2} + 2B_5; \quad f_2 = -A_5 \frac{\zeta^2}{R^2} + B_5, \tag{7c}$$

$$A_5 = \left(\frac{15c_2^2}{s^2R^2} + \frac{15c_2}{sR} + \frac{sR}{c_2} + 6 \right) \frac{e^{-sR/c_2}}{R^3} - \frac{c_2^2}{c_1^2} \left(\frac{15c_1^2}{s^2R^2} + \frac{15c_1}{sR} + \frac{s^2R^2}{c_1^2} + \frac{4sR}{c_1} + 9 \right) \frac{e^{-sR/c_1}}{R^3}, \tag{7d}$$

$$B_5 = \left(\frac{3c_2^2}{s^2R^2} + \frac{3c_2}{sR} + 1 \right) \frac{e^{-sR/c_2}}{R^3} - \frac{c_2^2}{c_1^2} \left(\frac{3c_1^2}{s^2R^2} + \frac{3c_1}{sR} + \frac{sR}{c_1} + 2 \right) \frac{e^{-sR/c_1}}{R^3}, \tag{7e}$$

where $R = \{r^2 + (z+\zeta)^2\}^{1/2}$ is the distance from the virtue source point $(0, 0, -\zeta)$ to the point $\boldsymbol{\xi}(x, y, z)$:

$$\sigma_{zr}^{1} = \mu K_1 \left(2B_1 \frac{c^2}{R_1^2} - D_1 \right), \tag{8a}$$

$$D_1 = \left(\frac{6c_2^2}{s^2R_1^2} + \frac{6c_2}{sR_1} + \frac{sR_1}{c_2} + 3 \right) \frac{re^{-sR_1/c_2}}{R_1^3} - \frac{c_2^2}{c_1^2} \left(\frac{6c_1^2}{s^2R_1^2} + \frac{6c_1}{sR_1} + 2 \right) \frac{re^{-sR_1/c_1}}{R_1^3}, \tag{8b}$$

$$\sigma_{zr}^{5} = 2\mu K_5 \int_c^\infty f_3 \mathrm{d}\zeta, \tag{9a}$$

$$f_3 = -A_5 \frac{\zeta r}{R^2}. \tag{9b}$$

As shown in (6a)–(9b), it is difficult to analytically determine where σ_{zz} or σ_{zr} equals zero on the free surface. To further simplify the expressions, let the depth of the source point be zero; that is, $c = 0$. The expressions for the stress components can be rewritten as follows:

$$\sigma_{zz}^{1} = \sigma_{zz}^{2} = \sigma_{zz}^{3} = \sigma_{zz}^{4} = \sigma_{zz}^{6} = 0, \tag{10}$$

$$\sigma_{zz}^{5} = \lambda K_5 \int_0^\infty f_1 \mathrm{d}\zeta + (\lambda + 2\mu) K_5 \int_0^\infty f_2 \mathrm{d}\zeta, \tag{11}$$

$$\sigma_{zr}^{1} = \mu K_1 (-E_1); \quad \sigma_{zr}^{2} = \mu K_2 (-E_1), \tag{12a}$$

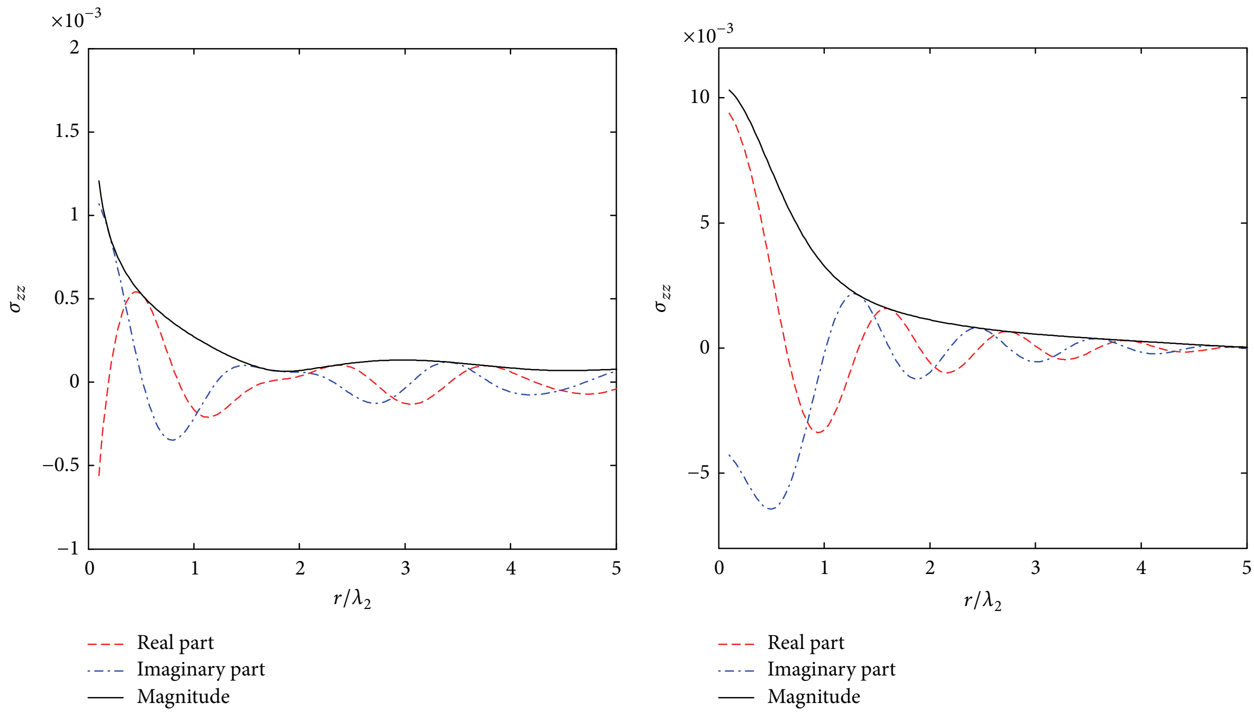
$$K_2 = \frac{3-4\nu}{4\pi\mu}, \tag{12b}$$

$$E_1 = \left(\frac{6c_2^2}{s^2r^2} + \frac{6c_2}{sr} + \frac{sr}{c_2} + 3 \right) \frac{e^{-sr/c_2}}{r^2} - \frac{c_2^2}{c_1^2} \left(\frac{6c_1^2}{s^2r^2} + \frac{6c_1}{sr} + 2 \right) \frac{e^{-sr/c_1}}{r^2}, \tag{12c}$$

$$\sigma_{zr}^{3} = \sigma_{zr}^{4} = \sigma_{zz}^{6} = 0, \tag{13}$$

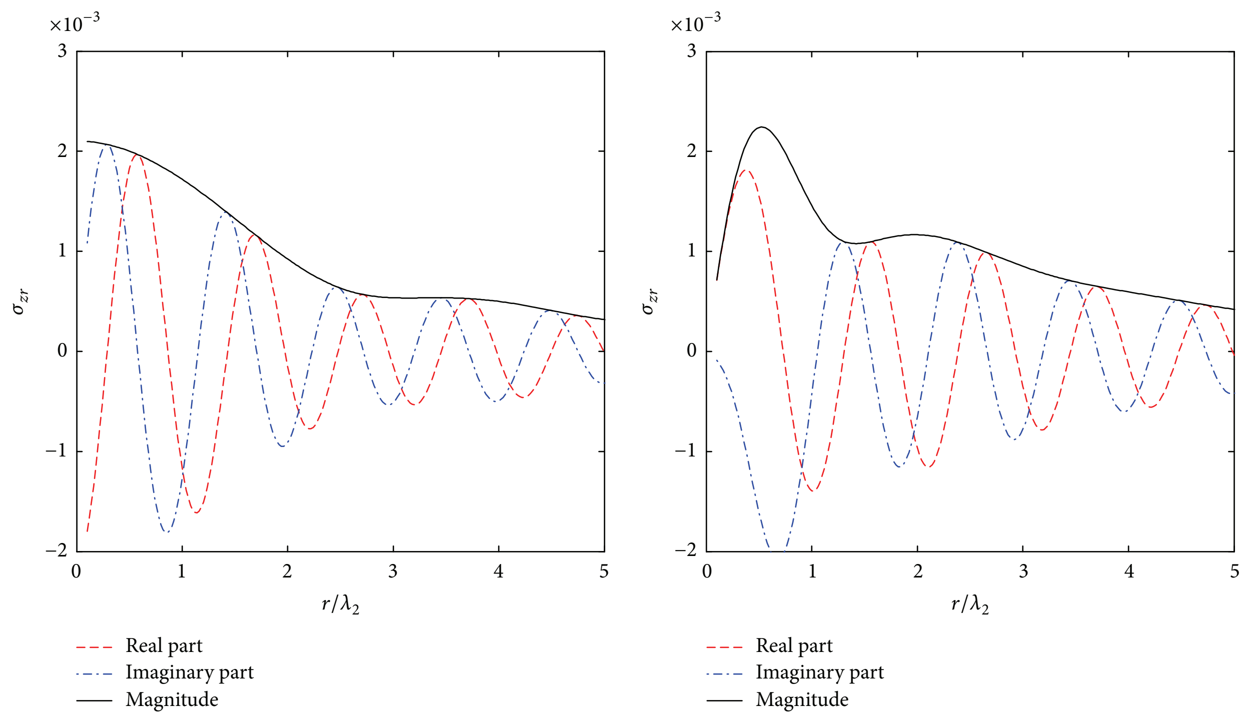
$$\sigma_{zr}^{5} = 2\mu K_5 \int_0^\infty f_3 \mathrm{d}\zeta. \tag{14}$$

From (10) and (11), it can be seen that although σ_{zz} of the other five problems are zero, the value of σ_{zz}^5 cannot be evaluated analytically since it involves nonintegrable integrands. As for σ_{zr} given by (12a)–(14), σ_{zr}^1, σ_{zr}^2, and σ_{zr}^5 are nonzero. Whether they can be cancelled out when summing up cannot be determined analytically either.



(a) (b)

FIGURE 6: σ_{zz} on the free surface for different source depths: (a) $c = 0$ and (b) $c = \lambda_2$.



(a) (b)

FIGURE 7: σ_{zr} on the free surface for different source depths: (a) $c = 0$ and (b) $c = \lambda_2$.

With (6a)–(14) in hand, σ_{zz} and σ_{zr} on the free surface are computed numerically using MATLAB. The distributions of σ_{zz} and σ_{zr} on the free surface for different source depths are displayed in Figures 6 and 7, respectively, from which it can be seen clearly that the normal and shear stresses are nonzero. Therefore, the boundary conditions for the free surface given by (3) are not satisfied, which means that the superposition of the six full-space elastodynamic solutions does not solve the dynamic half-space problem.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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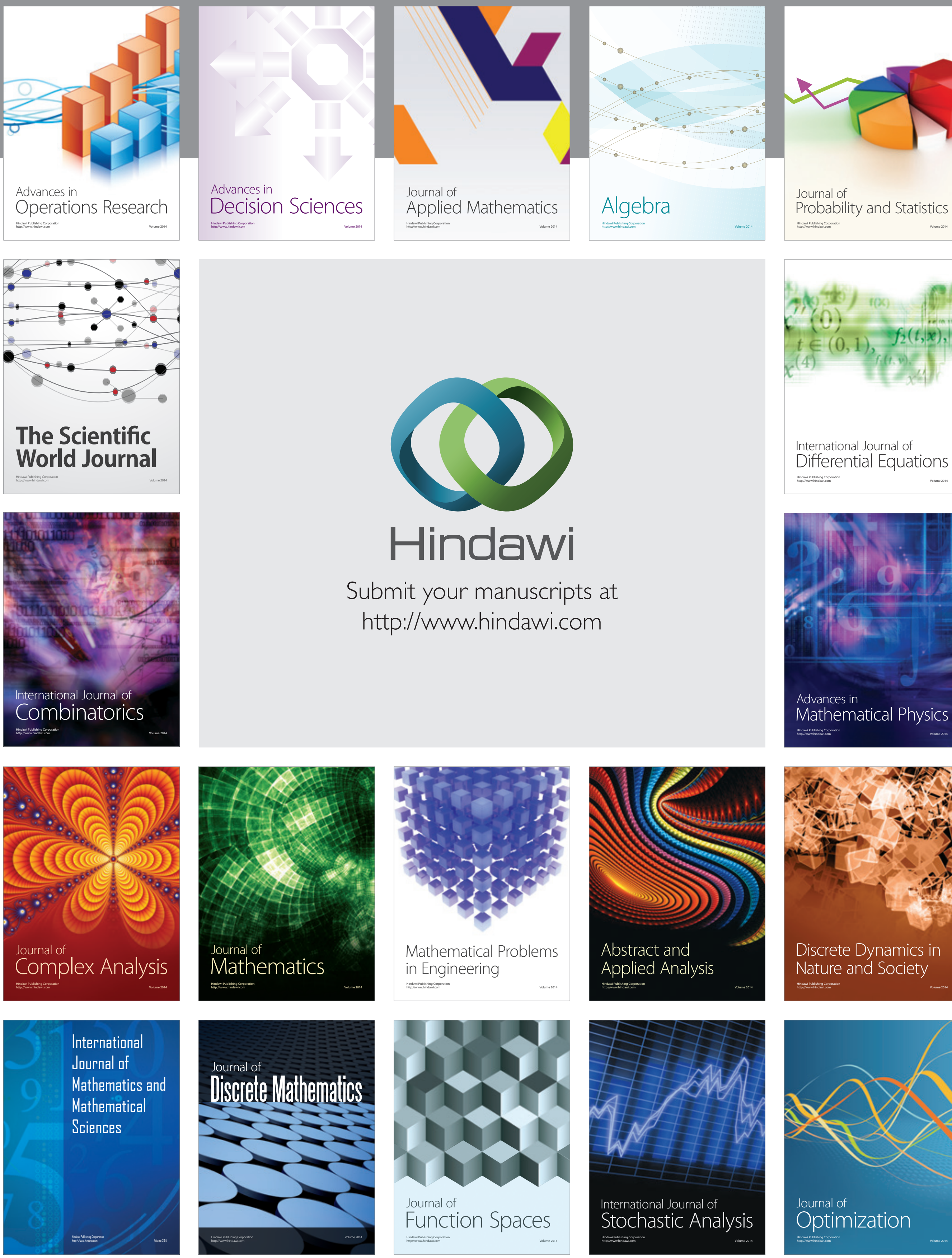
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