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*Research Article*

# Minimum 2-Tuple Dominating Set of an Interval Graph

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The $k$-tuple domination problem, for a fixed positive integer $k$, is to find a minimum size vertex subset such that every vertex in the graph is dominated by at least $k$ vertices in this set. The case when $k = 2$ is called 2-tuple domination problem or double domination problem. In this paper, the 2-tuple domination problem is studied on interval graphs from an algorithmic point of view, which takes $O(n^2)$ time, $n$ is the total number of vertices of the interval graph.

## 1. Introduction

An undirected graph $G = (V, E)$ is an *interval graph* if the vertex set $V$ can be put into one-to-one correspondence with a set of intervals $I$ on the real line $R$ such that two vertices are adjacent in G if and only if their corresponding intervals have nonempty intersection. The set $I$ is called an *interval representation* of G and G is referred to as the *intersection graph* of $I$ [1]. Let $I = \{i_1, i_2, \ldots, i_n\}$, where $i_c = [a_c, b_c]$ for $1 \leq c \leq n$, be the interval representation of the graph $G$, $a_c$ is the left endpoint, and $b_c$ is the right end point of the interval $i_c$. Without any loss of generality, assume the following:

(a) an interval contains both its endpoints and that no two intervals share a common endpoint [1],

(b) intervals and vertices of an interval graph are one and the same thing,

(c) the graph $G$ is connected, and the list of sorted endpoints is given

(d) the intervals in $I$ are indexed by increasing right endpoints, that is, $b_1 < b_2 < \cdots < b_n$.

In a graph $G$, a vertex is said to *dominate* itself and all of its neighbors. A *dominating set* of $G = (V, E)$ is a subset $D$ of $V$ such that every vertex in $V$ is dominated by at least one vertex in $D$. The *domination number* $\gamma(G)$ is the minimum size of a dominating set of $G$. For a fixed positive integer $k$, a *$k$-tuple dominating set* of $G = (V, E)$ is a subset $D$ of $V$ such that every vertex in $V$ is dominated by at least $k$ vertices of $D$. As introduced by Harary and Haynes [2], a *$k$-tuple dominating set* $D$ is a set $D \subseteq V$ for which $|N[v] \cap D| \geq k$ for every $v \in V$, where $N[v] = \{v\} \cup \{u \in V : (u, v) \in E\}$ is the *closed neighborhood* of the vertex $v$. Note that we must have the minimum degree greater than or equal to $k - 1$ for a $k$-tuple dominating set to exist. The *$k$-tuple domination number* $\gamma_{\times k}(G)$ is the minimum cardinality of $k$-tuple dominating set of $G$. When $k = 2$, this is called *double domination* [3].

A 2-tuple dominating set $D$ is said to be *minimal* if there does not exist any $D' \subset D$ such that $D'$ is a 2-tuple dominating set of $G$. A 2-tuple dominating set $D$, denoted by $\gamma_{\times 2}(G)$, is said to be *minimum*, if it is minimal as well as it gives 2-tuple domination number.

In graph theory, a connected component of an undirected graph is a subgraph in which any two vertices are connected to each other by paths. For a graph $G$, if the subgraph $G$ itself is a connected component, then the graph $G$ is called *connected*, else the graph $G$ is called *disconnected* and each connected component subgraph is called its *component*. Removal of a vertex $v$ from a graph $G$ means the removal of vertex $v$ and edges incident to $v$. A *cut vertex* of a connected graph $G$ is a vertex of $G$ whose removal renders $G$ disconnected. Pal et al. [4] described an algorithm for computing cut vertices and blocks on interval graphs.

A graph $G$ is *vertex domination-critical* if for any vertex $v$ of $G$, the domination number of $G - v$ is less than the domination number of $G$. If such a graph $G$ has domination number $\gamma$, it is called *$\gamma$-critical*. Brigham et al. [5] studied $\gamma$-critical graphs and posed the following questions.

(1) If $G$ is a $\gamma$-critical graph, is $|V| \geq (\delta + 1)(\gamma - 1) + 1$?

(2) If a $\gamma$-critical graph $G$ has $(\Delta + 1)(\gamma - 1) + 1$ vertices, is $G$ regular?

(3) Does $i = \gamma$ for all $\gamma$-critical graphs?

(4) Let $d$ be the diameter of the $\gamma$-critical graph $G$. Does $d \leq 2(\gamma - 1)$ always hold?

Later in this paper, it has been proved that for some vertex (or cut vertex) $v$ of $G$, $G - v$ and $G$ have the same domination number $\gamma_{\times 2}(G)$.

### *1.1. Survey of Related Works*

Various works have been found on interval graphs. Interval graphs are useful in modeling resource allocation problems in operations research. A. Pal and M. Pal [6] have studied about interval graphs. So many algorithms and results of various parameters on interval graphs have been found in [4, 7–12]. The *domination* is one of the parameters in graphs which has a great importance in modern circuit designing systems. Chang et al. [13] have extensively studied about domination in graphs. Also domination and its variations can be found in [14–17]. Another type of dominating set has been widely studied in [18] which is a *total dominating set*. Henning has worked on graphs with *large total domination number* in [19]. For a domination number, Sumner and Blitch [20] studied graphs where the addition of any edge changed the domination number. They called graphs with this property *domination edge critical*. Brigham et al. [5] and Fulman et al. [21] have worked on *vertex domination-critical graphs*. Wojcicka [22] have found some results on Hamiltonian properties of *domination-critical* *graphs*. The total domination edge critical graphs, that is, graphs where the addition of any edge decreased the total domination number were studied by Haynes et al. in [23–26]. Among the variations of domination, the $k$-tuple domination was introduced in [3]. The case when $k = 2$ was called double domination in [3], where exact values of the double domination numbers for some special graphs are obtained. In the same paper, various bounds of double and $k$-tuple domination numbers are available in terms of the other parameters.

## 2. Interval Graph and Some of Its Properties

Let $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, $|V| = n$, $|E| = m$ be a connected interval graph in which the vertices are given in the sorted order of the right endpoints of the interval representation of the graph. Intervals are labeled according to increasing order of their endpoints. This labeling is referred to as IG ordering. Let $(u, v)$ or $(v, u)$ denote the existence of an adjacency relation between two vertices $u, v$. It is assumed that $(u, u)$ is always true, that is, $(u, u) \in E$. If $[a_u, b_u]$ and $[a_v, b_v]$ are two end points of the vertices $u$ and $v$, respectively, then $u, v$ are adjacent if at least one of the following conditions hold:

(i) $a_v < a_u < b_v$,

(ii) $a_v < b_u < b_v$,

(iii) $a_u < a_v < b_u$,

(iv) $a_u < b_v < b_u$.

The following lemma is true for a given interval graph, $G = (V, E)$.

**Lemma 2.1** (see [27]). *If the vertices $u, v, w \in V$ are such that $u < v < w$ in the IG ordering and $u$ is adjacent to $w$, then $v$ is also adjacent to $w$.*

For each vertex $v \in V$, let $H(v)$ represent the highest numbered adjacent vertex of $v$. If no adjacent vertex of $v$ exists with higher IG number than $v$, then $H(v)$ is assumed to be $v$. In other words, $H(v) = \max\{u : (v, u) \in E, u \geq v\}$.

Throughout this paper, we use the notation $D$ for 2-tuple dominating set. For the purpose to find $D$ of the interval graph $G = (V, E)$, we consider a function $f : V(G) \to \{0, 1\}$ which is defined by $f(v) = 1$ if $v \in D$, otherwise, $f(v) = 0$. We define the function $f$ so that for $S \subseteq V(G)$, $f(S) = \sum_{v \in S} f(v)$. The weight of the function $f$ is $w(f) = f(V(G))$. Also, $w_i(f)$ is defined as $w_i(f) = f(N[i]) = \sum_{v \in N[i]} f(v)$, for all $i = 1, 2, 3, \ldots, n$.

## 3. Algorithm for 2-Tuple Domination

In a connected interval graph, the vertices are ordered by IG ordering. First of all, we treat none of a vertex of $V(G)$ as a member of dominating set $D$. Then, insert vertices one by one by testing their consistency. If a vertex $v$ is dominated by at least two vertices then leave it, otherwise, take the highest numbered adjacent vertex (vertices) from $N[v]$ as member(s) of dominating set $D$.

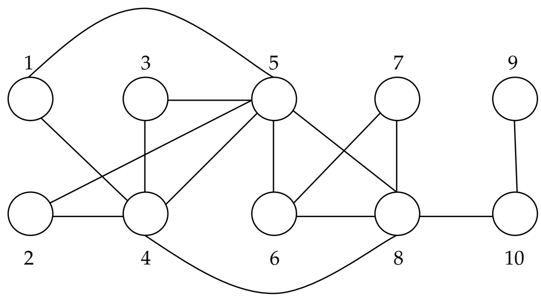


**Figure 1:** An interval graph $G = (V, E)$.

**Table 1:** Computation of all $p$th numbered adjacent vertices.

| $M_i(v)$ | $v$ | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| $M_0(v)$ | 5 | 5 | 5 | 8 | 8 | 8 | 8 | 10 | 10 | 10 |
| $M_1(v)$ | 4 | 4 | 4 | 5 | 6 | 7 | 7 | 8 | 9 | 9 |
| $M_2(v)$ | 1 | 2 | 3 | 4 | 5 | 6 | 6 | 7 | — | 8 |
| $M_3(v)$ | — | — | — | 3 | 4 | 5 | — | 6 | — | — |
| $M_4(v)$ | — | — | — | 2 | 3 | — | — | 5 | — | — |
| $M_5(v)$ | — | — | — | 1 | 2 | — | — | 4 | — | — |
| $M_6(v)$ | — | — | — | — | 1 | — | — | — | — | — |

Let us associate a new term $M_i(v)$ for a vertex $v \in V$, for all $i = 0, 1, 2, \ldots, k$ ($k = |N(v)|$) to each adjacent vertices of $v$ in order to set IG ordering of intervals in the following way:

$$M_i(v) = \max\left\{ N[v] - \bigcup_{j=0}^{i-1} M_i(v) \right\} \tag{3.1}$$

$$\text{with } M_0(v) = \max\{N[v]\}.$$

Basically, $M_0(v) = H(v)$, the highest numbered adjacent vertex of $v$ [28]. In connection with the name of $H(v)$, we call this $M_i(v)$ as the $p$th *numbered adjacent vertex of* $v$ through Definition 3.1.

*Definition 3.1* ($p$th numbered adjacent vertex). Let $u, v \in V$. If for some $i$ ($i = 0, 1, 2, \ldots, |N(v)|$), $|N(v)| - i = p$ such that $u = M_i(v)$, then $u$ is called the $p$th numbered adjacent vertex of $v$.

From the definition, it is easily seen that, for a vertex $v$, $M_i(v)$ exists for maximum possible $i = |N(v)|$, that is, degree of the vertex $v$. Therefore, in a graph, the maximum possible $i$ occurs in the degree of the graph, that is, $\Delta = \max\{\deg(v) : v \in V\}$. An illustration of the computations of all $M_i(v)$ for the graph of Figure 1 are shown in Table 1.

Now, we describe an algorithm to find two sets of vertices $D$ and $L$ depending only on $M_0(i)$ and $M_1(i)$.

**Input:** An interval graph $G = (V, E)$ with IG ordering vertex set $V = \{1, 2, 3, \ldots, n\}$.
**Output:** 2-tuple dominating set $D$ and 2-tuple domination number $\gamma_{\times 2}(G)(= |D|)$.
**Step 1:** Set $f(j) = 0,\ \forall j = 1, 2, \ldots, n$; //Assume that no vertices are the members of $D$.//
**Step 2:** Set $i = 1$, $D = \emptyset$ and $L = \emptyset$;
  **Step 2.1:** Compute $w_i(f) = \sum_{v \in N[i]} f(v)$;
  **Step 2.2:** If $w_i(f) = 0$ then //At least the vertex $i$ is not adjacent to any of the vertices of $D$.//
    Set $f(M_0(i)) = 1$ and $f(M_1(i)) = 1$;
    $D = D \cup \{M_0(i)\} \cup \{M_1(i)\}$ and $L = L \cup \{i\}$;
   else if $w_i(f) = 1$ then //At least the vertex $i$ is connected to one of the vertex of $D$.//
    If $f(M_0(i)) = 1$ then
     Set $f(M_1(i)) = 1$;
     $D = D \cup \{M_1(i)\}$;
    else
     Set $f(M_0(i)) = 1$;
     $D = D \cup \{M_0(i)\}$;
    end if;
    $L = L \cup \{i\}$;
   else
    Goto Step 2.3;
   end if;
  **Step 2.3:** Calculate $i = i + 1$ and go to Step 2.1 and continue until $i > n$;
**end 2DIG.**

**Algorithm 1:** Algorithm 2DIG.

Actually, the Algorithm 2DIG (Algorithm 1) gives the set $D$ which is the minimum 2-tuple dominating set and $|D|$, the 2-tuple domination number of the interval graph $G = (V, E)$. Before going to prove this result, we first verify Algorithm 2DIG in Figure 2. Here, we denote the set $L$ as the set of leading vertices corresponding to the 2-tuple dominating set $D$.

In Algorithm 2DIG, at $i$th iteration, if $w_i(f) = 0$, then $i$ is a member of $L$ and $i$ is said to be the *leading vertex of order* 2 corresponding to the vertices $M_0(i)$ and $M_1(i)$ of $D$, and if $w_i(f) = 1$, then $i$ is said to be the *leading vertex of order* 1 corresponding to the vertex $M_0(i)$ or $M_1(i)$ of $D$, otherwise, $i$ does not belong to $L$.

Therefore, we conclude that if $l_1 \in L$, then $l_1$ is adjacent to exactly two vertices of $D$.

### *3.1. Verification of the Algorithm*

Suppose we are to find 2-tuple dominating set $D$ and 2-tuple domination number $|D|$ of the interval graph $G = (V, E)$, where $V = \{1, 2, \ldots, 10\}$ shown in Figure 1. First, set $f(j) = 0$, for all $j \in V$. In Step 2, set $i = 1$, $D = \emptyset$ and $L = \emptyset$, that is, initially $D$ and $L$ are empty. Step 2 repeats for $n$ times. Here, $n = 10$, number of vertices in the graph. We illustrate the iterations in the following way.

*Iteration 1.* For the first iteration $i = 1$, $N[1] = \{1, 4, 5\}$. Calculate $w_1(f) = f(1) + f(4) + f(5) = 0$. The first condition of *if-end if* is satisfied. Since $w_1(f) = 0$, we find $M_0(1) = 5$ and $M_1(1) = 4$. Then, set $f(5) = 1$ and $f(4) = 1$. Also, set $D = \emptyset \cup \{4, 5\} = \{4, 5\}$, $L = \{1\}$, and $i = i + 1 = 2$.

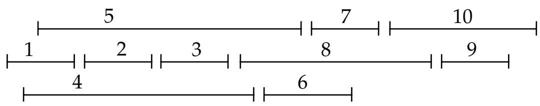


**Figure 2:** An interval representation of Figure 1.

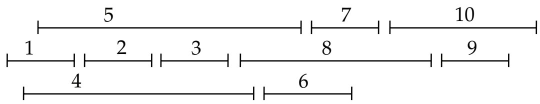


**Figure 3:** Finding $D$ by Algorithm 2DIG.

*Iteration 2.* $N[2] = \{2, 4, 5\}$. $w_2(f) = f(N[2]) = 2$. The vertex 2 is dominated by two vertices 4 and 5 of $D$. So, in this iteration, $D$ could not be calculated. Hence, $D$ and $L$ remain the same and $i$ is being increased to 3.

*Iteration 3.* $N[3] = \{3, 4, 5\}$. $w_3(f) = f(N[3]) = 2$. In this iteration, also $D$ and $L$ remain unchanged. The iteration number $i$ is being increased to 4.

*Iteration 4.* Here, $N[4] = \{1, 2, 3, 4, 5, 8\}$ and $w_4(f) = f(N[4]) = 2$. So, $D$ and $L$ are the same as the previous iteration. Set $i = 5$.

*Iteration 5.* In this iteration, $N[5] = \{1, 2, 3, 4, 5, 6, 8\}$ and $w_5(f) = f(N[5]) = 2$, and hence no change occurs. $i$ is being increased to 6.

*Iteration 6.* $N[6] = \{5, 6, 7, 8\}$ and $w_6(f) = f(N[6]) = 1$. So, domination criteria are not satisfied here. The *else-if* condition of *if-end if* is satisfied. Now, we check either $f(M_0(6)) = 1$ or not. We see that $f(M_0(6)) = f(8) = 0$, and hence set $f(8) = 1$. Update $D$ by $D \cup \{8\} = \{4, 5, 8\}$ and $L$ by $L \cup \{6\} = \{1, 6\}$. $i$ is being increased to 7.

*Iteration 7.* $N[7] = \{6, 7, 8\}$ and $w_7(f) = f(N[7]) = 1$. Here, also domination criteria are not satisfied. As $f(M_0(7)) = f(8) = 1$, set $f(M_1(7)) = f(7) = 1$ and $D = D \cup \{7\} = \{4, 5, 8, 7\}$ with $L = L \cup \{7\} = \{1, 6, 7\}$. $i$ is being increased to 8.

*Iteration 8.* In this iteration, that is, for $i = 8$, $N[8] = \{4, 5, 6, 7, 8, 10\}$. $w_8(f) = 4$. Hence, $D$ and $L$ remain unchanged and $i$ is being increased to 9.

*Iteration 9.* At ninth iteration, $i = 9$. Here, $N[9] = \{9, 10\}$ $w_9(f) = 0$. Then, $D = D \cup \{9, 10\} = \{4, 5, 8, 7, 9, 10\}$ and $L = L \cup \{9\} = \{1, 6, 7, 9\}$ with $f(9) = 1$ and $f(10) = 1$. Set $i = 10$.

*Iteration 10.* For $i = 10$, $N[10] = \{8, 9, 10\}$ $w_{10}(f) = 3$. Hence, $D$ and $L$ remain unchanged. As there are 10 vertices in Figure 3, so, this is the last iteration.

So, by the Algorithm 2DIG, we get $D = \{4, 5, 8, 7, 9, 10\}$, that is, $D = \{4, 5, 7, 8, 9, 10\}$ and $L = \{1, 6, 7, 9\}$. Therefore, $|D|$ = cardinality of $D = 6$. In Figure 3, thick lines represent the members of $D$.

## 4. Proof of Correctness and Time Complexity

Here, we will prove that $D$ is a minimum 2-tuple dominating set.

**Theorem 4.1.** *The set $D$ is a minimal 2-tuple dominating set.*

*Proof.* Let $D = \{d_1, d_2, \ldots, d_k\}$ be the 2-tuple dominating set obtained by Algorithm 2DIG. We are to prove that this $D$ is minimal 2-tuple dominating set, that is, there does not exist any $D' \subset D$ such that $D'$ is a 2-tuple dominating set.

Suppose, there exists a $D' \subset D$ such that $D'$ is a 2-tuple dominating set. Since $D' \subset D$, there must exist at least one member of $D$, say $d_i$, such that $d_i \notin D'$. Let the leading vertex corresponding to $d_i$ be $k$, then $w_k(f) = 2$. Again, since $D'$ is a 2-tuple dominating set and $d_i \notin D'$, $f(d_i) = 0$ and $w_k(f) = 1$ with respect to the 2-tuple dominating set $D'$. Therefore, $k$ is dominated by only one vertex of $D'$, which is a contradiction of our assumption that $D'$ is a 2-tuple dominating set. Thus, $D$ is minimal 2-tuple dominating set. □

**Theorem 4.2.** *The 2-tuple domination number of the given interval graph is the cardinality of the 2-tuple dominating set $D$, that is, $\gamma_{\times 2}(G) = |D|$.*

*Proof.* Let $L$ be the set of leading vertices corresponding to the minimal 2-tuple dominating set $D$ of $G$. Suppose there exists another minimal 2-tuple dominating set $D'$ such that $|D'| < |D|$.

Without loss of generality, we assume that $l_1$ is the leading vertex of order 2 corresponding to the two vertices $d_1$ and $d_2$ of $D$. Then, $l_1$ is adjacent to exactly two vertices $d_1$, $d_2$ $(d_1 < d_2)$ of $D$. Also $d_2 = M_0(l_1)$, the highest numbered adjacent vertex to $l_1$. So, there does not exist any vertex $v > d_2$ in $V$ such that $l_1$ is adjacent to $v$. If $d_1, d_2 \notin D'$, then there exist at least two vertices, say, $d'_1, d'_2 \in D'$ such that $d'_1 < d'_2 < d_1 < d_2$, where each $d'_1$ and $d'_2$ are adjacent to $l_1$. If $|D| = 2$, then we have $|D'| = 2$. So, $\gamma_{\times 2}(G) = |D'| = |D|$. For $|D| > 2$, consider the following two cases.

*Case 1.* Let $l_2$ be the leading vertex of order 1 corresponding to a vertex $d_3 \in D$. Since $l_2$ is of order 1, either $d_3 = M_0(l_2)$ or $d_3 = M_1(l_2)$ (by Algorithm 2DIG) and $l_2$ is adjacent to $d_2$ but not adjacent to $d_1$. If $l_2$ is adjacent to $d_1$, then $l_2$ is adjacent to three vertices $d_1, d_2$, and $d_3$ of $D$ (not exactly two), a contradiction. Hence, $l_2$ is not adjacent to the vertices $d'_1$ and $d'_2$. As, $(d'_1, l_2) \in E$ or $(d'_2, l_2) \in E$ implies $(d_1, l_2) \in E$. Therefore, $l_2 \in V$ is not dominated by at least two vertices of $D'$ and hence there exist at least two vertices $d'_3, d'_4 \in D'$, where each $d'_3$ and $d'_4$ are adjacent to $l_2$. Hence, $|D'| \geq |D|$.

*Case 2.* Let $l_2$ be the leading vertex of order 2 corresponding to the vertices $d_3, d_4 \in D$ $(d_3 < d_4)$. Then, $l_2$ is not adjacent to any vertex higher than $d_4$. Also, $l_2$ is not adjacent to $d'_1$ and $d'_2$ as $l_2$ is not adjacent to $d_2$. Therefore, if $D'$ is a 2-tuple dominating set, $l_2$ must be dominated by at least two vertices of $D'$, say $d'_3$ and $d'_4$. Hence, $|D'| \geq |D|$.

Thus, there does not exist any $D'$ such that $|D'| < |D|$, that is, $D$ is minimum and hence $\gamma_{\times 2}(G) = |D|$. □

Henceforth, $D$ means the minimum 2-tuple dominating set and $L$ is the set of leading vertices corresponding to $D$.

**Theorem 4.3.** *The 2-tuple dominating set of an interval graph can be computed sequentially in $O(n^2)$ time.*

*Proof.* Let the processor take unit time to perform a single instruction. Step 1 of Algorithm 2DIG takes $O(n)$ time. The algorithm consists of a loop from Step 2.1 to Step 2.3. This loop carry over $n$ times. Within this loop, we see that a loop occurs, which is terminated after $O(|N[i]|)$ times. It is clear that $|N[i]| \leq p \leq n$, $p$ is the upper bound of $|N[i]|$, for fixed $i$. In the worst case, we assume the loop runs over $n$ times. So the total time complexity of Step 2 is $O(n^2)$. Hence, the overall time complexity of the Algorithm 2DIG is of $O(n^2)$. □

## 5. Some Important Results Related to Minimum 2-Tuple Domination

In this section, we present some important results related to minimum 2-tuple domination on interval graphs. For a given interval graph $G$, let a tree $T(G) = (V, E')$ be defined such that $E' = \{(u, H(u)) : u \in V, u \neq n\}$, let $n$ be the root of $T(G)$. This tree is called the interval tree. The various properties of interval tree are available in [6, 10, 28].

The following lemma is true for every connected interval graph.

**Lemma 5.1** (see [28]). *For a connected interval graph, there exists a unique interval tree $T(G)$.*

For each vertex $v$ of interval tree, level$(v)$ is the distance of $v$ from the vertex $n$ in the tree. The height $h$ of the tree $T(G)$ is defined by

$$h = \max\{\text{level}(v) : v \in V\}. \tag{5.1}$$

We have found a result for the minimum 2-tuple dominating set $D$ in terms of the height $h$ of interval tree $T(G)$ stated as follows.

**Lemma 5.2.** *Let $T(G)$ be the interval tree of the interval graph G with height $h$, then*

$$|D| \geq \begin{cases} 2\left\lceil \dfrac{h}{3} \right\rceil, & \text{where } h \neq 3m \text{ for some } m \in \mathbb{N}, \\ 2\left(\dfrac{h}{3} + 1\right), & \text{where } h = 3m \text{ for some } m \in \mathbb{N}, \end{cases} \tag{5.2}$$

*where $\mathbb{N}$ is the set of natural numbers.*

*Proof.* From the definition of interval tree $T(G)$, we know that the vertex 1 of $V$ is at level $h$. By the property of interval tree $T(G)$, we know that any vertex at level $l$ is not adjacent with a vertex at level $l-2$ and level $(u) \geq$ level $(v)$, for every $u < v$, $u, v \in V$ [8]. Therefore, it is clear that the neighbors of the vertex $v$ of level $l$ are either at level $l$ or at level $l-1$.

Let $D = \{v_1, v_2, \ldots, v_k\}$ such that $v_1 < v_2 < \cdots < v_k$. As the vertices at level $h$ are not adjacent with the vertices at level $h-2$ or at level greater than $h-2$, two vertices $v_1$, $v_2$ of $D$ must be taken from the level $h$ or $h-1$. For the least possible $D$, we assume that $v_2$ is at level $h-1$ and consequently $v_3$ is either at level $h-1$ or $h-2$ or $h-3$. If $v_2$ is at level $h$, then possibility of having $v_3$ is either at $h$ or at $h-1$ or at $h-2$ which decreases the level from earlier level and hence the number of vertices of $D$ may increase. So this last case is excluded from our result as the result demands the lower bound of $D$. Also, in further cases, we neglect such cases for the same reason. Thus, we take $v_3$ at level $h-3$, $v_4$ at level $h-4$, $v_5$ at level $h-6$, $v_6$ at level $h-7$, $v_7$ at level $h-9$, and so on. That is, $v_{2k+1}$ at level $h-3k$ and $v_{2k+2}$ at level $h-3k-1$, for $k = 0, 1, 2, 3, \ldots$. So for each $k$ there are two vertices from the consecutive levels $h-3k$ and $h-3k-1$.

Now, if $h = 3m$, for some $m \in \mathbb{N}$, then $h - 3k$ is the last level, that is, level 0 of $T(G)$. So,

$$h - 3k = 0, \text{ this gives } k = \frac{h}{3}. \tag{5.3}$$

Thus, there are $((h/3) + 1)$ consecutive levels and hence the least value of $|D|$ is $2((h/3) + 1)$. If $h \neq 3m$, for some $m \in \mathbb{N}$, then $h - 3k$ is not at the last level of $T(G)$. So one vertex is required at level $h - 3k - 1$ or $h - 3k - 2$. In this case, $k = \lceil h/3 \rceil - 1$. So there are $2\lceil h/3 \rceil$ consecutive levels and hence the least value of $|D|$ is $2\lceil h/3 \rceil$.

Therefore,

$$|D| \geq \begin{cases} 2\left\lceil \dfrac{h}{3} \right\rceil, & \text{where } h \neq 3m \text{ for some } m \in \mathbb{N}, \\ 2\left(\dfrac{h}{3} + 1\right), & \text{where } h = 3m \text{ for some } m \in \mathbb{N}. \end{cases} \tag{5.4}$$

□

Here, we are going to prove a result that removal of a vertex $v$ from graph $G$, $G - v$ and $G$ have the same minimum 2-tuple dominating set $D$.

**Lemma 5.3.** *Let $v \notin L \cup D$ and $G' = (V', E')$, where $V' = V - \{v\}$ and $E' = \{(i, j) \in E : i \in V', j \in V'\}$. Then*

(i) *the minimum 2-tuple dominating set of $G'$ and $G$ is $D$, if $G'$ is connected,*

(ii) *if $G'$ is disconnected with $k$ components (blocks), say, $G_1, G_2, \ldots, G_k$, then there must exist minimum 2-tuple dominating sets $D_1, D_2, \ldots, D_k$ of $G_1, G_2, \ldots, G_k$ such that $D = D_1 \cup D_2 \cup \cdots \cup D_k$ and $D_i$'s are pairwise disjoint.*

*Proof.* (i) Suppose $G'$ is connected. Since, $v \notin L \cup D$, that is, $v \in V - L \cup D$. By Algorithm 2DIG, at $k$th iteration, say, either *if* or *else-if* condition is satisfied for a vertex $k$ of $V$, then $k \in L \cup D$, otherwise, $k \in V - L \cup D$. In this case, $v \in V - L \cup D$, at $v$-th iteration, *else* condition is satisfied for the vertex $v$ which has no effect on $L$ and $D$. Hence, if the vertex is being deleted from the graph $G$, then the new induced subgraph $G' = G - \{v\}$ has the same 2-tuple dominating set $D$ as $G$.

(ii) Let $G'$ be disconnected and $G' = G_1 \cup G_2 \cup \cdots \cup G_k$, where $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2), \ldots, G_k = (V_k, E_k)$. Let us decompose $D$ into disjoint subsets $D_1, D_2, \ldots, D_k$ such that $D_1 \subseteq V_1$, $D_2 \subseteq V_2, \ldots, D_k \subseteq V_k$, where $V = V_1 \cup V_2 \cup \cdots V_k \cup \{v\}$, that is, $D = D_1 \cup D_2 \cup \cdots \cup D_k$. As $D$ is obtained by Algorithm 2DIG and $v$ has no effect on $D$, then $v$ has no effect on $D_1, D_2, \ldots, D_k$, and they are also obtained by Algorithm 2DIG. Therefore, $D_1, D_2, \ldots, D_k$ are minimum 2-tuple dominating sets of interval graphs $G_1, G_2, \ldots, G_k$, respectively. □

The generalized form of the Lemma 5.3 is as follows.

**Corollary 5.4.** *Let $S = \{v \in V : v \notin L \cup D\}$ and $G' = (V', E')$, where $V' = V - S$ and $E' = \{(i, j) \in E : i \in V', j \in V'\}$. Then*

(i) *the minimum 2-tuple dominating set of $G'$ is also $D$, if $G'$ is connected,*

(ii) *if $G'$ is disconnected with $k$ components (blocks), say, $G_1, G_2, \ldots, G_k$, then there must exist minimum 2-tuple dominating sets $D_1, D_2, \ldots, D_k$ of $G_1, G_2, \ldots, G_k$ such that $D = D_1 \cup D_2 \cup \cdots \cup D_k$ and $D_i$'s are pairwise disjoint.*

*Proof.* (i) By Lemma 5.3, we have seen that the deletion of $v \notin L \cup D$ does not change the minimum 2-tuple dominating set $D$. Let $G^1$ be a graph obtained after the deletion of $v_1 \in S$, so $D$ is also the 2-tuple dominating set of $G^1$. Again, $v_2 \in S$ is being deleted from the graph $G^1$ and the graph $G^2$ is obtained. It also has the same 2-tuple dominating set $D$ as of $G$. Proceeding in this way, we obtain the graph $G'$ which has same 2-tuple dominating set as $G$.
(ii) The proof of this case follows from (ii) of Lemma 5.3. □

In Lemma 5.3 and Corollary 5.4, the graph $G'$ is a subgraph of the graph $G$ induced by $V'$ whose vertex set is $V'$ and edge set is the set of those edges of $G$ that have both ends in $V'$. By keeping the statement of Corollary 5.4 in mind, we define new terms 2-*tuple base graph* and *redundant vertex* as follows.

*Definition 5.5* (2-tuple base graph). Let a graph $G' = (V', E')$ be induced subgraph of the graph $G = (V, E)$, where $V' \subseteq V$, $E' \subseteq E$. The graph $G'$ is called the 2-tuple base graph of the graph $G$ if the vertex set $V' = L \cup D$ and edge set $E' = \{(i, j) \in E : i \in V', j \in V'\}$, where $L$ is the set of leading vertices corresponding to minimum 2-tuple dominating set $D$ of $G$.

*Note 1.* If $V' = V = L \cup D$, then the graph $G'$ is the same as $G$ and hence the graph $G$ is the 2-tuple base graph of the graph itself.

The 2-tuple base graph and its interval representation of the graph of Figure 1 are given in Figures 4 and 5, respectively. Note that, in case of 2-tuple base graph, $L \cup D = V'$, but in case of original graph, in general, $L \cup D \neq V$.

*Definition 5.6* (Redundant vertex). Let $G = (V, E)$ be a given interval graph. A vertex $v \in V$ is said to be redundant in $G$, if the minimum 2-tuple dominating set $D$ of $G - v$ is same as of $G$.

An important conclusion is drawn about 2-tuple base graph as follows.

**Lemma 5.7.** *Every interval graph has a unique 2-tuple base graph.*

*Proof.* Suppose there exist two distinct 2-tuple base graphs $G' = (V', E')$ and $G'' = (V'', E'')$ of the interval graph G. Then, either (i) $V' \neq V''$ or (ii) $E' \neq E''$. Let $L$ be the set of leading vertices corresponding to the minimum 2-tuple dominating set of $G$. Since $G'$ is the 2-tuple base graph of $G$, $L \cup D = V'$ and $E' = \{(i, j) \in E : i \in V', j \in V'\}$. Again, $G''$ is the 2-tuple base graph of $G$. Then, $L \cup D = V''$ and $E'' = \{(i, j) \in E : i \in V'', j \in V''\}$. So $L \cup D = V' = V''$ and hence $E' = E''$, which is a contradiction of our assumption. Therefore, $G'$ and $G''$ are same. □

Now we define a relation between two interval graphs and it is proved that the relation is an equivalence relation.

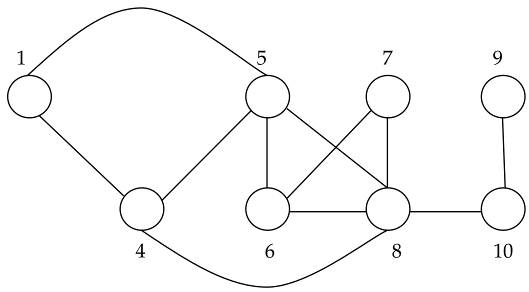


**Figure 4:** 2-tuple base graph of the graph $G$.

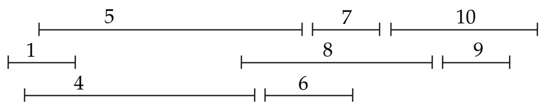


**Figure 5:** Interval representation of the 2-tuple base graph of the graph $G$.

**Lemma 5.8.** *Let $G_I$ be the set of all interval graphs. Let a relation, denoted by $\approx$, and defined by $G_1 \approx G_2 \Rightarrow G_1$ and $G_2$ have same 2-tuple base graph, for all $G_1, G_2 \in G_I$. Then, the relation $\approx$ is an equivalence relation.*

*Proof.* The relation is an equivalence relation since the following properties hold as well.

*Reflexive*

Since every graph has unique 2-tuple base graph, the same graph has the same 2-tuple base graph. Therefore, $G_1 \approx G_1$.

*Symmetric*

$G_1 \approx G_2 \Rightarrow G_2 \approx G_1$, for all $G_1, G_2 \in G_I$. Since, $G_1 \approx G_2$ means $G_1$ and $G_2$ have the same 2-tuple base graph, then $G_2$ and $G_1$ have the same 2-tuple base graph, that is, $G_2 \approx G_1$.

*Transitive*

If $G_1 \approx G_2$ and $G_2 \approx G_3$ holds for all $G_1, G_2, G_3 \in G_I$, then $G_1 \approx G_3$ holds. Let us consider $G_1$, $G_2$ have the same 2-tuple base graph $G'$ and $G_2$, $G_3$ have the same 2-tuple base graph $G''$. However we know every graph has unique 2-tuple base graph, $G_2$ cannot have distinct 2-tuple base graph $G'$ and $G''$. So $G_2$ and $G_3$ have the 2-tuple base graph $G'$ same as each of $G_1$, $G_2$. So $G_1$ and $G_3$ have the same 2-tuple base graph $G'$. So, the transitive property holds for each of $G_I$.

Since all the properties of equivalence relation hold good in $G_I$, then the relation $\approx$ defined on $G_I$ is an equivalence relation. □

*Definition 5.9* (2-tuple equivalent). An interval graph $G_1$ is said to be 2-tuple equivalent to an interval graph $G_2$ if $G_1 \approx G_2$, that is, $G_1$, $G_2$ have the same 2-tuple base graph.

*Definition 5.10* (2-tuple equivalent class). Let $G_I$ be a set of interval graphs. $G_I$ is said to be 2-tuple equivalent class if any interval graph of $G_I$ is 2-tuple equivalent to each and every graph of $G_I$.

From the above notions, we have an important result about 2-tuple equivalent class.

**Lemma 5.11.** *Equivalence relation, defined on interval graphs, makes the partition of the set of interval graphs into 2-tuple equivalent classes.*

*Proof.* This result directly follows from the abstract algebra that every equivalence relation defined on a set makes the partition of the set into equivalent classes. Hence, the result follows. Particularly, the partitions can be found by the 2-tuple base graph. That is, we are trying to say that among all interval graphs, for each 2-tuple base graph, there is a 2-tuple equivalent class. □

Next, we have an another important result regarding the leading vertex corresponding to 2-tuple dominating set $D$.

**Lemma 5.12.** *For an interval graph* $G = (V, E)$,

$$|D| = |L| + n_2, \tag{5.5}$$

*where* $n_2$ *is the number of leading vertices of order 2.*

*Proof.* Let $n_1$ be the number of leading vertices of order 1 and let $n_2$ be the number of leading vertices of order 2. By definition of leading vertex, a leading vertex of order 1 corresponds to a single vertex of $D$ and leading vertex of order 2 corresponds to two vertices of $D$. Since there are $n_1$ leading vertices of order 1, then $D$ has $n_1$ vertices and also there are $n_2$ leading vertices of order 2, so $D$ has $n_1 + 2n_2$ vertices. Therefore, $|D| = n_1 + 2n_2$. Now, $n_1 + n_2 = |L|$. Thus, $|D| = |L| + n_2$. □

## 6. Conclusion

In this paper, we have traced out to find the minimum 2-tuple dominating set on interval graphs. The algorithm we have designed in this paper can be generalized to find minimum $k$-tuple dominating set and $k$-tuple domination number. Further investigations can be done by generalizing our Algorithm 2DIG to find $k$-tuple dominating set of an interval graph. We think it will reduce the next researcher's labour.

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