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*Review Article*

# Quantum Analysis on Time Behavior of a Lengthening Pendulum

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Quantum properties of a lengthening pendulum are studied under the assumption that the length of the string increases at a steady rate. Advanced analysis for various physical problems in several types of quantum states, such as propagators, Wigner distribution functions, energy eigenvalues, probability densities, and dispersions of physical quantities, is carried out using quantum wave functions of the system. In particular, the time behavior of Gaussian-type wave packets is investigated in detail. The probability density for a Gaussian wave packet displaced in the positive $\theta$ at $t = 0$ oscillates back and forth from the center ($\theta = 0$). This phenomenon is very similar to the classical motion of the pendulum. As a consequence, we can confirm that there is a correspondence between its quantum and classical behaviors. When we analyze a dynamical system in view of quantum mechanics, the quantum and classical correspondence is very important in order for the associated quantum theory to be valid and viable.

## 1. Introduction

If the Hamiltonian of a system is dependent on time, it is classified as a time-dependent Hamiltonian system (TDHS). Typical examples of TDHSs include a damped harmonic oscillator [1–3], a harmonic oscillator driven by a time-varying force [4, 5], and a harmonic oscillator with time-dependent parameters such as time-dependent mass [6–9] and/or frequency [10–12]. A powerful method for solving quantum solutions of a TDHS is the invariant operator method [13–17].

Among various types of TDHSs, we study quantum dynamical properties of a lengthening pendulum [18–23] in this paper. This kind of pendulum is an interesting topic as a fundamental nonstationary mechanical system and its study may enable us to acquire elementary methods for manipulating more complicated general dynamical systems. The derivation of quantum mechanical solutions of such time-varying swinging objects and the analysis of them with high precision demands exquisite theory of quantum physics and chaos based on advanced mathematical techniques.

Much attention has been paid to the classical and quantum problems of the lengthening pendulum up to now. The time variation of angular and linear amplitudes has been analyzed by Brearley for a simple lengthening pendulum under the assumption that the amplitude of the oscillation is small [18]. The characteristics of quantum wave functions in Fock state for the lengthening pendulum have been investigated by Um et al. [21]. The research group of McMillan et al. [23] studied a parametric amplification of its oscillatory motion and the physical mechanism associated with the energy transfer to the pendulum via a time-varying parameter (the length of pendulum) under the assumption that the lengthening pendulum is driven by an arbitrary force. Nevertheless, as far as we know, the exact analytical solutions have not been obtained yet due to the difficulty in mathematical treatment of the differential equation describing the motion of the pendulum, which involves a sinusoidal function. The derivation of quantum solutions of the system with higher accuracy than what we can do by hand requires the aid of computer programs such as MATLAB [23].

The pendulum or the lengthening thereof not only is a good mathematical model for practicing a nontrivial problem in mechanics but also can possibly be applied to many actual systems. A gravitational pendulum which has been used as a seismic sensor of translation and rotation is one of its well-known applications [24]. Usually seismologists use a kind of sensing device (pendulum) in order to detect earthquake waves, which is more complicated than the simple pendulum. The theory of the lengthening pendulum can also be applied to analyzing a drifting spiral motion of a charged particle in mirror-confined plasma [25]. In this case, a position-dependent magnetic field, which is a dense and weak magnetic field, is responsible for the change of the rotational radius along its spiral motion.

We will study in this work several quantum states of the lengthening pendulum beyond the Fock state. We will compare quantum motions of the pendulum with its classical behaviors. Quantized energy and its dispersion will be investigated via a rigorous development of the quantum theory associated with the pendulum.

This paper is organized as follows. The fundamental mechanics of the lengthening pendulum is surveyed in Section 2 and various quantum mechanical properties of the system are analyzed in the subsequent sections. Section 3 is devoted to the study of the time behavior of the quantized energy. Path integral formulation of quantum mechanics for the system is treated in Section 4. Wigner distribution function (WDF) of the system is investigated in Section 5. Quantum properties of the superposition of two neighboring states are discussed in Section 6. The correspondence between classical and quantum mechanics for the pendulum is demonstrated in Section 7 through Gaussian wave packet description of the system. The concluding remarks are given in Section 8 which is the last section.

## 2. Fundamentals of the Lengthening Pendulum

Now we are carrying out a survey of the fundamentals of the lengthening pendulum. Let us consider a pendulum of a relatively massive object hung by a vertical string from a fixed ceiling and assume that it swings from its fixed equilibrium position. Suppose that the length of the pendulum increases with a constant rate $k$ from its initial length $l_0$.

$$l(t) = l_0 + kt. \tag{1}$$

In case $k$ is sufficiently small, the pendulum undergoes an adiabatic change and such case will only be considered in this review.

We do not regard the quantum effects of the $l$-component for the sake of simplicity. Then, the Hamiltonian considered up to the second order in $\theta$ is given by [21]

$$\widehat{H}\left(\widehat{\theta}, \widehat{p}_\theta\right) = \frac{\widehat{p}_\theta^2}{2ml^2(t)} + \frac{1}{2}mgl(t)\widehat{\theta}^2 + \mathscr{H}(t), \tag{2}$$

where $\mathscr{H}(t) = mk^2/2 - mgl(t)$. A classical solution of the system can be represented as [22]

$$\theta(t) = cM(t)\,e^{i[\gamma(t)+c_0]}. \tag{3}$$

Here, $c$ and $c_0$ are arbitrary constants and

$$\begin{aligned} M(t) &= \frac{1}{\sqrt{l(t)}}\left[J_1^2(\xi(t)) + N_1^2(\xi(t))\right]^{1/2}, \\ \gamma(t) &= \tan^{-1}\frac{N_1(\xi(t))}{J_1(\xi(t))}, \end{aligned} \tag{4}$$

where $J$ and $N$ are Bessel functions of the first kind and the second kind, respectively, and $\xi(t) = 2[gl(t)]^{1/2}/k$. A more general solution for this system is given in [21].

The construction of the annihilation and creation operators is useful for developing a quantum theory of a TDHS. According to the invariant operator theory [15], the formulae of them are somewhat different from those of the simple harmonic oscillator. For instance, the annihilation operator is given by [21]

$$\widehat{a} = \sqrt{\frac{1}{2\hbar\dot{\gamma}(t)ml^2(t)}}\left[ml^2(t)\left(\dot{\gamma}(t) - i\frac{\dot{M}(t)}{M(t)}\right)\widehat{\theta} + i\widehat{p}_\theta\right]. \tag{5}$$

Of course, the Hermitian adjoint, $\widehat{a}^\dagger$, of the above equation plays the role of the creation operator. These operators satisfy the boson commutation relation, such that $[\widehat{a}, \widehat{a}^\dagger] = 1$.

If we use the property of this commutation relation, we can easily derive the wave functions in Fock state, which are represented in terms of some time functions $\phi_n(\theta, t)$ and time-dependent phases $\epsilon_n(t)$ [21, 22]:

$$\psi_n(\theta, t) = \phi_n(\theta, t)\,e^{i\epsilon_n(t)}, \tag{6}$$

where the quantum number is given by $n = 1, 2, 3, \ldots$. (All subsequent $n$ will follow this convention unless we particularly specify it.) The explicit forms of $\phi_n(\theta, t)$ and $\epsilon_n(t)$ are given by

$$\begin{aligned} \phi_n(\theta, t) &= \frac{1}{\sqrt{2^n n!}}\left(\frac{\Lambda(t)}{\pi}\right)^{1/4} H_n\left(\sqrt{\Lambda(t)}\theta\right) e^{-(1/2)\Lambda(t)\kappa_1^*(t)\theta^2}, \\ \epsilon_n(t) &= -\left(n + \frac{1}{2}\right)\gamma(t), \end{aligned} \tag{7}$$

where $H_n$ is $n$th-order Hermite polynomial and

$$\Lambda(t) = P(t)\frac{\sqrt{g}m[l(t)]^{3/2}}{2\hbar[J_1^2(\xi(t)) + N_1^2(\xi(t))]}, \qquad \kappa_1(t) = 1 + i\Pi(t), \tag{8}$$

with

$$\begin{aligned} P(t) &= J_1(\xi(t))[N_0(\xi(t)) - N_2(\xi(t))] \\ &\quad - N_1(\xi(t))[J_0(\xi(t)) - J_2(\xi(t))], \\ \Pi(t) &= [P(t)]^{-1}\left\{Q(t) - \frac{k[J_1^2(\xi(t)) + N_1^2(\xi(t))]}{\sqrt{gl(t)}}\right\}, \\ Q(t) &= J_1(\xi(t))[J_0(\xi(t)) - J_2(\xi(t))] \\ &\quad + N_1(\xi(t))[N_0(\xi(t)) - N_2(\xi(t))]. \end{aligned} \tag{9}$$

The wave functions given in (6) are the basic information necessary for studying quantum features of the system. We will investigate quantum characteristics of the pendulum for more complicated cases in the next section.

## 3. Quantized Energy

Bohr tried to merge quantum and classical mechanics by introducing a correspondence principle between them. Even though the results of quantum and classical descriptions for a system more or less overlap under particular limits, their underlying principles are quite different from each other. There are significant differences between consequences of the two theories when the quantum number is sufficiently low, whereas those with a higher quantum number look very similar. It is well known that we can compute the probability for finding the pendulum at a particular angle from the square of the given wave function. In cases of higher energy states, the most probable angle of the pendulum will shift away from the center ($\theta = 0$) and the intervals between any two adjacent peaks in the graph of probability densities in Fock state become narrow. Due to this trend associated with the probability density, the behavior of the quantum pendulum in a high energy limit may look like that of the counterpart classical one.

Another difference between them is that while the classical probability density function is confined within the two classical turning points, the quantum probabilities extend beyond the classically allowed angle. For more detailed investigations about this consequence, we consider the quantum and the classical energies of the system that are given by

$$E_n = \left(n + \frac{1}{2}\right)\frac{\hbar}{2}\left[\frac{\hbar\Lambda(t)}{ml^2(t)}|\kappa_1(t)|^2 + \frac{mgl(t)}{\hbar\Lambda(t)}\right] + \mathscr{H}(t), \tag{10}$$

$$E_{\rm cl} = \frac{1}{2}ml^2(t)\dot{\theta}^2 + \frac{1}{2}mgl(t)\theta^2 + \mathscr{H}(t). \tag{11}$$

Let us evaluate the probability that the pendulum stays outside the classically allowed region. If we denote the amplitude of angle as $\Theta$, the maximum classical potential energy is given by $mgl(t)\Theta^2/2 + \mathscr{H}(t)$. Then, regarding the classical turning point of the pendulum associated with the ground state wave packet, we equate the ground state energy $E_0$ with the maximum classical potential energy:

$$E_0 = \frac{1}{2}mgl(t)\Theta^2 + \mathscr{H}(t). \tag{12}$$

Thus, by inserting (10) with $n = 0$ into the left-hand side of this equation, the classical amplitude can be evaluated to be

$$\Theta = \left[\frac{\hbar}{2mgl(t)}\left(\frac{\hbar\Lambda(t)}{ml^2(t)}|\kappa_1(t)|^2 + \frac{mgl(t)}{\hbar\Lambda(t)}\right)\right]^{1/2}. \tag{13}$$

The probability density outside the classically allowed region is given by

$$P_{\rm outside} = 2\int_\Theta^\pi |\psi_0(\theta,t)|^2 d\theta \simeq 2\int_\Theta^\infty |\psi_0(\theta,t)|^2 d\theta. \tag{14}$$

Here, factor 2 is multiplied considering the equal contribution of the probability from outside of the opposite turning angle, $[-\pi, -\Theta]$. In addition, we have supposed that the oscillating amplitude is sufficiently small in order to replace the integral interval $[\Theta, \pi]$ in (14) with $[\Theta, \infty]$. This extension of the scope of integration will also be equally applied in further similar calculations. By substituting (6) with $n = 0$ into (14), we have

$$P_{\rm outside} = \sqrt{\frac{4}{\pi}}\left[1 - {\rm erf}\left(\sqrt{\Lambda(t)}\Theta\right)\right]. \tag{15}$$

This is the probability that the quantum pendulum remains within the classically forbidden regions.

In fact, the description of quantized energy for a TDHS, such as (10), is a delicate problem. While some authors [1, 3, 11, 21, 22] including ours have investigated quantized energy levels for specific TDHSs, there is another opinion [26, 27] that such energy levels do not exist for the case of TDHSs. Hence, explicit demonstrations of the existence of quantized energy levels may be an interesting research topic for further study in the future in this field.

## 4. Propagator

The path integral formulation of quantum mechanics was found by Feynman [28] as an alternative quantum description of dynamical systems. Historically, this achievement was partly inspired by Dirac's idea of a quantum mechanical description [29]. The techniques of path integrals continued to be advanced until now, leading to providing many useful tools for solving problems in quantum mechanics. One of the efficient uses of the path integral method is to evaluate a propagator. This achieved great success in both quantum mechanics and quantum field theory, because a propagator involves all the information of the quantum system. Through the analysis of the propagator, we can confirm how to propagate quantum wave packets from an initial angle and time $(\theta', t')$ to a final angle and time $(\theta, t)$.

In terms of wave functions, the propagator is given by

$$K\left(\theta, t; \theta', t'\right) = \sum_{n=0}^{\infty} \psi_n(\theta, t)\, \psi_n^*\left(\theta', t'\right). \quad (16)$$

Let us calculate the complete analytical form of the propagator of the system using this definition. By inserting (6) with (7), we have

$$K = \left(\frac{\Lambda(t)\Lambda(t')}{\pi^2}\right)^{1/4} e^{-(1/2)[\Lambda(t)\kappa_1^*(t)\theta^2 + \Lambda(t')\kappa_1(t')\theta'^2]} e^{-(i/2)[\gamma(t)-\gamma(t')]} \sum_{n=0}^{\infty} \frac{e^{-in[\gamma(t)-\gamma(t')]}}{2^n n!} H_n\left(\sqrt{\Lambda(t)}\theta\right) H_n\left(\sqrt{\Lambda(t')}\theta'\right). \quad (17)$$

From the use of Mehler's formula [30]

$$\sum_{n=0}^{\infty} \frac{(z/2)^n}{n!} H_n(x) H_n(y) = \frac{1}{(1-z^2)^{1/2}} \exp\left[\frac{2xyz - \left(x^2+y^2\right)z^2}{1-z^2}\right], \quad (18)$$

it becomes

$$K = \frac{\sqrt[4]{\Lambda(t)\Lambda(t')}}{\sqrt{2\pi i \sin[\gamma(t)-\gamma(t')]}} \cdot \exp\left(-i\frac{\sqrt{\Lambda(t)\Lambda(t')}\theta\theta'}{\sin[\gamma(t)-\gamma(t')]}\right) \cdot e^{(i/2)\{\Lambda(t)[\Pi(t)+\cot[\gamma(t)-\gamma(t')]]\theta^2 - \Lambda(t')[\Pi(t')-\cot[\gamma(t)-\gamma(t')]]\theta'^2\}}. \quad (19)$$

We can use this result in estimating the evolution of quantum states under the given Hamiltonian of the pendulum and in studying the dynamics of the pendulum with varying length. More precisely, the propagator enables us to know how the initial state $\psi(\theta', t')$ evolves to an arbitrary later state $\psi(\theta, t)$ via the following equation

$$\psi(\theta, t) = \int_{-\pi}^{\pi} d\theta' K\left(\theta, t; \theta', t'\right) \psi\left(\theta', t'\right). \quad (20)$$

The propagator plays an important role in modern physics.

## 5. Wigner Distribution Function

The investigation of the probability distribution is useful for understanding the novel features of quantum mechanics. The WDF [31] may be the most useful one among various distribution functions that are needed when we study quantum statistical mechanics. Many quantum mechanical characteristics of a dynamical system, which are absent in classical mechanics, can be derived from WDF.

The WDFs in phase space are given by

$$W_n(\theta, p_\theta, t) = \frac{1}{\pi\hbar} \int_{-\pi}^{\pi} \psi_n^*(\theta+\alpha, t)\, \psi_n(\theta-\alpha, t)\, e^{2ip_\theta\alpha/\hbar} d\alpha. \quad (21)$$

We replace the range of the integration $[-\pi, \pi]$ with $[-\infty, \infty]$ for the same reason as that of the previous integration in (14). Using the integral formula of the form [32]

$$\int_{-\infty}^{\infty} dz e^{-z^2} H_n(z+a+b) H_n(z+a-b) = 2^n \pi^{1/2} n! L_n\left[2\left(b^2-a^2\right)\right], \quad (22)$$

we can evaluate (21), leading to

$$W_n(\theta, p_\theta, t) = \frac{(-1)^n}{\pi\hbar} e^{-Y(\theta, p_\theta, t)} L_n[2Y(\theta, p_\theta, t)], \quad (23)$$

where

$$Y(\theta, p_\theta, t) = \Lambda(t)\left[\theta^2 + \left(\Pi(t)\theta - \frac{p_\theta}{\hbar\Lambda(t)}\right)^2\right]. \quad (24)$$

This can be used to study the probability distribution for $\theta$ and $p_\theta$ of the pendulum within a quantum mechanical view point. Figure 1 shows the WDF for several values of $t$. Exact marginal distributions can be achievable with the use of a WDF, whereas it cannot be done by means of the Husimi distribution function [33]. It is allowed that the WDF takes negative values as well as positive ones in phase space. The negativity of the WDF is, in fact, one of the primary nonclassical effects of quantum mechanics [34, 35]. The WDF, given in (23), enables us to study the quantum statistical properties of the system in configuration space. Some imaginary distribution functions that can be used in a wide branch of physics were also proposed in the literature by other authors [36–40]. The width of $W_n$ in the angle space becomes small while that in the angular momentum space becomes large as time goes by.

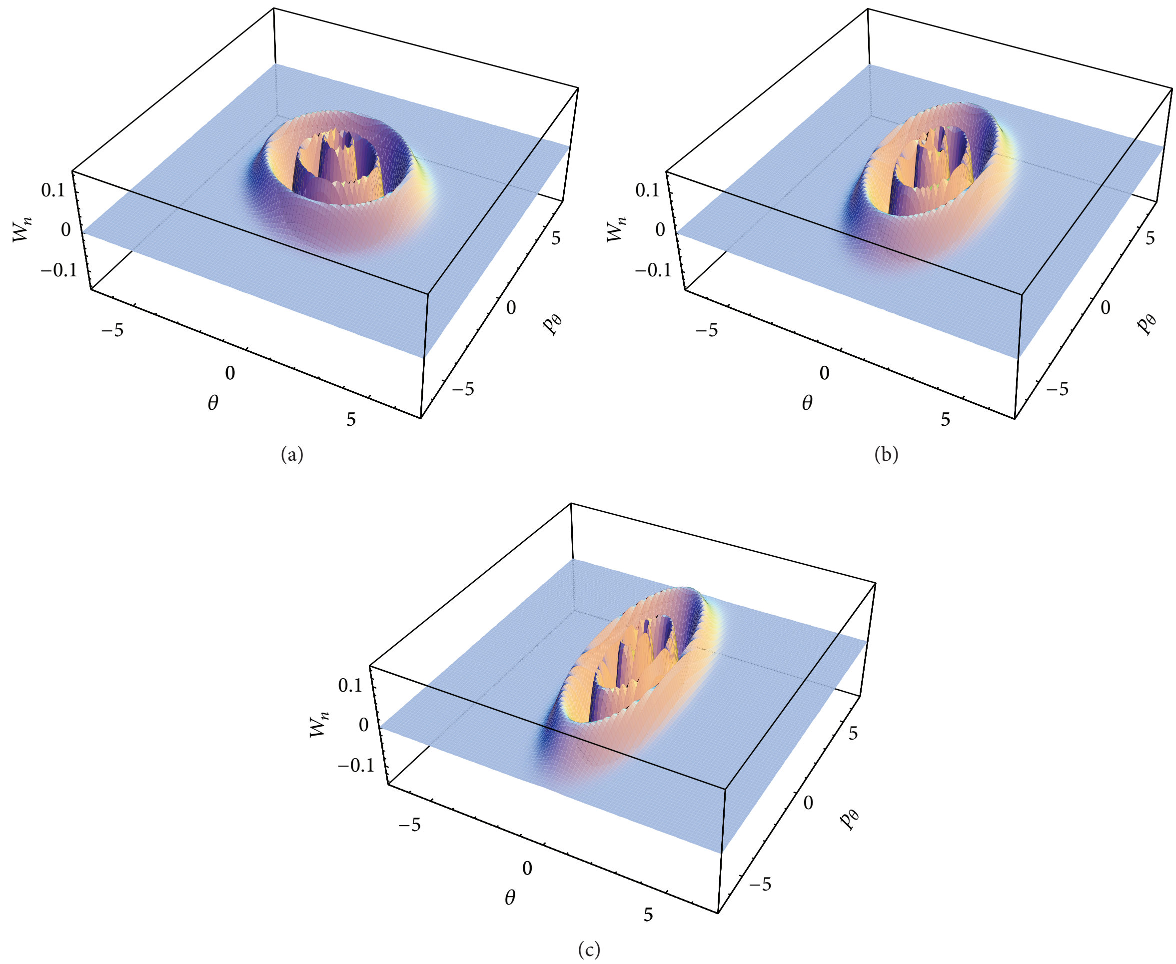


FIGURE 1: The WDF given in (23) as a function of angle $\theta$ and angular momentum $p_\theta$ where $t = 0$ for (a), $t = 2.5$ for (b), and $t = 5$ for (c). We have used $\hbar = 1$, $k = 0.2$, $m = 1$, $l_0 = 1$, $g = 1$, and $n = 5$. The width of $W_n$ associated with $\theta$ becomes small with time while that associated with $p_\theta$ becomes large.

## 6. Superposition of Two Neighboring States

The general solution of the Schrödinger equation is a linear combination of separated solutions given in (6):

$$\psi(\theta,t) = \sum_{n=0}^{\infty} c_n \psi_n(\theta,t), \quad (25)$$

where $\sum_n |c_n|^2 = 1$. To study the energy spectrum in the state, it is necessary to evaluate the eigenvalue equation of the Hamiltonian, which is

$$\widehat{H}\psi(\theta,t) = E\psi(\theta,t). \quad (26)$$

It is well known that quantum mechanics predicts only the probability for happening of particular results among a large number of mutually independent possible outcomes in physical systems. An arbitrary state function representing any dynamical state can be expanded in terms of Fock state wave functions. As a first example, let us consider the case that (25) is a superposition of two neighboring states:

$$\psi(\theta,t) = \frac{1}{\sqrt{2}} e^{i\delta_0} \left[\psi_n(\theta,t) + e^{i\delta}\psi_{n+1}(\theta,t)\right], \quad (27)$$

where phases $\delta_0$ and $\delta$ are arbitrary real constants. The expectation values of canonical variables in this state can be easily evaluated to be [41]

$$\begin{aligned} \langle \psi | \widehat{\theta} | \psi \rangle &= \sqrt{n+1} \left( \frac{\hbar}{2\dot{\gamma}(t) m l^2(t)} \right)^{1/2} \\ &\quad \cdot \cos[\gamma(t) - \delta], \\ \langle \psi | \widehat{p}_\theta | \psi \rangle &= \sqrt{n+1} \left( \frac{\hbar m l^2(t)}{2\dot{\gamma}(t)} \right)^{1/2} \\ &\quad \cdot \left( \frac{\dot{M}(t)}{M(t)} \cos[\gamma(t) - \delta] - \dot{\gamma}(t) \sin[\gamma(t) - \delta] \right). \end{aligned} \quad (28)$$

From these equations, we can confirm that the expectation values of $\widehat{\theta}$ and $\widehat{p}_\theta$ oscillate with time.

Through the similar procedure, the expectation value of the energy operator can also be obtained as

$$E = (n+1)\frac{\hbar}{2}\left(\frac{\hbar\Lambda(t)}{ml^2(t)}\left|\kappa_1(t)\right|^2 + \frac{mgl(t)}{\hbar\Lambda(t)}\right) + \mathscr{H}(t). \quad (29)$$

This is the same as the average energy of the $n$th state and $(n+1)$th state energies with the same proportion. This energy corresponds to that of an instantaneous eigenstate and varies with time like a classical one. However, this reduces to a constant value when $k = 0$, which is associated with the results of standard quantum mechanics for the simple pendulum represented in many text books in this field.

## 7. Gaussian Wave Packets

As a next example, we consider the case that the initial state function is given by

$$\psi(\theta,0) = \sqrt[4]{\frac{\eta^2}{\pi}} e^{-(1/2)\eta^2\theta^2}, \quad (30)$$

where $\eta$ is some constant which is real. In view of quantum theory, the probabilistic predictions are always obtained by squaring the modulus of a probability amplitude. The probability amplitudes that the oscillator stays in $n$th eigenstates are

$$c_n = \int_{-\pi}^{\pi} \psi_n^*(\theta,0)\,\psi(\theta,0)\,d\theta. \quad (31)$$

Then, using (6), (30), and (31) after replacing the interval of the integration $[-\pi,\pi]$ with $[-\infty,\infty]$, we obtain that

$$c_n = \frac{\sqrt{n!}}{2^{n/2}(n/2)!}\left[\frac{2\sqrt{\Lambda(0)}\eta}{\Lambda(0)\kappa_1(0)+\eta^2}\left(\frac{\Lambda(0)\kappa_1^*(0)-\eta^2}{\Lambda(0)\kappa_1(0)+\eta^2}\right)^n\right]^{1/2} \cdot e^{i(n+1/2)\gamma(0)}, \quad (32)$$

for $n = 0, 2, 4, \ldots$. On the other hand, $c_n = 0$ for $n = 1, 3, 5, \ldots$. Hence, odd order amplitudes do not contribute to the state function.

One may also possibly consider other types of wave packets described as a superposition of Fock state wave functions. Now, we are going to choose a state which seems appropriate for studying whether there is a quantum and classical correspondence concerning the behavior of the lengthening pendulum. At the early era of quantum mechanics, Schrödinger tried to explain the relation between quantum and classical mechanics by introducing coherent wave packets [42]. In order to find the possibility for the existence of a quantum solution relevant to a particular wave packet whose center oscillates with the same period as the classical motion, let us consider a state function that is expressed at $t = 0$ as

$$\psi(\theta,0) = \sqrt[4]{\frac{\Lambda(0)}{\pi}} e^{-(1/2)\Lambda(0)(\theta-\beta)^2}, \quad (33)$$

where $\beta$ is a real constant with the dimension of angle. This is the same as (30) except that $\eta^2$ is replaced by $\Lambda(0)$ and the center of the initial wave packet is displaced by an amount $\beta$. In this case, the probability amplitudes $c_n$ can also be obtained using the same method as that of the first case (the nondisplaced Gaussian case) performed with (31), such that

$$c_n = \frac{1}{\sqrt{2^n n!}}\frac{\beta^n}{[\kappa_2(0)/2]^{n+1/2}} \cdot e^{(1/2)[1/\kappa_2(0)-\Lambda(0)]\beta^2 + i(n+1/2)\gamma(0)}, \quad (34)$$

where $\kappa_2(t) = 2 + i\Pi(t)$. Here we have supposed that the increase of the length of string is sufficiently slow. With the help of (6) and (34) the time evolution of the state function (25) becomes

$$\psi(\theta,t) = \sqrt[4]{\frac{\Lambda(t)}{\pi}}\sqrt{\frac{2}{\kappa_2(0)}}\exp\left[\frac{2\beta}{\kappa_2(0)} \cdot e^{i[\gamma(0)-\gamma(t)]}\left(\sqrt{\Lambda(t)}\theta - \frac{1}{2}\frac{\beta}{\kappa_2(0)}e^{i[\gamma(0)-\gamma(t)]}\right)\right] \cdot e^{(1/2)[1/\kappa_2(0)-\Lambda(0)]\beta^2} e^{-(1/2)\Lambda(t)\kappa_1(t)\theta^2} e^{(i/2)[\gamma(0)-\gamma(t)]}. \quad (35)$$

The corresponding probability density is obtained by squaring the above equation. Thus, a direct evaluation gives

$$\left|\psi(\theta,t)\right|^2 = \sqrt{\frac{\Lambda(t)}{\pi}}\frac{2}{\left|\kappa_2(0)\right|} \cdot \exp\left[-\Lambda(t)\left(\theta - \frac{\beta[X(t)+X^*(t)]}{\sqrt{\Lambda(t)}}\right)^2\right] \cdot e^{[4/|\kappa_2(0)|^2-\Lambda(0)]\beta^2}, \quad (36)$$

where $X(t) = e^{i[\gamma(0)-\gamma(t)]}/\kappa_2(0)$. The wave packet equation (36) is a time-dependent Gaussian shape. We depicted this wave packet in Figure 2. The wave packet not only converges to the center ($\theta = 0$) but also oscillates back and forth around the center as time goes by. The amplitude of such oscillation is determined depending on the magnitude of $\beta$. This quantum behavior is very similar to the classical motion of the pendulum. The width of the wave packet is $1/\sqrt{2\Lambda(t)}$. The probabilities for the contribution of $n$th quantum state $\psi_n(\theta,t)$ to the state function are $P_n = |c_n|^2$. Thus, using (34), we have

$$P_n = \frac{1}{2^n n!}\frac{\beta^{2n}}{[|\kappa_2(0)|/2]^{2n+1}} e^{[2/|\kappa_2(0)|^2-\Lambda(0)]\beta^2}. \quad (37)$$

These probabilities do not vary with time. The mean value of quantum numbers which contribute to the state function is $\overline{n} = \sum_n nP_n$. Using (37), we see that this can be evaluated to be

$$\overline{n} = \frac{\beta^2}{2[|\kappa_2(0)|/2]^3} e^{[4/|\kappa_2(0)|^2-\Lambda(0)]\beta^2}. \quad (38)$$

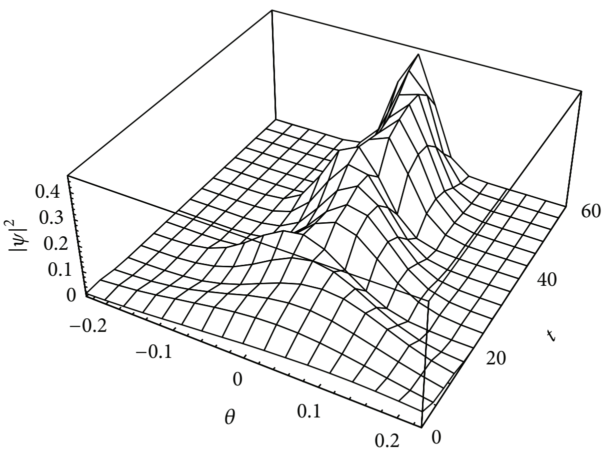


Figure 2: The probability density $|\psi|^2$ given in (36) as a function of angle $\theta$ and time $t$ for $\hbar = 1$, $k = 1$, $m = 1$, $l_0 = 6$, $g = 1$, and $\beta = 0.5$. We see that the wave packet oscillates around the center like a classical one.

Thus, for the case that $|\beta|$ is large, the contribution of high energy states is also large.

The dispersion of $n$ is given by $\Delta n = [\sum_n n^2 P_n - (\sum_n nP_n)^2]^{1/2}$. This can be easily calculated as

$$\Delta n = \left\{ \frac{1}{4} \left\{ \frac{|\kappa_2(0)|}{2} - \exp\left[ \left( \frac{4}{|\kappa_2(0)|^2} - \Lambda(0) \right) \beta^2 \right] \right\} \frac{\beta^4}{\left[ |\kappa_2(0)|^2/4 \right]^3} + \frac{\beta^2}{2\left[ |\kappa_2(0)|/2 \right]^3} \right\}^{1/2} e^{(1/2)[4/|\kappa_2(0)|^2 - \Lambda(0)]\beta^2}. \quad (39)$$

Figure 3(a) reveals that $\Delta n$ increases as $\beta$ grows.

We can also evaluate the dispersion of energy from $\Delta E = [\sum_n E_n^2 P_n - (\sum_n E_n P_n)^2]^{1/2}$. Through a mathematical procedure similar to that of the calculation of $\Delta n$, we have

$$\Delta E = \frac{\hbar\beta}{|\kappa_2(0)|^2} \left[ \frac{4\beta^2}{|\kappa_2(0)|^2} \left( \frac{\hbar\Lambda(t)}{ml^2(t)} |\kappa_1(t)|^2 + \frac{mgl(t)}{\hbar\Lambda(t)} \right) \left( \frac{|\kappa_2(0)|}{2} - e^{(4/|\kappa_2(0)|^2 - \Lambda(0))\beta^2} \right) + \frac{8\mathscr{H}(t)}{\hbar} \left( \frac{|\kappa_2(0)|}{2} - 1 \right) + 2\left( |\kappa_2(0)| - 1 \right) \right]^{1/2} \cdot e^{[2/|\kappa_2(0)|^2 - \Lambda(0)/2]\beta^2}. \quad (40)$$

From Figure 3(b), we can confirm that $\Delta E$ increases as $\beta$ grows. However, $\Delta E$ decreases with time.

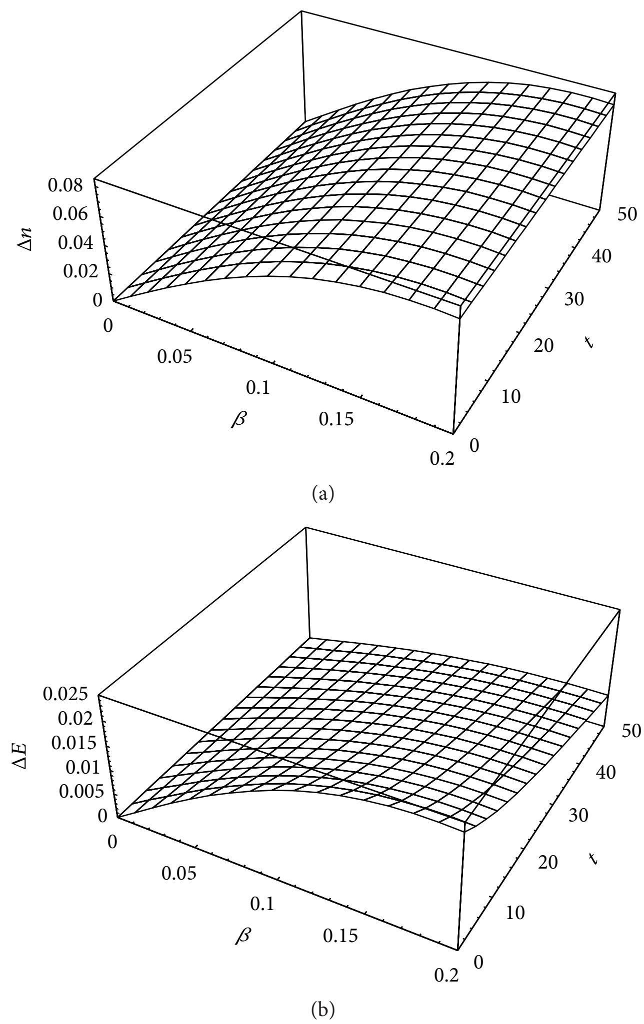


Figure 3: The dispersion $\Delta n$ (39) and $\Delta E$ (40) as a function of $\beta$ and time $t$ for $\hbar = 1$, $k = 0.5$, $m = 1$, $l_0 = 10$, and $g = 1$. Note that $\Delta n$ increases as $\beta$ grows.

## 8. Conclusion

On the basis of the Schrödinger solutions that can be obtained by taking advantage of the ladder operators $\hat{a}$ and $\hat{a}^\dagger$, various quantum states of the lengthening pendulum have been investigated. We obtained the energy eigenvalues, propagator, WDFs, uncertainties, and probability densities in several types of quantum states. The quantum behavior of the pendulum was analyzed rigorously and compared to that of the classical one.

The propagator given in (19) entirely determines the quantum behavior of the lengthening pendulum in a gravitation. If we know the wave function at an initial angle and time, it is possible to estimate probability amplitude for another angle at a subsequent time using the propagator. Wigner distribution function of the system was derived as shown in (23). This can be a negative value in some regions of phase space as well as a positive one, which signifies a novel nonclassical feature of the quantum system. We can confirm from Figure 1 that the width of $W_n$ associated with $\theta$ decreases as time goes by while that associated with $p_\theta$ increases. This appearance is due to the fact that the uncertainty in $\theta$ decreases with time while that in $p_\theta$ increases. For the variation of the uncertainty for $\theta$ and $p_\theta$, you can refer to [41].

As the length of string increases, mechanical energy cannot be conserved because some amount of work is extracted from the system [22]. This is the reason why the amplitude decreases with time, not only for the classical pendulum but for the quantum one as well.

From (28), we see that the expectation values of canonical variables in the superposition of the two neighboring states oscillate with time. The quantum energy in this case varies with time as shown in (29). For the state that is described by (30), the odd order probability amplitudes disappear since (30) is an even function about $\theta$.

To investigate the existence of the quantum and classical correspondence for the behavior of the lengthening pendulum, we have supposed that the form of the state function at $t = 0$ is given by (33). The initial angle in this case is displaced toward a positive $\theta$ by an amount $\beta$. The resulting probability density is represented in (36) which is a Gaussian shape. From Figure 2, we can confirm that this probability density oscillates back and forth from the center depending on the magnitude of $\beta$, while its width becomes narrower over time as the quantum energy decays. So to speak, its quantum behavior is very similar to that of the classical oscillation of the pendulum. Therefore, the correspondence between quantum and classical behaviors holds.

Indeed, the quantum and classical correspondence for a certain quantum theory is a crucial requirement for the validity of the associated quantum theory. The invariant operator theory we used here is sound and enables us to obtain reasonable quantum solutions for a time-dependent dynamical system. Further, we know that it sometimes gives the exact quantum solutions for a system of which the classical solutions are completely known.

## Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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