Research Article

Common Fixed-Point Theorems in Complete Generalized Metric Spaces

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We introduce the notions of the $\mathcal{W}$ function and $\mathcal{S}$ function, and then we prove two common fixed point theorems in complete generalized metric spaces under contractive conditions with these two functions. Our results generalize or improve many recent common fixed point results in the literature.

1. Introduction and Preliminaries

In 2000, Branciari [1] introduced the following notion of a generalized metric space where the triangle inequality of a metric space had been replaced by an inequality involving three terms instead of two. Later, many authors worked on this interesting space (e.g., [2–7]).

Let (X, d) be a generalized metric space. For $\gamma > 0$ and $x \in X$, we define that

$$B_\gamma(x) := \{y \in X \mid d(x, y) < \gamma\}. \tag{1.1}$$

Branciari [1] also claimed that $\{B_\gamma(x) : \gamma > 0, \ x \in X\}$ is a basis for a topology on X, d is continuous in each of the coordinates, and a generalized metric space is a Hausdorff space. We recall some definitions of a generalized metric space as follows.

Definition 1.1 (See [1]). Let X be a nonempty set and $d : X \times X \to [0, \infty)$ be a mapping such that for all $x, y \in X$ and for all distinct point $u, v \in X$ each of them different from x and y,

one has the following:

   (i) $d(x, y) = 0$ if and only if $x = y$,

   (ii) $d(x, y) = d(y, x)$,

   (iii) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$ (rectangular inequality).

Then (X, d) is called a generalized metric space (or shortly g.m.s).

   We present an example to show that not every generalized metric on a set X is a metric on X.

Example 1.2. Let $X = \{t, 2t, 3t, 4t, 5t\}$ with $t > 0$ be a constant, and we define $d : X \times X \rightarrow [0, \infty)$ by

   (1) $d(x, x) = 0$, for all $x \in X$,

   (2) $d(x, y) = d(y, x)$, for all $x, y \in X$,

   (3) $d(t, 2t) = 3\gamma$,

   (4) $d(t, 3t) = d(2t, 3t) = \gamma$,

   (5) $d(t, 4t) = d(2t, 4t) = d(3t, 4t) = 2\gamma$,

   (6) $d(t, 5t) = d(2t, 5t) = d(3t, 5t) = d(4t, 5t) = (3/2)\gamma$,

where $\gamma > 0$ is a constant. Then let (X, d) be a generalized metric space, but it is not a metric space, because

$$d(t, 2t) = 3\gamma > d(t, 3t) + d(3t, 2t) = 2\gamma. \tag{1.2}$$

Definition 1.3 (See [1]). Let (X, d) be a g.m.s, $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is g.m.s convergent to x if and only if $d(x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. We denote by $x_n \rightarrow x$ as $n \rightarrow \infty$.

Definition 1.4 (See [1]). Let (X, d) be a g.m.s, $\{x_n\}$ be a sequence in X and $x \in X$. We say that $\{x_n\}$ is g.m.s Cauchy sequence if and only if, for each $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $n > m > n_0$.

Definition 1.5 (See [1]). Let (X, d) be a g.m.s. Then X is called complete g.m.s if every g.m.s Cauchy sequence is g.m.s convergent in X.

   In this paper, we also recall the concept of compatible mappings and prove two common fixed point theorems which incorporated the compatible map concept followed. In 1986, Jungck [8] introduced the below concept of compatible mappings.

Definition 1.6 (See [8]). Let (X, d) be a g.m.s, and let $S, F{:}X \rightarrow X$ be two single-valued functions. We say that S and F are compatible if

$$\lim_{n \to \infty} d(SFx_n, FSx_n) = 0, \tag{1.3}$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n \to \infty} d(Fx_n, Sx_n) = 0$.

   In particular, $d(SFx, FSx) = 0$ if $d(Fx, Sx) = 0$ by taking $x_n = x$ for all $n \in \mathbb{N}$.

Later, many authors studied this subject (compatible mappings), and many results on fixed points and common fixed points are proved (see, e.g., [9–14]).

2. Main Results

In this paper, we first introduce the below concept of the $\mathcal{W}$ function.

Definition 2.1. We call $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ a $\mathcal{W}$ function if the function φ satisfies the following conditions:

(φ_1) $\varphi(t) < t$ for all $t > 0$ and $\varphi(0) = 0$,

(φ_2) $\lim_{t_n \to t} \inf \varphi(t_n) < t$ for all $t > 0$.

Lemma 2.2. *Let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a $\mathcal{W}$ function. Then $\lim_{n \to \infty} \varphi^n(t) = 0$ for all $t > 0$, where $\varphi^n(t)$ denotes the nth iteration of φ.*

Proof. Let $t > 0$ be fixed. If $\varphi^{n_0}(t) = 0$ for some $n_0 \in \mathbb{N}$, then

$$\varphi^{n_0+1}(t) = \varphi(\varphi^{n_0}(t)) = \varphi(0) = 0. \tag{2.1}$$

It follows that $\varphi^{n_0+k}(t) = 0$ for all $k \in \mathbb{N}$, and so we get that $\lim_{n \to \infty} \varphi^n(t) = 0$.

If $\varphi^n(t) > 0$ for all $n \in \mathbb{N}$, then we put $\alpha_n = \varphi^n(t)$. Thus,

$$\alpha_{n+1} = \varphi^{n+1}(t) = \varphi(\varphi^n(t)) = \varphi(\alpha_n). \tag{2.2}$$

Since φ is a $\mathcal{W}$ function, we have that $\alpha_{n+1} = \varphi(\alpha_n) < \alpha_n$. Therefore, the sequence $\{\alpha_n\}$ is strictly decreasing and bounded from below, and so there exists an $\gamma \geq 0$ such that $\lim_{n \to \infty} \alpha_n = \gamma^+$. We claim that $\gamma = 0$. If not, suppose that $\gamma > 0$, then we have that

$$\gamma = \lim_{n \to \infty} \alpha_{n+1} = \lim_{n \to \infty} \inf \varphi(\alpha_n) = \lim_{\alpha_n \to \gamma^+} \inf \varphi(\alpha_n) < \alpha, \tag{2.3}$$

a contradiction. So we obtain that $\gamma = 0$, that is, $\lim_{n \to \infty} \varphi^n(t) = 0$. □

We now state the main common fixed-point theorem for the $\mathcal{W}$ function in a complete g.m.s, as follows.

Theorem 2.3. *Let (X, d) be a Hausdorff and complete g.m.s, and let $\varphi : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be a $\mathcal{W}$ function. Let $S, T, F, G : X \rightarrow X$ be four single-valued functions such that for all $x, y \in X$,*

$$d(Sx, Ty) \leq \varphi(\max\{d(Fx, Gy), d(Fx, Sx), d(Gy, Ty)\}). \tag{2.4}$$

Assume that $T(X) \subset F(X)$ and $S(X) \subset G(X)$, and the pairs $\{S, F\}$ and $\{T, G\}$ are compatible. If F or G is continuous, then S, T, F, and G have a unique common fixed point in X.

Proof. Given that $x_0 \in X$. Define the sequence $\{x_n\}$ recursively as follows:

$$Gx_{2n+1} = Sx_{2n} = z_{2n}, \qquad Fx_{2n+2} = Tx_{2n+1} = z_{2n+1}. \tag{2.5}$$

Step 1. We will prove that

$$\lim_{n \to \infty} d(z_n, z_{n+1}) = 0. \tag{2.6}$$

Using (2.4), we have that for each $n \in \mathbb{N}$

$$
\begin{aligned}
d(z_{2n}, z_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\
&\le \varphi(\max\{d(Fx_{2n}, Gx_{2n+1}), d(Fx_{2n}, Sx_{2n}), d(Gx_{2n+1}, Tx_{2n+1})\}) \\
&\le \varphi(\max\{d(z_{2n-1}, z_{2n}), d(z_{2n-1}, z_{2n}), d(z_{2n}, z_{2n+1})\}),
\end{aligned} \tag{2.7}
$$

and so we can conclude that

$$d(z_{2n}, z_{2n+1}) \le \varphi(d(z_{2n-1}, z_{2n})). \tag{2.8}$$

Similarly, we also conclude that

$$d(z_{2n+1}, z_{2n+2}) \le \varphi(d(z_{2n}, z_{2n+1})). \tag{2.9}$$

Generally, we have that for each $n \in \mathbb{N}$

$$d(z_n, z_{n+1}) \le \varphi(d(z_{n-1}, z_n)). \tag{2.10}$$

By induction, we get that

$$d(z_n, z_{n+1}) \le \varphi(d(z_{n-1}, z_n)) \le \varphi^2(d(z_{n-2}, z_{n-1})) \le \cdots \le \varphi^n(d(z_0, z_1)). \tag{2.11}$$

By Lemma 2.2, we obtained that $\lim_{n \to \infty} d(z_n, z_{n+1}) = 0$.
   We claim that $\{z_n\}$ is g.m.s Cauchy. We claim that the following result holds.

Step 2. Claim that, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that if $m, n \ge n_0$, then $d(z_m, z_n) < \varepsilon$.
   Suppose that the above statesment is false. Then there exists $\varepsilon > 0$ such that, for any $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$ with $m_k > n_k \ge k$ satisfying that

(a) m_k is even and n_k is odd,

(b) $d(z_{n_k}, z_{m_k}) \ge \varepsilon$,

(c) m_k is the smallest even number such that the condition (b) holds.

Taking into account (b) and (c), we have that

$$
\begin{aligned}
\varepsilon &\le d(z_{n_k}, z_{m_k}) \\
&\le d(z_{n_k}, z_{m_k-2}) + d(z_{m_k-2}, z_{m_k-1}) + d(z_{m_k-1}, z_{m_k}) \\
&\le \varepsilon + d(z_{m_k-2}, z_{m_k-1}) + d(z_{m_k-1}, z_{m_k}).
\end{aligned} \tag{2.12}
$$

Letting $k \to \infty$, we get the following:

$$\lim_{k \to \infty} d(z_{n_k}, z_{m_k}) = \varepsilon, \tag{2.13}$$

$$\varepsilon \leq d(z_{n_k-1}, z_{m_k-1})$$
$$\leq d(z_{n_k-1}, z_{m_k-3}) + d(z_{m_k-3}, z_{m_k-2}) + d(z_{m_k-2}, z_{m_k-1}) \tag{2.14}$$
$$\leq \varepsilon + d(z_{m_k-3}, z_{m_k-2}) + d(z_{m_k-2}, z_{m_k-1}).$$

Letting $k \to \infty$, we get the following:

$$\lim_{k \to \infty} d(z_{n_k-1}, z_{m_k-1}) = \varepsilon. \tag{2.15}$$

Using (2.4), (2.13), and (2.15), we have

$$
\begin{aligned}
d(z_{n_k}, z_{m_k}) &= d(Sx_{n_k}, Tx_{m_k}) \\
&\leq \varphi(\max\{d(Fx_{n_k}, Gx_{m_k}), d(Fx_{n_k}, Sx_{n_k}), d(Gx_{n_k}, Tx_{n_k})\}) \\
&= \varphi(\max\{d(z_{n_k-1}, z_{m_k-1}), d(z_{n_k-1}, z_{n_k}), d(z_{m_k-1}, z_{m_k})\}) \\
&= \varphi(\max\{d(z_{n_k-1}, z_{m_k-1}), c_{n_k-1}, c_{m_k-1}\}),
\end{aligned}
\tag{2.16}
$$

taking $\lim_{k \to \infty} \inf$, we get that $\varepsilon < \varepsilon$, a contradiction. So $\{z_n\}$ is g.m.s Cauchy. Since X is complete, there exists $z \in X$ such that $\lim_{n \to \infty} z_n = z$. So we have

$$d(Fx_{2n}, z) \longrightarrow 0, \qquad d(Gx_{2n+1}, z) \longrightarrow 0, \qquad d(Sx_{2n}, z) \longrightarrow 0, \qquad d(Tx_{2n+1}, z) \longrightarrow 0, \tag{2.17}$$

as $n \to \infty$.

Step 3. We will show that z is a common fixed point of S, T, F, and G.
  Assume that F is continuous. Then we have

$$d\left(F^2 x_{2n}, Fz\right) \longrightarrow 0, \qquad d(FSx_{2n}, Fz) \longrightarrow 0, \tag{2.18}$$

as $n \to \infty$. By the rectangular property, we have

$$d(SFx_{2n}, Fz) \leq d(SFx_{2n}, FSx_{2n}) + d\left(FSx_{2n}, F^2 x_{2n}\right) + d\left(F^2 x_{2n}, Fz\right). \tag{2.19}$$

Since S and F are compatible and $d(Sx_{2n}, Fx_{2n}) \to 0$ as $n \to \infty$, we conclude that

$$d(SFx_{2n}, FSx_{2n}) \longrightarrow 0, \tag{2.20}$$

as $n \to \infty$. Taking into account (2.18), (2.19), and (2.20), we have that

$$d(SFx_{2n}, Fz) \longrightarrow 0, \tag{2.21}$$

as $n \to \infty$. Since

$$d(SFx_{2n}, Tx_{2n+1}) \leq \varphi\left(\max\left\{d\left(F^2 x_{2n}, Gx_{2n+1}\right), d\left(F^2 x_{2n}, SFx_{2n}\right), d(Gx_{2n+1}, Tx_{2n+1})\right\}\right), \tag{2.22}$$

for each $n \in \mathbb{N}$. Taking $\lim_{n \to \infty}$ and taking into account (2.17), (2.18), (2.19), (2.20), and (2.21), we get that

$$d(Fz, z) \leq \varphi(\max\{d(Fz, z), d(Fz, Fz), d(z, z)\}) = \varphi(\max\{d(Fz, z), 0, 0\}) < d(Fz, z), \tag{2.23}$$

and this is a contradiction unless $d(Fz, z) = 0$, that is, $Fz = z$.

On the same way, we have that, for each $n \in \mathbb{N}$,

$$d(Sz, Tx_{2n+1}) \leq \varphi(\max\{d(Fz, Gx_{2n+1}), d(Fz, Sz), d(Gx_{2n+1}, Tx_{2n+1})\}). \tag{2.24}$$

letting $n \to \infty$, we obtained the following:

$$d(Sz, z) \leq \varphi(\max\{d(z, z), d(z, Sz), d(z, z)\}) = \varphi(\max\{0, d(z, Sz), 0\}) < d(Sz, z), \tag{2.25}$$

and this is a contradiction unless $d(Sz, z) = 0$, that is, $Sz = z$.

Since $S(X) \subset G(X)$, put $z' \in X$ such that $Gz' = z = Sz$. Then $TGz' = Tz$ and using (2.4),

$$\begin{aligned}
d(z, Tz') &= d(Sz, Tz') \\
&\leq \varphi(\max\{d(Fz, Gz'), d(Fz, Sz), d(Gz', Tz')\}) \\
&= \varphi(\max\{d(z, z), d(z, z), d(z, Tz')\}) \\
&< d(z, Tz'),
\end{aligned} \tag{2.26}$$

and this is a contradiction unless $d(z, Tz') = 0$, that is, $Tz' = z$ and so $d(Tz', Gz') = d(z, z) = 0$.

Since T and G are compatible and $d(Tz', Gz') = 0$, we have that

$$d(Tz, Gz) = d(TGz', GTz') = 0, \tag{2.27}$$

which implies that $Tz = Gz$. Using (2.4), we also have

$$
\begin{aligned}
d(z,Tz) &= d(Sz,Tz) \\
&\quad \times \varphi(\max\{d(Fz,Gz),d(Fz,Sz),d(Gz,Tz)\}) \\
&= \varphi(\max\{d(z,Tz),d(z,z),d(Tz,Tz)\}) \\
&< d(z,Tz),
\end{aligned}
\tag{2.28}
$$

and this is a contradiction unless $d(z,Tz) = 0$, that is, $Tz = z$.

From above argument, we get that

$$
Sz = Tz = z = Fz = Gz,
\tag{2.29}
$$

and so z is a common fixed point of S,T,F, and G.

Step 4. Finally, to prove the uniqueness of the common fixed point of S,T,F and G, let y be another common fixed point of S,T,F, and G. Then using (2.4), we have

$$
\begin{aligned}
d(y,z) &= d(Sy,Tz) \\
&\quad \times \varphi(\max\{d(Fy,Gz),d(Fy,Sy),d(Gz,Tz)\}) \\
&= \varphi(\max\{d(y,z),d(y,y),d(z,z)\}) \\
&< d(y,z),
\end{aligned}
\tag{2.30}
$$

and this is a contradiction unless $d(y,z) = 0$, that is, $y = z$. Hence z is the unique common fixed point of S,T,F, and G in X. $\qquad\square$

We give the following example to illustrate Theorem 2.3.

Example 2.4. Let $X = \{t,2t,3t,4t,5t\}$ with $t > 0$ is a constant, and we define $d : X \times X \to [0,\infty)$ by

(1) $d(x,x) = 0$, for all $x \in X$,

(2) $d(x,y) = d(y,x)$, for all $x,y \in X$,

(3) $d(t,2t) = 3\gamma$,

(4) $d(t,3t) = d(2t,3t) = \gamma$,

(5) $d(t,4t) = d(2t,4t) = d(3t,4t) = 2\gamma$,

(6) $d(t,5t) = d(3t,5t) = \gamma$ and $d(2t,5t) = d(4t,5t) = 2\gamma$,

where $\gamma > 0$ is a constant.

If $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$, $\varphi(t) = (4/5)t$, then φ is a $\mathcal{W}$ function. We next define $S, T, F, G : X \to X$ by

$$S(x) = \begin{cases} 3t & \text{if } x \neq 4t, \\ 5t & \text{if } x = 4t, \end{cases}$$

$$T(x) = \begin{cases} 3t & \text{if } x \neq 4t, \\ t & \text{if } x = 4t, \end{cases}$$

$$G(x) = I(x) = \text{the identity mapping,} \qquad (2.31)$$

$$F(x) = \begin{cases} 3t & \text{if } x = 3t, \\ t & \text{if } x = t, 2t, 5t, \\ 2t & \text{if } x = 4t. \end{cases}$$

Then all conditions of Theorem 2.3 are satisfied, and $3t$ is a unique common fixed point of S, T, F, and G.

For the case $G = F = I$ (the identity mapping) and $S = T$, we are easy to get the below fixed-point theorem.

Theorem 2.5. *Let (X, d) be a Hausdorff and complete g.m.s, and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a $\mathcal{W}$ function. Let $T : X \to X$ be a single-valued function such that for all $x, y \in X$,*

$$d(Tx, Ty) \leq \varphi(\max\{d(x, y), d(x, Tx), d(y, Ty)\}). \qquad (2.32)$$

Then T has a unique fixed point in X.

We next introduce the below concept of the $\mathcal{S}$ function.

Definition 2.6. We call $\phi : \mathbb{R}^{+3} \to \mathbb{R}^+$ a $\mathcal{S}$ function if the function ϕ satisfies the following conditions:

(ϕ_1) ϕ is a strictly increasing, continuous function in each coordinate,

(ϕ_2) for all $t > 0$, $\phi(t, t, t) < t$, $\phi(t, 0, 0) < t$, $\phi(0, t, 0) < t$, and $\phi(0, 0, t) < t$.

Example 2.7. Let $\phi : \mathbb{R}^{+3} \to \mathbb{R}^+$ denote that

$$\phi(t_1, t_2, t_3) = k \cdot \max\{t_1, t_2, t_3\}, \qquad \text{for } k \in (0, 1). \qquad (2.33)$$

Then ϕ is a $\mathcal{S}$ function.

We now state the main common fixed point theorem for the $\mathcal{S}$ function in a complete g.m.s.

Theorem 2.8. *Let (X, d) be a Hausdorff and complete g.m.s, and let $\varphi : \mathbb{R}^{+3} \to \mathbb{R}^+$ be a $\mathcal{S}$ function. Let $S, T, F, G : X \to X$ be four single-valued functions such that for all $x, y \in X$,*

$$d(Sx, Ty) \leq \phi(d(Fx, Gy), d(Fx, Sx), d(Gy, Ty)). \tag{2.34}$$

Assume that $T(X) \subset F(X)$ and $S(X) \subset G(X)$, and the pairs $\{S, F\}$ and $\{T, G\}$ are compatible. If F or G is continuous, then S, T, F, and G have a unique common fixed point in X.

Proof. Given $x_0 \in X$. Define the sequence $\{x_n\}$ recusively as follows:

$$Gx_{2n+1} = Sx_{2n} = z_{2n}, \qquad Fx_{2n+2} = Tx_{2n+1} = z_{2n+1}. \tag{2.35}$$

Step 1. We will prove that

$$\lim_{n \to \infty} d(z_n, z_{n+1}) = 0, \tag{2.36}$$

Using (2.34) and the definition of the $\mathcal{S}$ function, we have that for each $n \in \mathbb{N}$

$$\begin{aligned} d(z_{2n}, z_{2n+1}) &= d(Sx_{2n}, Tx_{2n+1}) \\ &\leq \phi(d(Fx_{2n}, Gx_{2n+1}), d(Fx_{2n}, Sx_{2n}), d(Gx_{2n+1}, Tx_{2n+1})) \\ &\leq \phi(d(z_{2n-1}, z_{2n}), d(z_{2n-1}, z_{2n}), d(z_{2n}, z_{2n+1})), \end{aligned} \tag{2.37}$$

and so we can conclude that

$$d(z_{2n}, z_{2n+1}) \leq \phi(d(z_{2n-1}, z_{2n}), d(z_{2n-1}, z_{2n}), d(z_{2n-1}, z_{2n})) < d(z_{2n-1}, z_{2n}). \tag{2.38}$$

Similarly, we also conclude that

$$d(z_{2n+1}, z_{2n+2}) \leq \phi(d(z_{2n}, z_{2n+1}), d(z_{2n}, z_{2n+1}), d(z_{2n}, z_{2n+1})) < d(z_{2n}, z_{2n+1}). \tag{2.39}$$

Generally, we have that for each $n \in \mathbb{N}$

$$d(z_n, z_{n+1}) \leq \phi(d(z_{n-1}, z_n), d(z_{n-1}, z_n), d(z_{n-1}, z_n)) < d(z_{n-1}, z_n). \tag{2.40}$$

Now, for each $m \in \mathbb{N}$, if we denote $c_m = d(z_m, z_{m+1})$, then $\{c_m\}$ is a strictly decreasing sequence. Thus, it must converge to some c with $c \geq 0$. We claim that $c = 0$. If not, suppose that $c > 0$, then

$$c \leq c_{m+1} \leq \phi(c_m, c_m, c_m). \tag{2.41}$$

Passing to the limit, as $m \to \infty$, we have that $c \leq c < \phi(c, c, c) < c$, which is a contradiction. So we get $c = 0$.

We claim that $\{z_n\}$ is g.m.s Cauchy. We claim that the following result holds.

Step 2. Claim that, for every $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that if $m, n \geq n_0$, then $d(z_m, z_n) < \varepsilon$.

Suppose that the above statesment is false. Then there exists $\varepsilon > 0$ such that, for any $k \in \mathbb{N}$, there are $m_k, n_k \in \mathbb{N}$ with $m_k > n_k \geq k$ satisfying that

(d) m_k is even and n_k is odd,

(e) $d(z_{n_k}, z_{m_k}) \geq \varepsilon$,

(f) m_k is the smallest even number such that the condition (e) holds.

Taking into account (e) and (f), we have that

$$
\begin{aligned}
\varepsilon &\leq d(z_{n_k}, z_{m_k}) \\
&\leq d(z_{n_k}, z_{m_k-2}) + d(z_{m_k-2}, z_{m_k-1}) + d(z_{m_k-1}, z_{m_k}) \\
&\leq \varepsilon + d(z_{m_k-2}, z_{m_k-1}) + d(z_{m_k-1}, z_{m_k}).
\end{aligned}
\tag{2.42}
$$

Letting $k \to \infty$, we get the following:

$$
\lim_{k \to \infty} d(z_{n_k}, z_{m_k}) = \varepsilon,
\tag{2.43}
$$

$$
\begin{aligned}
\varepsilon &\leq d(z_{n_k-1}, z_{m_k-1}) \\
&\leq d(z_{n_k-1}, z_{m_k-3}) + d(z_{m_k-3}, z_{m_k-2}) + d(z_{m_k-2}, z_{m_k-1}) \\
&\leq \varepsilon + d(z_{m_k-3}, z_{m_k-2}) + d(z_{m_k-2}, z_{m_k-1}).
\end{aligned}
\tag{2.44}
$$

Letting $k \to \infty$, we get the following:

$$
\lim_{k \to \infty} d(z_{n_k-1}, z_{m_k-1}) = \varepsilon.
\tag{2.45}
$$

Using (2.34), (2.43), and (2.45), we have

$$
\begin{aligned}
d(z_{n_k}, z_{m_k}) &= d(Sx_{n_k}, Tx_{m_k}) \\
&\leq \phi(d(Fx_{n_k}, Gx_{m_k}), d(Fx_{n_k}, Sx_{n_k}), d(Gx_{n_k}, Tx_{n_k})) \\
&= \phi(d(z_{n_k-1}, z_{m_k-1}), d(z_{n_k-1}, z_{n_k}), d(z_{m_k-1}, z_{m_k})) \\
&= \phi(d(z_{n_k-1}, z_{m_k-1}), c_{n_k-1}, c_{m_k-1}),
\end{aligned}
\tag{2.46}
$$

taking $k \to \infty$, we get that $\varepsilon \leq \phi(\varepsilon, 0, 0) < \varepsilon$, a contradiction. So $\{z_n\}$ is g.m.s Cauchy. Since X is complete, there exists $z \in X$ such that $\lim_{n \to \infty} z_n = z$. So we have

$$
d(Fx_{2n}, z) \longrightarrow 0, \qquad d(Gx_{2n+1}, z) \longrightarrow 0, \qquad d(Sx_{2n}, z) \longrightarrow 0, \qquad d(Tx_{2n+1}, z) \longrightarrow 0,
\tag{2.47}
$$

as $n \to \infty$.

Step 3. We will show that z is a common fixed point of S, T, F, and G.

Assume that F is continuous. Then, we have

$$d\left(F^2 x_{2n}, Fz\right) \longrightarrow 0, \qquad d(FSx_{2n}, Fz) \longrightarrow 0, \tag{2.48}$$

as $n \to \infty$. By the rectangular property, we have

$$d(SFx_{2n}, Fz) \le d(SFx_{2n}, FSx_{2n}) + d\left(FSx_{2n}, F^2 x_{2n}\right) + d\left(F^2 x_{2n}, Fz\right). \tag{2.49}$$

Since S and F are compatible and $d(Sx_{2n}, Fx_{2n}) \to 0$ as $n \to \infty$, we conclude that

$$d(SFx_{2n}, FSx_{2n}) \longrightarrow 0, \tag{2.50}$$

as $n \to \infty$. Taking into account (2.48), (2.49), and (2.50), we have that

$$d(SFx_{2n}, Fz) \longrightarrow 0, \tag{2.51}$$

as $n \to \infty$. Since

$$d(SFx_{2n}, Tx_{2n+1}) \le \phi\left(d\left(F^2 x_{2n}, Gx_{2n+1}\right), d\left(F^2 x_{2n}, SFx_{2n}\right), d(Gx_{2n+1}, Tx_{2n+1})\right), \tag{2.52}$$

for each $n \in \mathbb{N}$. Letting $n \to \infty$ and taking into account (2.47), (2.48), (2.49), (2.50), and (2.51), we get that

$$d(Fz, z) \le \phi(d(Fz, z), d(Fz, Fz), d(z, z)) = \phi(d(Fz, z), 0, 0) < d(Fz, z), \tag{2.53}$$

and this is a contradiction unless $d(Fz, z) = 0$, that is, $Fz = z$.

On the same way, we have that, for each $n \in \mathbb{N}$,

$$d(Sz, Tx_{2n+1}) \le \phi(d(Fz, Gx_{2n+1}), d(Fz, Sz), d(Gx_{2n+1}, Tx_{2n+1})). \tag{2.54}$$

Taking $\lim_{n \to \infty}$, we obtained that

$$d(Sz, z) \le \phi(d(Fz, z), d(Fz, Sz), d(z, z)) = \phi(d(0, d(z, Sz), 0)) < d(Sz, z), \tag{2.55}$$

and this is a contradiction unless $d(Sz, z) = 0$, that is, $Sz = z$.

Since $S(X) \subset G(X)$, put $z' \in X$ such that $Gz' = z = Sz$. Then $TGz' = Tz$ and using (2.34),

$$
\begin{aligned}
d(z, Tz') &= d(Sz, Tz') \\
&\leq \phi(d(Fz, Gz'), d(Fz, Sz), d(Gz', Tz')) \\
&= \phi(d(z, z), d(z, z), d(z, Tz')) \\
&< d(z, Tz'),
\end{aligned}
\tag{2.56}
$$

and this is a contradiction unless $d(z, Tz') = 0$, that is, $Tz' = z$ and so $d(Tz', Gz') = d(z, z) = 0$.

Since T and G are compatible and $d(Tz', Gz') = 0$, we have that

$$
d(Tz, Gz) = d(TGz', GTz') = 0,
\tag{2.57}
$$

which implies that $Tz = Gz$. Using (2.34), we also have

$$
\begin{aligned}
d(z, Tz) &= d(Sz, Tz) \\
&\leq \phi(d(Fz, Gz), d(Fz, Sz), d(Gz, Tz)) \\
&= \phi(d(z, Tz), d(z, z), d(Tz, Tz)) \\
&< d(z, Tz),
\end{aligned}
\tag{2.58}
$$

and this is a contradiction unless $d(z, Tz) = 0$, that is, $Tz = z$.

From above argument, we get that

$$
Sz = Tz = z = Fz = Gz,
\tag{2.59}
$$

and so z is a common fixed point of S, T, F, and G.

Step 4. Finally, to prove the uniqueness of the common fixed point of S, T, F and G, let y be another common fixed point of S, T, F, and G. Then using (2.34), we have

$$
\begin{aligned}
d(y, z) &= d(Sy, Tz) \\
&\leq \phi(d(Fy, Gz), d(Fy, Sy), d(Gz, Tz)) \\
&= \phi(d(y, z), d(y, y), d(z, z)) \\
&< d(y, z),
\end{aligned}
\tag{2.60}
$$

and this is a contradiction unless $d(y, z) = 0$, that is, $y = z$. Hence z is the unique common fixed point of S, T, F, and G in X. $\qquad\square$

Using Example 2.4, we get the following example to illustrate Theorem 2.8.

Example 2.9. Let $X = \{t, 2t, 3t, 4t, 5t\}$ with $t > 0$ be a constant, and we define $d : X \times X \rightarrow [0, \infty)$ by

(1) $d(x, x) = 0$, for all $x \in X$,

(2) $d(x, y) = d(y, x)$, for all $x, y \in X$,

(3) $d(t, 2t) = 3\gamma$,

(4) $d(t, 3t) = d(2t, 3t) = \gamma$,

(5) $d(t, 4t) = d(2t, 4t) = d(3t, 4t) = 2\gamma$,

(6) $d(t, 5t) = d(3t, 5t) = \gamma$ and $d(2t, 5t) = d(4t, 5t) = 2\gamma$,

where $\gamma > 0$ is a constant.

If $\phi : \mathbb{R}^{+3} \rightarrow \mathbb{R}^{+}$, $\phi(t) = (8/9) \cdot \max\{t_1, t_2, t_3\}$, then ϕ is a $\mathcal{S}$ function. We next define $S, T, F, G : X \rightarrow X$ by

$$
S(x) = \begin{cases} 3t & \text{if } x \neq 4t, \\ 5t & \text{if } x = 4t, \end{cases}
$$

$$
T(x) = \begin{cases} 3t & \text{if } x \neq 4t, \\ t & \text{if } x = 4t, \end{cases}
$$

$$
G(x) = I(x) = \text{the identity mapping}, \tag{2.61}
$$

$$
F(x) = \begin{cases} 3t & \text{if } x = 3t, \\ t & \text{if } x = t, 2t, 5t, \\ 2t & \text{if } x = 4t. \end{cases}
$$

Then all conditions of Theorem 2.8 are satisfied, and $3t$ is a unique common fixed point of S, T, F, and G.

For the case $G = F = I$ (the identity mapping) and $S = T$, we are easy to get the below fixed-point theorem.

Theorem 2.10. *Let (X, d) be a Hausdorff and complete g.m.s, and let $\phi : \mathbb{R}^{+3} \rightarrow \mathbb{R}^{+}$ be a $\mathcal{S}$ function. Let $T : X \rightarrow X$ be a single-valued function such that for all $x, y \in X$,*
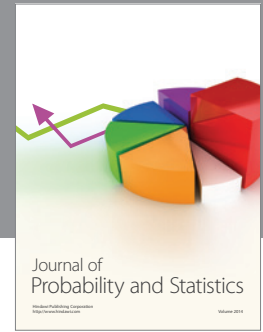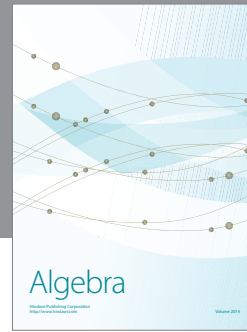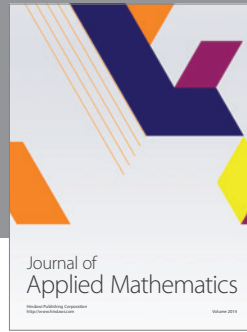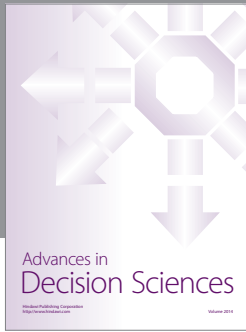
$$
d(Tx, Ty) \leq \phi(d(x, y), d(x, Tx), d(y, Ty)). \tag{2.62}
$$

Then T has a unique fixed point in X.

Acknowledgments

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