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Research Article

Convergence Theorems of Common Elements for Pseudocontractive Mappings and Monotone Mappings

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An algorithm for treating pseudocontractive mappings and monotone mappings is proposed. Convergence analysis of algorithm is investigated in the framework of Hilbert spaces.

1. Introduction

The motivation for common element problem is mainly due to its possible applications to mathematical modeling of concrete complex problems. The common element problems include mini-max problems, complementarily problems, equilibrium problems, common fixed point problems, and variational inequalities as special cases; see [1–7] and the references therein. It is well-known that the convex feasibility problem is a special case of the common zero (fixed) points of nonlinear mappings. And many important problems have reformulations which require finding zero points, for instance, evolution equations, complementarily problems, mini-max problems, and variational inequalities and optimization. For studying zero points of monotone mappings, the most well-known algorithm is the proximal point algorithm; see [8, 9] and the references therein. Regularization methods recently have been investigated for treating zero points of monotone mappings; see [2, 5, 6, 9] and references therein.

In 2010, Takahashi et al. [6] studied zero point problems of the sum of two monotone mappings and fixed point problems of a nonexpansive mapping based on the following iterative algorithm:

$$\begin{gathered} x_1 = x \in C, \\ y_n = \alpha_n x + (1-\alpha_n) J_{r_n}(I - r_n A) x_n, \\ x_{n+1} = \beta_n x_n + (1-\beta_n) T y_n, \quad \forall n \ge 1, \end{gathered} \tag{1}$$

where C is a nonempty closed convex subset of a real Hilbert space H, $\{\alpha_n\}$ and $\{\beta_n\}$ are real number sequences in $(0, 1)$, $\{r_n\}$ is a positive sequence, $T : C \to C$ is a nonexpansive mapping, $A : C \to H$ is an inverse strongly monotone mapping, $B : H \to 2^H$ is a maximal monotone mapping, and $J_{r_n} = (I + r_n B)^{-1}$, where I is the identity mapping. They proved that the sequence $\{x_n\}$ generated in (1) converges strongly to some $z \in F(T) \cap (A + B)^{-1}(0)$ provided that the control sequences satisfy some restrictions, where $F(T)$ is the set fixed points of T.

In 2014, Shahzad and Zegeye [5] considered an iterative method for a common point of fixed points of Lipschitzian pseudocontractive mappings and zeros of sum of two monotone mappings based on the projection method in a real Hilbert space. To be more precise, they investigated the following algorithm:

$$\begin{gathered} x_0 \in C, \\ y_n = (1-\beta_n) x_n + \beta_n T x_n, \\ x_{n+1} \\ = P_C\left[(1-\alpha_n)\left(\theta_n x_n + \delta_n T y_n + \gamma_n J_{r_n}(I - \lambda_n A) x_n\right)\right], \end{gathered} \tag{2}$$

where C is a nonempty closed convex subset of a real Hilbert space H, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\theta_n\}$, $\{\delta_n\}$, and $\{\gamma_n\}$ are real number sequences in $(0, 1)$, $\{r_n\}$ is a positive sequence, $T : C \to C$ is a Lipschitzian pseudocontractive mapping, $A : C \to H$ is an inverse strongly monotone mapping, $B : H \to 2^H$

is a maximal monotone mapping, and $J_{r_n} = (I + r_n B)^{-1}$. They proved that the sequence $\{x_n\}$ generated in (2) converges strongly to the minimum-norm point $x \in F(T) \cap (A+B)^{-1}(0)$ provided that the control sequences satisfy some restrictions.

In this paper, we are concerned with the problem of finding a common element in the intersection $F(T_1) \cap F(T_2) \cap (A + B)^{-1}(0)$, where $F(T_i)$ denotes the fixed point set of the pseudocontractive mapping T_i, $i = 1, 2$, and $(A + B)^{-1}(0)$ denotes the zero point set of the sum of an inverse strongly monotone mapping A and a maximal monotone mapping B. Applications to a common element of the set of common fixed points of Lipschitzian pseudocontractive mappings and solutions of variational inequality for α-inverse strongly monotone mappings are included. Our theorems improve and extend those announced by Shahzad and Zegeye [5], Takahashi et al. [6], and other authors with the related interest.

2. Preliminaries

Let H be a real Hilbert space with the inner product $\langle \cdot, \cdot \rangle$ and the norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H and let P_C be the metric projection from H onto C. Let $T : C \to C$ be a mapping. In this paper, we use $F(T)$ to denote the fixed point set of T; that is, $F(T) = \{x \in C : x = Tx\}$.

Recall that T is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{3}$$

T is said to be a γ-strictly pseudocontractive mapping if there exists $\gamma \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \gamma \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{4}$$

Note that the class of γ-strictly pseudocontractive mappings includes the class of nonexpansive mappings as a special case. T is said to be a pseudocontractive mapping if

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \|(I - T)x - (I - T)y\|^2, \quad \forall x, y \in C. \tag{5}$$

We note that inequalities (4) and (5) can be equivalently written as

$$\langle x - y, Tx - Ty \rangle \le \|x - y\|^2 - \gamma \|(I - T)x - (I - T)y\|^2 \tag{6}$$

for some $\gamma > 0$ and

$$\langle x - y, Tx - Ty \rangle \le \|x - y\|^2, \quad \forall x, y \in C, \tag{7}$$

respectively. Note that the class of γ-strictly pseudocontractive mappings is contained in the class of pseudocontractive mappings. We note that the inclusion is proper. We remark that T is a γ-strictly pseudocontractive mapping if and only if $I - T$ is a γ-inverse strongly monotone mapping and T is a pseudocontractive mapping if and only if $I - T$ is a monotone mapping.

Let $A : C \to H$ be a mapping and $A^{-1}0$ stands for the zero point set of A; that is, $A^{-1}0 = \{x \in C : Ax = 0\}$. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \ge 0, \quad \forall x, y \in C. \tag{8}$$

A is said to be α-inverse strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \ge \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{9}$$

It is not hard to see that α-inverse strongly monotone mappings are Lipschitz continuous with constant $L = 1/\alpha$; that is, $\|Ax - Ay\| \le (1/\alpha)\|x - y\|$ for all $x, y \in C$.

Recall that the classical variational inequality, denoted by $\mathrm{VI}(C, A)$, is to find $u \in C$ such that

$$\langle Au, v - u \rangle \ge 0, \quad \forall v \in C. \tag{10}$$

A multivalued mapping $B : H \to 2^H$ with the domain $D(B) = \{x \in H : Bx \ne \phi\}$ and the range $R(B) = \{Bx : x \in D(B)\}$ is said to be monotone if, for $x_1 \in D(B)$, $x_2 \in D(B)$, $y_1 \in Bx_1$, and $y_2 \in Bx_2$, we have $\langle x_1 - x_2, y_1 - y_2 \rangle \ge 0$. A monotone mapping B is said to be maximal if its graph $G(B) = \{(x, y) : y \in Bx\}$ is not properly contained in the graph of any other monotone mapping. Let $B : H \to 2^H$ be a maximal monotone mapping. Then we can define, for each $\lambda > 0$, a nonexpansive single-valued mapping $J_\lambda : H \to H$ by $J_\lambda = (I + \lambda B)^{-1}$. It is called the resolvent of B. We know that $B^{-1}0 = F(J_\lambda)$ for all $\lambda > 0$ and J_λ is firmly nonexpansive.

Lemma 1. *Let H be a real Hilbert space. Then, for any given $x, y \in H$, the following inequality holds:*

$$\|x + y\|^2 \le \|x\|^2 + 2 \langle y, x + y \rangle. \tag{11}$$

Lemma 2 (see [10]). *Let C be a convex subset of a real Hilbert space H. Let $x \in H$. Then $x_0 = P_C x$ if and only if*

$$\langle z - x_0, x - x_0 \rangle \le 0, \quad \forall z \in C. \tag{12}$$

Lemma 3 (see [2]). *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \to H$ be a mapping and let $B : H \to 2^H$ be a maximal monotone mapping. Then $F(J_\lambda(I - \lambda A)) = (A + B)^{-1}0$.*

Lemma 4 (see [11]). *Let H be a Hilbert space. Let $B_1 : D(B_1) \subseteq H \to 2^H$ and let $B_2 : D(B_2) \subseteq H \to 2^H$ be maximal monotone mappings. Suppose that $D(A) \cap \mathrm{int}(D(B)) \ne \phi$. Then $B_1 + B_2$ is a maximal monotone mapping.*

Lemma 5 (see [4]). *Let $\{a_n\}$ be a sequence of real numbers. Assume that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} < a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all sufficiently large numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1}, \qquad a_k \le a_{m_k+1}. \tag{13}$$

Lemma 6 (see [12]). *Let H be a real Hilbert space. Then, for all $x_i \in H$ and $\alpha_i \in [0,1]$ for $i = 1,2,\ldots,n$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, the following equality holds:*

$$\begin{aligned} &\left\|\alpha_0 x_0 + \alpha_1 x_1 + \cdots + \alpha_n x_n\right\|^2 \\ &\quad = \sum_{i=0}^{n} \alpha_i \left\|x_i\right\|^2 - \sum_{0\le i,j\le n} \alpha_i, \alpha_j \left\|x_i - x_j\right\|^2. \end{aligned} \tag{14}$$

Lemma 7 (see [7]). *Let C be a nonempty closed convex subset of a real Hilbert Hilbert and let $T : C \to C$ be a continuous pseudocontractive mapping. Then*

(i) *$F(T)$ is a closed convex subset of C;*

(ii) *$(I-T)$ is demiclosed at zero; that is, if $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $Tx_n - x_n \to 0$ as $n \to \infty$, then $x = Tx$.*

Lemma 8 (see [13]). *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n) a_n + \alpha_n \delta_n, \quad n \ge n_0, \tag{15}$$

where $\{\alpha_n\} \subset (0,1)$ and $\{\delta_n\} \subset \mathbb{R}$ satisfy the following conditions: $\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, and $\limsup_{n\to\infty}\delta_n \le 0$. Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Theorem 9. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2 : C \to C$ be Lipschitzian pseudocontractive mappings with Lipschitz constants L_1 and L_2, respectively. Let $A : C \to H$ be an α-inverse strongly monotone mapping and let B be a maximal monotone mapping such that the domain of B is subset of C. Assume that $\mathcal{F} = F(T_1) \cap F(T_2) \cap (A+B)^{-1}0 \neq \phi$. Let $J_{\lambda_n} = (I + \lambda_n B)^{-1}$, where $\{\lambda_n\}$ is a positive real number sequence. Given $x_1, u \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:*

$$\begin{aligned} z_n &= (1-c_n) x_n + c_n T_2 x_n, \\ y_n &= (1-\beta_n) x_n + \beta_n T_1 x_n, \\ x_{n+1} &= P_C \Big[\alpha_n u + (1-\alpha_n) \\ &\qquad \times \big(\theta_n x_n + \delta_n T_1 y_n + \gamma_n T_2 z_n \\ &\qquad\quad + \xi_n J_{\lambda_n} (I - \lambda_n A) x_n\big)\Big]. \end{aligned} \tag{16}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{c_n\}$, $\{\theta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$, $\{\xi_n\}$, and $\{\lambda_n\}$ satisfy the following restrictions:

(a) $0 < a < \lambda_n < b < 2\alpha$;

(b) $0 < c \le \theta_n, \delta_n, \gamma_n, \xi_n \le d < 1$ *and* $\theta_n + \delta_n + \gamma_n + \xi_n = 1$;

(c) $0 < \alpha_n < e < 1$, $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(d) $\delta_n + \gamma_n + \xi_n \le \beta_n$, $c_n \le \beta < 1/(\sqrt{1+L^2}+1)$, *for all* $n \ge 1$,

for some real numbers $a, b, c, d, e, \beta > 0$, where $L = \max\{L_1, L_2\}$. Then $\{x_n\}$ converges strongly to some point $\overline{x}$, where $\overline{x} = P_{\mathcal{F}} u$.

Proof. First, we show that $I - \lambda_n A$ is nonexpansive.

Indeed, we have

$$\begin{aligned} &\left\|(I-\lambda_n A)x - (I-\lambda_n A)y\right\|^2 \\ &\quad = \|x-y\|^2 - 2\lambda_n\langle x-y, Ax-Ay\rangle \\ &\qquad + \lambda_n^2\|Ax-Ay\|^2 \\ &\quad \le \|x-y\|^2 - \lambda_n(2\alpha-\lambda_n)\|Ax-Ay\|^2. \end{aligned} \tag{17}$$

It follows from restriction (a) that $I - \lambda_n A$ is nonexpansive.

Let $p \in \mathcal{F}$. It follows from (5), (16), and Lemmas 3 and 6 that

$$\begin{aligned} &\|x_{n+1} - p\|^2 \\ &= \Big\|P_C\big[\alpha_n u + (1-\alpha_n) \\ &\qquad \times \big(\theta_n x_n + \delta_n T_1 y_n + \gamma_n T_2 z_n \\ &\qquad\quad + \xi_n J_{\lambda_n}(I-\lambda_n A)x_n\big)\big] - p\Big\|^2 \\ &\le \Big\|\alpha_n u + (1-\alpha_n) \\ &\qquad \times \big(\theta_n x_n + \delta_n T_1 y_n + \gamma_n T_2 z_n \\ &\qquad\quad + \xi_n J_{\lambda_n}(I-\lambda_n A)x_n\big) - p\Big\|^2 \\ &\le \alpha_n\|u-p\|^2 + (1-\alpha_n) \\ &\quad \times \Big\|\theta_n(x_n - p) + \delta_n(T_1 y_n - p) \\ &\qquad + \gamma_n(T_2 z_n - p) \\ &\qquad + \xi_n\big(J_{\lambda_n}(I-\lambda_n A)x_n - p\big)\Big\|^2 \\ &\le \alpha_n\|u-p\|^2 + (1-\alpha_n) \\ &\quad \times \Big[\theta_n\|x_n - p\|^2 + \delta_n\|T_1 y_n - p\|^2 \\ &\qquad + \gamma_n\|T_2 z_n - p\|^2 \\ &\qquad + \xi_n\left\|J_{\lambda_n}(I-\lambda_n A)x_n - p\right\|^2\Big] \\ &\quad - (1-\alpha_n)\theta_n\delta_n\|T_1 y_n - x_n\|^2 \\ &\quad - (1-\alpha_n)\theta_n\gamma_n\|T_2 z_n - x_n\|^2 \\ &\quad - (1-\alpha_n)\theta_n\xi_n\left\|J_{\lambda_n}(I-\lambda_n A)x_n - x_n\right\|^2 \end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n\|u-p\|^2 + (1-\alpha_n)(\theta_n+\xi_n)\|x_n-p\|^2 \\
&\quad + (1-\alpha_n)\delta_n\left[\|y_n-p\|^2 + \|y_n-T_1y_n\|^2\right] \\
&\quad + (1-\alpha_n)\gamma_n\left[\|z_n-p\|^2 + \|z_n-T_2z_n\|^2\right] \\
&\quad - (1-\alpha_n)\theta_n\delta_n\|T_1y_n-x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\gamma_n\|T_2z_n-x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\xi_n\left\|J_{\lambda_n}(I-\lambda_nA)x_n-x_n\right\|^2.
\end{aligned} \tag{18}$$

It follows from (5), (16), and Lemma 6 that

$$\begin{aligned}
&\|z_n-p\|^2 \\
&\quad = \|(1-c_n)(x_n-p)+c_n(T_2x_n-p)\|^2 \\
&\quad = (1-c_n)\|x_n-p\|^2 \\
&\qquad + c_n\|T_2x_n-p\|^2 - c_n(1-c_n)\|x_n-T_2x_n\|^2 \\
&\quad \leq (1-c_n)\|x_n-p\|^2 \\
&\qquad + c_n\left[\|x_n-p\| + \|x_n-T_2x_n\|^2\right] \\
&\qquad - c_n(1-c_n)\|x_n-T_2x_n\|^2 \\
&\quad = \|x_n-p\|^2 + c_n^2\|x_n-T_2x_n\|^2, \\
&\|y_n-p\|^2 \\
&\quad = \|(1-\beta_n)(x_n-p)+\beta_n(T_1x_n-p)\|^2 \\
&\quad \leq \|x_n-p\|^2 + \beta_n^2\|x_n-T_1x_n\|^2.
\end{aligned} \tag{19}$$

Similarly, we have that

$$\begin{aligned}
&\|y_n-T_1y_n\|^2 \\
&\quad = \|(1-\beta_n)(x_n-T_1y_n)+\beta_n(T_1x_n-T_1y_n)\|^2 \\
&\quad = (1-\beta_n)\|x_n-T_1y_n\|^2 \\
&\qquad + \beta_n\|T_1x_n-T_1y_n\|^2 \\
&\qquad - \beta_n(1-\beta_n)\|x_n-T_1x_n\|^2 \\
&\quad \leq (1-\beta_n)\|x_n-T_1y_n\|^2 + \beta_nL^2\|x_n-y_n\|^2 \\
&\qquad - \beta_n(1-\beta_n)\|x_n-T_1x_n\|^2 \\
&\quad = (1-\beta_n)\|x_n-T_1y_n\|^2 \\
&\qquad - \beta_n\left(1-\beta_n^2L^2-\beta_n\right)\|x_n-T_1x_n\|^2. \\
&\|z_n-T_2z_n\|^2 \\
&\quad = \|(1-c_n)(x_n-T_2z_n)+c_n(T_2x_n-T_2z_n)\|^2 \\
&\quad \leq (1-c_n)\|x_n-T_2z_n\|^2 \\
&\qquad - c_n\left(1-c_n^2L^2-c_n\right)\|x_n-T_2x_n\|^2.
\end{aligned} \tag{20}$$

Substituting (19) and (20) into (18), we obtain that

$$\begin{aligned}
&\|x_{n+1}-p\|^2 \\
&\quad \leq \alpha_n\|u-p\|^2 \\
&\qquad + (1-\alpha_n)(\theta_n+\xi_n)\|x_n-p\|^2 \\
&\qquad + (1-\alpha_n)\delta_n\left[\|x_n-p\|^2 + \beta_n^2\|x_n-T_1x_n\|^2\right] \\
&\qquad + (1-\alpha_n)\delta_n\left[(1-\beta_n)\|x_n-T_1y_n\|^2\right. \\
&\qquad\qquad \left.-\beta_n\left(1-\beta_n^2L^2-\beta_n\right)\|x_n-T_1x_n\|^2\right] \\
&\qquad + (1-\alpha_n)\gamma_n\left[\|x_n-p\|^2 + c_n^2\|x_n-T_2x_n\|^2\right] \\
&\qquad + (1-\alpha_n)\gamma_n\left[(1-c_n)\|x_n-T_2z_n\|^2\right. \\
&\qquad\qquad \left.-c_n\left(1-c_n^2L^2-c_n\right)\|x_n-T_2x_n\|^2\right] \\
&\qquad - (1-\alpha_n)\theta_n\delta_n\|T_1y_n-x_n\|^2 \\
&\qquad - (1-\alpha_n)\theta_n\gamma_n\|T_2z_n-x_n\|^2 \\
&\qquad - (1-\alpha_n)\theta_n\xi_n\left\|J_{\lambda_n}(I-\lambda_nA)x_n-x_n\right\|^2 \\
&\quad = \alpha_n\|u-p\|^2 + (1-\alpha_n)\|x_n-p\|^2 \\
&\qquad - (1-\alpha_n)\delta_n\beta_n\left(1-2\beta_n-\beta_n^2L^2\right)\|x_n-T_1x_n\|^2 \\
&\qquad - (1-\alpha_n)\gamma_nc_n\left(1-2c_n-c_n^2L^2\right)\|x_n-T_2x_n\|^2 \\
&\qquad + (1-\alpha_n)\delta_n(\delta_n+\gamma_n+\xi_n-\beta_n)\|T_1y_n-x_n\|^2 \\
&\qquad + (1-\alpha_n)\gamma_n(\delta_n+\gamma_n+\xi_n-c_n)\|T_2z_n-x_n\|^2 \\
&\qquad - (1-\alpha_n)\theta_n\xi_n\left\|J_{\lambda_n}(I-\lambda_nA)x_n-x_n\right\|^2.
\end{aligned} \tag{21}$$

In view of restriction (d), we find that

$$\begin{gathered}
1-2\beta_n-\beta_n^2L^2 \geq 1-2\beta-\beta^2L^2 > 0, \\
1-2c_n-c_n^2L^2 \geq 1-2\beta-\beta^2L^2 > 0, \\
\delta_n+\gamma_n+\xi_n-\beta_n \leq 0, \\
\delta_n+\gamma_n+\xi_n-c_n \leq 0,
\end{gathered} \tag{22}$$

for all $n \geq 1$. It follows from (21) and (22) that

$$\|x_{n+1}-p\|^2 \leq \alpha_n\|u-p\|^2 + (1-\alpha_n)\|x_n-p\|^2. \tag{23}$$

Putting $M = \max\{\|u-p\|^2, \|x_1-p\|^2\}$, we find that $\|x_n-p\|^2 \leq M$ for all $n \geq 1$.

Indeed, it is clear that $\|x_2 - p\|^2 \leq M$. Suppose that $\|x_m - p\| \leq M$ for some positive integer m. It follows that

$$\begin{aligned} \|x_{m+1} - p\|^2 &\leq \alpha_m \|u - p\|^2 + (1-\alpha_m)\|x_m - p\|^2 \\ &\leq \alpha_m M + (1-\alpha_m) M \\ &= M. \end{aligned} \tag{24}$$

This finds that $\{x_n\}$ is bounded and hence $\{y_n\}$ and $\{z_n\}$ are bounded.

Let $w_n = \theta_n x_n + \delta_n T_1 y_n + \gamma_n T_2 z_n + \xi_n J_{\lambda_n}(I - \lambda_n A)x_n$. Then we see that $x_{n+1} = P_C(\alpha_n u + (1-\alpha_n)w_n)$. Put $\overline{x} = P_{\mathcal{F}} u$. Using (16), (19), and (20) and Lemmas 1 and 6, we find that

$$\begin{aligned}
&\|x_{n+1} - \overline{x}\|^2 \\
&\quad \leq \|\alpha_n(u - \overline{x}) + (1-\alpha_n)(w_n - \overline{x})\|^2 \\
&\leq (1-\alpha_n)\|w_n - \overline{x}\|^2 \\
&\quad + 2\alpha_n \langle u - \overline{x}, \alpha_n(u - \overline{x}) + (1-\alpha_n)(w_n - \overline{x})\rangle \\
&\leq (1-\alpha_n)\theta_n\|x_n - \overline{x}\|^2 \\
&\quad + (1-\alpha_n)\delta_n\|T_1 y_n - \overline{x}\|^2 \\
&\quad + (1-\alpha_n)\gamma_n\|T_2 z_n - \overline{x}\|^2 \\
&\quad + (1-\alpha_n)\xi_n\left\|J_{\lambda_n}(I - \lambda_n A)x_n - \overline{x}\right\|^2 \\
&\quad - (1-\alpha_n)\theta_n\delta_n\|T_1 y_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\gamma_n\|T_2 z_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\xi_n\left\|J_{\lambda_n}(I - \lambda_n A)x_n - x_n\right\|^2 \\
&\quad + 2\alpha_n^2\|u - \overline{x}\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle u - \overline{x}, w_n - \overline{x}\rangle \\
&\leq (1-\alpha_n)\theta_n\|x_n - \overline{x}\|^2 \\
&\quad + (1-\alpha_n)\delta_n\left[\|y_n - \overline{x}\|^2 + \|y_n - T_1 y_n\|^2\right] \\
&\quad + (1-\alpha_n)\gamma_n\left[\|z_n - \overline{x}\|^2 + \|z_n - T_2 z_n\|^2\right] \\
&\quad + (1-\alpha_n)\xi_n\|x_n - \overline{x}\|^2 \\
&\quad - (1-\alpha_n)\theta_n\delta_n\|T_1 y_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\gamma_n\|T_2 z_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\xi_n\left\|J_{\lambda_n}(I - \lambda_n A)x_n - x_n\right\|^2 \\
&\quad + 2\alpha_n^2\|u - \overline{x}\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle u - \overline{x}, w_n - \overline{x}\rangle \\
&\leq (1-\alpha_n)(\theta_n + \xi_n)\|x_n - \overline{x}\|^2 \\
&\quad + (1-\alpha_n)\delta_n\left[\|x_n - \overline{x}\|^2 + \beta_n^2\|x_n - T_1 x_n\|^2\right] \\
&\quad + (1-\alpha_n)\delta_n\left[(1-\beta_n)\|x_n - T_1 y_n\|^2 \right. \\
&\qquad \left. -\beta_n\left(1 - \beta_n^2 L^2 - \beta_n\right)\|x_n - T_1 x_n\|^2\right] \\
&\quad + (1-\alpha_n)\gamma_n\left[\|x_n - \overline{x}\|^2 + c_n^2\|x_n - T_2 x_n\|^2\right] \\
&\quad + (1-\alpha_n)\gamma_n\left[(1-c_n)\|x_n - T_2 z_n\|^2 \right. \\
&\qquad \left. - c_n\left(1 - c_n^2 L^2 - c_n\right)\|x_n - T_2 x_n\|^2\right] \\
&\quad - (1-\alpha_n)\theta_n\delta_n\|T_1 y_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\gamma_n\|T_2 z_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\theta_n\xi_n\left\|J_{\lambda_n}(I - \lambda_n A)x_n - x_n\right\|^2 \\
&\quad + 2\alpha_n^2\|u - \overline{x}\|^2 + 2\alpha_n(1-\alpha_n)\langle u - \overline{x}, w_n - \overline{x}\rangle,
\end{aligned} \tag{25}$$

which implies from (22) that

$$\begin{aligned}
&\|x_{n+1} - \overline{x}\|^2 \\
&\quad \leq (1-\alpha_n)\|x_n - \overline{x}\|^2 \\
&\quad - (1-\alpha_n)\delta_n\beta_n\left(1 - 2\beta_n - \beta_n^2 L^2\right)\|x_n - T_1 x_n\|^2 \\
&\quad + (1-\alpha_n)\delta_n(\delta_n + \xi_n + \gamma_n - \beta_n)\|T_1 y_n - x_n\|^2 \\
&\quad - (1-\alpha_n)\gamma_n c_n\left(1 - 2c_n - c_n^2 L^2\right)\|x_n - T_2 x_n\|^2 \\
&\quad + (1-\alpha_n)\gamma_n(\delta_n + \xi_n + \gamma_n - c_n)\|T_2 z_n - x_n\| \\
&\quad - (1-\alpha_n)\theta_n\xi_n\left\|J_{\lambda_n}(I - \lambda_n A)x_n - x_n\right\|^2 \\
&\quad + 2\alpha_n^2\|u - \overline{x}\|^2 + 2\alpha_n(1-\alpha_n)\langle u - \overline{x}, w_n - \overline{x}\rangle \\
&\leq (1-\alpha_n)\|x_n - \overline{x}\|^2 + 2\alpha_n^2\|u - \overline{x}\|^2 \\
&\quad + 2\alpha_n(1-\alpha_n)\langle u - \overline{x}, w_n - \overline{x}\rangle.
\end{aligned} \tag{26}$$

Now we consider two cases.

Case 1. Suppose that there exists $n_0 \in N$ such that $\{\|x_n - \overline{x}\|\}$ is decreasing for all $n \geq n_0$. Then we get that $\{\|x_n - \overline{x}\|\}$ is convergent. It follows from (22) and (26) that

$$\begin{gathered} x_n - T_1 x_n \longrightarrow 0, \qquad x_n - T_2 x_n \longrightarrow 0, \\ x_n - J_{\lambda_n}(I - \lambda_n A)x_n \longrightarrow 0, \end{gathered} \tag{27}$$

as $n \to \infty$. Also we obtain from (27) that

$$\begin{gathered} \|y_n - x_n\| = \beta_n\|x_n - T_1 x_n\| \longrightarrow 0, \\ \|z_n - x_n\| = c_n\|x_n - T_2 x_n\| \longrightarrow 0, \end{gathered} \tag{28}$$

as $n \to \infty$. In view of the Lipschitz continuity of T_1, T_2 and (27) and (28), we find that

$$\begin{aligned}\|T_1 y_n - x_n\| &\leq \|T_1 y_n - T_1 x_n\| + \|T_1 x_n - x_n\| \\ &\leq L\|y_n - x_n\| + \|T_1 x_n - x_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty,\end{aligned} \tag{29}$$

$$\begin{aligned}\|T_2 z_n - x_n\| &\leq \|T_2 z_n - T_2 x_n\| + \|T_2 x_n - x_n\| \\ &\leq L\|z_n - x_n\| + \|T_2 x_n - x_n\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{30}$$

It follows from (27), (29), and (30) that

$$\begin{aligned}\|w_n - x_n\| &\leq \delta_n \|T_1 y_n - x_n\| + \gamma_n \|T_2 z_n - x_n\| \\ &\quad + \xi_n \left\|J_{\lambda_n}(I - \lambda_n A)x_n - x_n\right\| \\ &\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{31}$$

Since $\{w_n\}$ is a bounded subset of H, we can choose a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} \rightharpoonup w$ and

$$\limsup_{n\to\infty} \langle u - \overline{x}, w_n - \overline{x}\rangle = \lim_{i\to\infty} \left\langle u - \overline{x}, w_{n_i} - \overline{x}\right\rangle. \tag{32}$$

It follows from (31) that $x_{n_i} \rightharpoonup w$. By (27) and Lemma 7, we obtain that $w \in F(T_1)$ and $w \in F(T_2)$.

Next, we show that $w \in (A+B)^{-1}0$.

Notice that

$$\begin{aligned}&\left\|J_{\lambda_n}(I - \lambda_n A)x_n - p\right\|^2 \\ &= \left\|J_{\lambda_n}(I - \lambda_n A)x_n - J_{\lambda_n}(I - \lambda_n A)p\right\|^2 \\ &\leq \left\|(I - \lambda_n A)x_n - (I - \lambda_n A)p\right\|^2 \\ &= \|x_n - p\|^2 - 2\lambda_n \langle x_n - p, Ax_n - Ap\rangle \\ &\quad + \lambda_n^2 \|Ax_n - Ap\|^2 \\ &\leq \|x_n - p\|^2 - 2\alpha\lambda_n \|Ax_n - Ap\|^2 + \lambda_n^2 \|Ax_n - Ap\|^2 \\ &= \|x_n - p\|^2 - \lambda_n (2\alpha - \lambda_n) \|Ax_n - Ap\|^2.\end{aligned} \tag{33}$$

It follows from (27) that

$$\begin{aligned}&\lambda_n (2\alpha - \lambda_n)\|Ax_n - Ap\|^2 \\ &\quad\leq \|x_n - p\|^2 - \left\|J_{\lambda_n}(I - \lambda_n A)x_n - p\right\| \\ &\quad= \left(\|x_n - p\| + \left\|J_{\lambda_n}(I - \lambda_n A)x_n - p\right\|\right) \\ &\qquad \times \left\|x_n - J_{\lambda_n}(I - \lambda_n A)x_n\right\| \\ &\quad\longrightarrow 0 \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{34}$$

Hence we get

$$\lim_{n\to\infty} \|Ax_n - Ap\|^2 = 0. \tag{35}$$

Putting $h_n = J_{\lambda_n}(I - \lambda_n A)x_n$, we find that $((x_{n_i} - h_{n_i})/\lambda_{n_i}) - Ax_{n_i} \in Bh_{n_i}$. Since B is monotone, we get that, for any $(u, v) \in G(B)$,

$$\left\langle h_{n_i} - u, \frac{x_{n_i} - h_{n_i}}{\lambda_{n_i}} - Ax_{n_i} - v\right\rangle \geq 0, \tag{36}$$

where $G(B) = \{(x, w) \in H \times H : x \in D(B), w \in Bx\}$. Since $\langle x_{n_i} - w, Ax_{n_i} - Aw\rangle \geq \alpha\|Ax_{n_i} - Aw\|^2$, $x_{n_i} \rightharpoonup w$, and $Ax_{n_i} \to Ap$ as $i \to \infty$, we have $Ax_{n_i} \to Aw$. Thus, letting $i \to \infty$, we obtain from (27) and (36) that $\langle w - u, -Aw - v\rangle \geq 0$. This means $-Aw \in Bw$, that is, $0 \in (A+B)w$. Hence we get $w \in (A+B)^{-1}0$. This implies from Lemma 2 that

$$\begin{aligned}\limsup_{n\to\infty} \langle u - \overline{x}, w_n - \overline{x}\rangle &= \lim_{i\to\infty} \left\langle u - \overline{x}, w_{n_i} - \overline{x}\right\rangle \\ &= \langle u - \overline{x}, w - \overline{x}\rangle \\ &\leq 0.\end{aligned} \tag{37}$$

On the other hand, we have from (26) that

$$\begin{aligned}\|x_{n+1} - \overline{x}\|^2 &\leq (1 - \alpha_n)\|x_n - \overline{x}\|^2 \\ &\quad + \alpha_n \left(2\alpha_n \|u - \overline{x}\|^2\right. \\ &\quad \left. + (1 - \alpha_n)\langle u - \overline{x}, w_n - \overline{x}\rangle\right).\end{aligned} \tag{38}$$

From Lemma 8 and (37), we find that $\lim_{n\to\infty}\|x_n - \overline{x}\| = 0$.

Case 2. Suppose that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that

$$\left\|x_{n_i} - \overline{x}\right\| < \left\|x_{n_i+1} - \overline{x}\right\|, \tag{39}$$

for all $i \in \mathbb{N}$. By Lemma 5, there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and

$$\left\|x_{m_k} - \overline{x}\right\| \leq \left\|x_{m_k+1} - \overline{x}\right\|, \qquad \left\|x_k - \overline{x}\right\| \leq \left\|x_{m_k+1} - \overline{x}\right\|, \tag{40}$$

for all $k \in \mathbb{N}$. From (22) and (26), we have $x_{m_k} - T_1 x_{m_k} \to 0$, $x_{m_k} - T_2 x_{m_k} \to 0$, and $x_{m_k} - J_{\lambda_{m_k}}(I - \lambda_{m_k} A)x_{m_k} \to 0$ as $k \to \infty$. Thus, like in Case 1, we obtain $w_{m_k} - x_{m_k} \to 0$ and

$$\limsup_{k\to\infty} \left\langle u - \overline{x}, w_{m_k} - \overline{x}\right\rangle \leq 0. \tag{41}$$

From (26) and (40), we have

$$\begin{aligned}&\alpha_{m_k}\left\|x_{m_k} - \overline{x}\right\|^2 \\ &\quad\leq \left\|x_{m_k} - \overline{x}\right\|^2 - \left\|x_{m_k+1} - \overline{x}\right\|^2 \\ &\qquad + 2\alpha_{m_k}\left(\alpha_{m_k}\|u - \overline{x}\|^2 + \left(1 - \alpha_{m_k}\right)\left\langle u - \overline{x}, w_{m_k} - \overline{x}\right\rangle\right) \\ &\quad\leq 2\alpha_{m_k}\left(\alpha_{m_k}\|u - \overline{x}\|^2 + \left(1 - \alpha_{m_k}\right)\left\langle u - \overline{x}, w_{m_k} - \overline{x}\right\rangle\right).\end{aligned} \tag{42}$$

Applying (41) and $\alpha_{m_k} > 0$, we have $\|x_{m_k} - \overline{x}\| \to 0$ as $k \to \infty$. It implies that $\|x_{m_k+1} - \overline{x}\| \to 0$ as $k \to \infty$. By (40), we have $x_k \to \overline{x}$ as $k \to \infty$.

Therefore, from the above two cases, we can conclude that the sequence $\{x_n\}$ converges strongly to $\overline{x} = P_{\mathcal{F}}u$. This completes the proof. □

From Lemma 4, we have the following result.

Corollary 10. *Let C be a nonempty closed convex subset of a real Hilbert space H such that* $\operatorname{int}(C) \neq \phi$. *Let $T_1, T_2 : C \to C$ be Lipschitzian pseudocontractive mappings with Lipschitz constants L_1 and L_2, respectively. Let $B_1 : D(B_1) \to 2^H$ and $B_2 : D(B_2) \to 2^H$ be maximal monotone mappings such that $D(B_1) \cap \operatorname{int}(D(B_2)) \neq \phi$. Assume that $\mathcal{F} = F(T_1) \cap F(T_2) \cap (B_1 + B_2)^{-1}(0) \neq \phi$. Let $J_{\lambda_n} = (I + \lambda_n(B_1 + B_2))^{-1}$, where $\{\lambda_n\}$ is a positive real number sequence. Given $x_1, u \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:*

$$\begin{aligned} z_n &= (1 - c_n) x_n + c_n T_2 x_n, \\ y_n &= (1 - \beta_n) x_n + \beta_n T_1 x_n, \\ x_{n+1} &= P_C \left[\alpha_n u + (1 - \alpha_n) \right. \\ &\quad \left. \times \left(\theta_n x_n + \delta_n T_1 y_n + \gamma_n T_2 z_n + \xi_n J_{\lambda_n} x_n\right)\right], \\ &\qquad \forall n \geq 1. \end{aligned} \tag{43}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{c_n\}$, $\{\theta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$, $\{\xi_n\}$, and $\{\lambda_n\}$ satisfy the following restrictions:

(a) $0 < a < \lambda_n < b < 1$;

(b) $0 < c \leq \theta_n, \delta_n, \gamma_n, \xi_n \leq d < 1$, *and* $\theta_n + \delta_n + \gamma_n + \xi_n = 1$;

(c) $0 < \alpha_n < e < 1$, $\lim_{n\to\infty}\alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(d) $\delta_n + \gamma_n + \xi_n \leq \beta_n$, $c_n \leq \beta < 1/(\sqrt{1 + L^2} + 1)$, *for all* $n \geq 1$,

for some real numbers $a, b, c, d, e > 0$, where $L = \max\{L_1, L_2\}$. Then $\{x_n\}$ converges strongly to some point $\overline{x}$, where $\overline{x} = P_{\mathcal{F}}u$.

Remark 11. If $T_1 = T$, $T_2 = I$ (the identity mapping), and $u = 0$, then Theorem 9 reduces to Theorem 3.1 of Shahzad and Zegeye [6]. Thus, Theorem 9 covers Theorem 3.1 of Shahzad and Zegeye [6] as a special case.

4. Applications

In this section, we will consider equilibrium problems and variational inequalities.

Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ denotes the set of real numbers. Recall the following equilibrium problem: find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{44}$$

We use $\mathrm{EP}(F)$ to denote the solution set of the equilibrium problem. To study the equilibrium problems, we assume that F satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x, y) + F(y, x) \geq 0$, for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\limsup_{t\downarrow 0} F(tz + (1 - t) x, y) \leq F(x, y); \tag{45}$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semicontinuous.

Lemma 12 (see [1]). *Let C be a nonempty closed convex subset of a real Hilbert space and let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{46}$$

Further, define

$$T_r(x) = \left\{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\right\}, \tag{47}$$

for all $r > 0$ and $x \in H$. Then the following hold:

(a) *T_r is single-valued;*

(b) *T_r is firmly nonexpansive; that is, for any $x, y \in H$,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle; \tag{48}$$

(c) $F(T_r) = \mathrm{EP}(F)$;

(d) $\mathrm{EP}(F)$ *is closed and convex.*

Lemma 13 (see [13]). *Let C be a nonempty closed convex subset of a real Hilbert space H, let F be a bifunction from $C\times C$ to $\mathbb{R}$ which satisfies (A1)–(A4), and let A_F be a multivalued mapping of H into itself defined by*

$$A_F x = \begin{cases} \{z \in H : F(x, y) \geq \langle y - x, z\rangle, \forall y \in C\}, & x \in C, \\ \phi, & x \notin C. \end{cases} \tag{49}$$

Then A_F is a maximal monotone mapping with the domain $D(A_F) \subset C$, $\mathrm{EP}(F) = A_F^{-1}0$, and

$$T_r(x) = (I + rA_F)^{-1} x, \quad \forall x \in H,\ r > 0, \tag{50}$$

where T_r is defined as in (47).

Now we consider an equilibrium problem. Using Lemmas 12 and 13, the following result holds.

Theorem 14. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2 : C \to C$ be Lipschitzian pseudocontractive mappings with Lipschitz constants L_1 and L_2, respectively. Assume that $\mathcal{F} = F(T_1) \cap F(T_2) \cap \mathrm{EP}(F) \neq \phi$.*

Given $x_1, u \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:

$$\begin{aligned} z_n &= (1-c_n)x_n + c_n T_2 x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_1 x_n, \\ u_n &\in C \quad \text{such that } F(u_n, v) + \frac{1}{r_n}\langle v-u_n, u_n-x_n\rangle \geq 0, \\ &\qquad \forall v \in C, \\ x_{n+1} &= P_C[\alpha_n u + (1-\alpha_n) \\ &\quad \times (\theta_n x_n + \delta_n T_1 y_n + \gamma_n T_2 z_n + \xi_n u_n)], \\ &\qquad \forall n \geq 1. \end{aligned} \tag{51}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{c_n\}$, $\{\theta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$, $\{\xi_n\}$, and $\{r_n\}$ satisfy the following restrictions:

(a) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}\|r_{n+1}-r_n\| = 0$;

(b) $0 < c \leq \theta_n, \delta_n, \gamma_n, \xi_n \leq d < 1$, *and* $\theta_n+\delta_n+\gamma_n+\xi_n = 1$;

(c) $0 < \alpha_n < e < 1$, $\lim_{n\to\infty}\alpha_n = 0$, *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(d) $\delta_n + \gamma_n + \xi_n \leq \beta_n$, $c_n < \beta < 1/(\sqrt{1+L^2}+1)$, *for all* $n \geq 1$,

for some real numbers $c, d, e > 0$, where $L = \max\{L_1, L_2\}$. Then $\{x_n\}$ converges strongly to some point $\overline{x}$, where $\overline{x} = P_{\mathcal{F}}u$.

Let $f : H \to (-\infty, +\infty]$ be a proper convex lower semicontinuous function. Then the subdifferential of ∂f of f is defined as follows:

$$\partial f(x) = \{y \in H : f(z) \geq f(x) + \langle z-x, y\rangle,\ z \in H\}, \quad \forall x \in H. \tag{52}$$

From Rockafellar [14], we find that ∂f is maximal monotone. It is easy to verify that $0 \in \partial f(x)$ if and only if $f(x) = \min_{y\in H} f(y)$. Let I_C be the indicator function of C; that is,

$$I_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & x \notin C. \end{cases} \tag{53}$$

Then $I_C : H \to (-\infty, +\infty]$ is a proper convex lower semicontinuous function and ∂I_C is a maximal monotone mapping.

Lemma 15 (see [6]). *Let C be a nonempty closed convex subset of a real Hilbert space H, let P_C be the metric projection from H onto C, and let ∂I_C be the subdifferential of I_C, where I_C is the indicator function of C and let $J_\lambda = (I + \lambda\partial I_C)^{-1}$. Then*

$$y = J_\lambda x \Longleftrightarrow y = P_C x, \quad x \in H,\ y \in C. \tag{54}$$

Now we consider a variational inequality problem.

Theorem 16. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T_1, T_2 : C \to C$ be Lipschitzian pseudocontractive mappings with Lipschitz constants L_1 and L_2, respectively. Let $A : C \to H$ be an α-inverse strongly monotone mapping. Assume that $F(T_1)\cap F(T_2)\cap \mathrm{VI}(C, A) \neq \phi$. Given $x_1, u \in C$, let $\{x_n\}$ be the sequence generated by the following algorithm:*

$$\begin{aligned} z_n &= (1-c_n)x_n + c_n T_2 x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_1 x_n, \\ x_{n+1} &= P_C[\alpha_n u + (1-\alpha_n) \\ &\quad \times (\theta_n x_n + \delta_n T_1 y_n + \gamma_n T_2 z_n \\ &\quad + \xi_n P_C(I-\lambda_n A)x_n)], \quad \forall n \geq 1. \end{aligned} \tag{55}$$

Assume that the sequences $\{\alpha_n\}$, $\{\beta_n\}$, $\{c_n\}$, $\{\theta_n\}$, $\{\delta_n\}$, $\{\gamma_n\}$, $\{\xi_n\}$, and $\{\lambda_n\}$ satisfy the following restrictions:

(a) $0 < a < \lambda_n < b < 2\alpha$;

(b) $0 < c \leq \theta_n, \delta_n, \gamma_n, \xi_n \leq d < 1$, *and* $\theta_n+\delta_n+\gamma_n+\xi_n = 1$;

(c) $0 < \alpha_n < e < 1$, $\lim_{n\to\infty}\alpha_n = 0$, *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(d) $\delta_n + \gamma_n + \xi_n \leq \beta_n$, $c_n < \beta < 1/(\sqrt{1+L^2}+1)$, *for all* $n \geq 1$,

for some real numbers $a, b, c, d, e > 0$, where $L = \max\{L_1, L_2\}$. Then $\{x_n\}$ converges strongly to some point $\overline{x}$, where $\overline{x} = P_{\mathcal{F}}u$.

Proof. Put $B = \partial I_C$ in Theorem 9. Then we get that

$$\begin{aligned} x \in (A + \partial I_C)^{-1}0 &\Longleftrightarrow 0 \in Ax + \partial I_C x \\ &\Longleftrightarrow -Ax \in \partial I_C x \\ &\Longleftrightarrow \langle Ax, y-x\rangle \geq 0 \\ &\Longleftrightarrow x \in \mathrm{VI}(C, A). \end{aligned} \tag{56}$$

From Lemma 15, we can conclude the desired conclusion immediately. □

Conflict of Interests

The author declares that there is no conflict of interests regarding the publication of this paper.

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