Tao et al. *Advances in Difference Equations* (2015) 2015:279
DOI 10.1186/s13662-015-0621-5

Advances in Difference Equations
a SpringerOpen Journal

RESEARCH Open Access

Existence of the nonoscillatory solutions of higher order neutral dynamic equations on time scales

Chunyan Tao[1], Taixiang Sun[1*] and Hongjian Xi[2]

*Correspondence: stxhql@gxu.edu.cn
[1]Guangxi Colleges and Universities Key Laboratory of Mathematics and Its Applications, Nanning, 530004, China
Full list of author information is available at the end of the article

Abstract
In this paper, we investigate the existence of the nonoscillatory solutions of the following higher order neutral dynamic equation: $\{r_n(t)[(r_{n-1}(t)(\cdots(r_1(t)(x(t)-q(t)x(\tau(t)))^\Delta)^\Delta\cdots)^\Delta)^\Delta]^\gamma\}^\Delta + f(t,x(\delta(t))) = 0$ for $t \in [t_0,\infty)_{\mathbb{T}}$, and obtain some necessary and sufficient conditions for the existence of nonoscillatory bounded solutions for this equation.

MSC: 34K11; 39A10; 39A99

Keywords: nonoscillatory solution; dynamic equation; time scale

1 Introduction

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. Thus the real numbers $\mathbb{R}$, the integers $\mathbb{Z}$ and the natural numbers $\mathbb{N}$ are examples of time scales. On a time scale $\mathbb{T}$, the forward jump operator, the backward jump operator and the graininess function are defined

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}, \qquad \rho(t) = \sup\{s \in \mathbb{T} : s < t\} \quad \text{and} \quad \mu(t) = \sigma(t) - t,$$

respectively.

In this paper, we investigate the existence of the nonoscillatory solutions of the following higher order neutral dynamic equation:

$$\begin{aligned} &\Bigl\{r_n(t)\Bigl[\bigl(r_{n-1}(t)\bigl(\cdots\bigl(r_1(t)\bigl(x(t)-q(t)x\bigl(\tau(t)\bigr)\bigr)^\Delta\bigr)^\Delta\cdots\bigr)^\Delta\bigr)^\Delta\Bigr]^\gamma\Bigr\}^\Delta \\ &\quad + f\bigl(t,x\bigl(\delta(t)\bigr)\bigr) = 0 \quad \text{for } t \in [t_0,\infty)_{\mathbb{T}}, \end{aligned} \tag{1.1}$$

where γ is the quotient of odd positive integers, $t_0 \in \mathbb{T}$, the time scale interval $[t_0,\infty)_{\mathbb{T}} = \{t \in \mathbb{T} : t \ge t_0\}$, $r_1 \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, [L,\infty))$ for some constant $L > 0$ and $r_k \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, (0,\infty))$ $(2 \le k \le n)$, $q \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}}, \mathbb{R})$, $\tau, \delta \in C(\mathbb{T},\mathbb{T})$ with $\lim_{t\to\infty}\tau(t) = \lim_{t\to\infty}\delta(t) = \infty$ and $f \in C([t_0,\infty)_{\mathbb{T}} \times \mathbb{R}, \mathbb{R})$ satisfying the following conditions:

(i) $uf(t,u) > 0$ for any $t \in [t_0,\infty)_{\mathbb{T}}$ and $u \neq 0$;

(ii) $f(t,u)$ is nondecreasing in u for any $t \in [t_0,\infty)_{\mathbb{T}}$.

Springer

© 2015 Tao et al. This article is distributed under the terms of the Creative Commons Attribution 4.0 International License (http://creativecommons.org/licenses/by/4.0/), which permits unrestricted use, distribution, and reproduction in any medium, provided you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license, and indicate if changes were made.

In the sequel, write

$$W_k(t,x(t)) = \begin{cases} x(t) & \text{if } k = 0, \\ r_k(t)W_{k-1}^{\Delta}(t,x(t)) & \text{if } 1 \le k \le n-1, \\ r_n(t)[W_{n-1}^{\Delta}(t,x(t))]^{\gamma} & \text{if } k = n \end{cases}$$

and

$$z(t) = x(t) - q(t)x(\tau(t)). \tag{1.2}$$

Then Eq. (1.1) reduces to the equation

$$W_n^{\Delta}(t,z(t)) + f(t,x(\delta(t))) = 0. \tag{1.3}$$

We can suppose that $\sup \mathbb{T} = \infty$ since we are interested in the oscillatory behavior of solutions near infinity. We say a nontrivial real-valued function $x \in C_{\mathrm{rd}}([T_x,\infty)_{\mathbb{T}},\mathbb{R})$ $(T_x \ge t_0)$ to be a solution of Eq. (1.1) if $W_n(t,z(t)) \in C_{\mathrm{rd}}^1([T_x,\infty)_{\mathbb{T}},\mathbb{R})$ and satisfies Eq. (1.1) on $[T_x,\infty)$. The solutions vanishing in some neighborhood of infinity will be excluded from our consideration. A solution x of Eq. (1.1) is said to be oscillatory if it is neither eventually positive nor eventually negative, otherwise it is called nonoscillatory.

The theory of time scale was initiated by Hilger's landmark paper [1] in order to create a theory that can unify discrete and continuous analysis, which has received a lot of attention. There exists a variety of interesting time scales, and they give rise to many applications (see [2]). We refer the reader to [3, 4] for further results on time scale calculus. In the thousands of papers in the literature, finding sufficient conditions for all solutions of an equation to be oscillatory has been a major focus of study. Necessary and sufficient conditions for the existence of a nonoscillatory bounded solution are more rare because it is much more difficult to find necessary and sufficient conditions for a solution of higher order equations.

In a number of papers, there has been much research activity concerning the oscillation and nonoscillation of solutions of various equations on time scales even today, many similar equations on time scales can be found in [5–29].

In [21] Zhu and Wang studied the existence of nonoscillatory solutions to the neutral dynamic equation

$$[x(t) + p(t)x(g(t))]^{\Delta} + f(t,x(h(t))) = 0.$$

Karpuz *et al.* [22] studied the asymptotic behavior of delay dynamic equations having the following form:

$$[x(t) + A(t)x(\alpha(t))]^{\Delta} + B(t)F(x(\beta(t))) - C(t)G(x(\gamma(t))) = \varphi(t).$$

In [25] Wu *et al.* investigated the oscillation of the following higher order dynamic equation:

$$\{r_n(t)[(r_{n-1}(t)(\cdots(r_1(t)x(t)^{\Delta})^{\Delta}\cdots)^{\Delta})^{\Delta}]^{\gamma}\}^{\Delta} + F(t,x(\tau(t))) = 0.$$

2 Auxiliary results

We state the following conditions, which are needed in the sequel.

(H_1) $\int_{t_0}^{\infty} \frac{1}{r_k(s)}\Delta s = \int_{t_0}^{\infty} [\frac{1}{r_n(s)}]^{\frac{1}{\gamma}}\Delta s = \infty$ $(1 \le k \le n-1)$.
(H_2) There exist constants $\alpha_1, \beta_1 \in [0,1)$ with $\alpha_1 + \beta_1 < 1$ such that $-\alpha_1 \le q(t) \le \beta_1$ for all $t \in [t_0,\infty)_{\mathbb{T}}$.
(H_3) There exist constants $\alpha_2, \beta_2 \in (1,\infty)$ such that $-\alpha_2 \le q(t) \le -\beta_2$ for all $t \in [t_0,\infty)_{\mathbb{T}}$.
(H_4) There exist constants $\alpha_3, \beta_3 \in (1,\infty)$ such that $\alpha_3 \le q(t) \le \beta_3$ for all $t \in [t_0,\infty)_{\mathbb{T}}$.

Let $BC_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ be the Banach space of all bounded rd-continuous functions on $[t_0,\infty)_{\mathbb{T}}$ with sup norm $\|x\| = \sup_{t\ge t_0}|x(t)|$. Let $X \subset BC_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$, we say that X is equi-continuous on $[a,b]_{\mathbb{T}}$ if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that for any $x \in X$ and $u,v \in [a,b]_{\mathbb{T}}$ with $|u-v| < \delta$, $|x(u)-x(v)| < \varepsilon$. X is said to be uniformly Cauchy if for any given $\varepsilon > 0$, there exists $t_1 > t_0$ such that for any $x \in X$, $|x(u)-x(v)| < \varepsilon$ for all $u,v \in [t_1,\infty)_{\mathbb{T}}$. $U : X \to BC_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is called completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Lemma 2.1 [26] *Let* $m \in \mathbf{N}$. *Then*
(1) $\liminf_{t\to\infty} W_m(t,x(t)) > 0$ *implies* $\lim_{t\to\infty} W_i(t,x(t)) = \infty$ *for all* $0 \le i \le m-1$;
(2) $\limsup_{t\to\infty} W_m(t,x(t)) < 0$ *implies* $\lim_{t\to\infty} W_i(t,x(t)) = -\infty$ *for all* $0 \le i \le m-1$.

Lemma 2.2 *Let* $x(t) \in C_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},(0,\infty))$ *be bounded for* $t \in [t_0,\infty)_{\mathbb{T}}$. *If* $W_n^{\Delta}(t,x(t)) < 0$ *on* $[t_1,+\infty)_{\mathbb{T}}$ *for some* $t_1 \ge t_0$, *then*

$$(-1)^{n-k} W_k\bigl(t,x(t)\bigr) > 0 \quad \text{for } t \ge t_1, k = 0,1,2,\ldots,n \tag{2.1}$$

and

$$\lim_{t\to\infty} W_k\bigl(t,x(t)\bigr) = 0, \quad k = 0,1,2,\ldots,n. \tag{2.2}$$

Proof Since $W_n^{\Delta}(t,x(t)) < 0$, $W_n(t,x(t))$ is strictly decreasing on $[t_1,\infty)_{\mathbb{T}}$. We claim that $W_n(t,x(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. If not, there exists $t_2 \in [t_1,\infty)_{\mathbb{T}}$ such that, for all $t \ge t_2$,

$$W_n\bigl(t,x(t)\bigr) \le W_n\bigl(t_2,x(t_2)\bigr) < 0.$$

Then by Lemma 2.1 we get $\lim_{t\to\infty} x(t) = -\infty$, which contradicts the fact that $x(t)$ is bounded. Then $W_n(t,x(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$, and $W_n^{\Delta}(t,x(t)) < 0$. Let

$$\lim_{t\to\infty} W_n\bigl(t,x(t)\bigr) = L_1 \ge 0 \quad \text{and} \quad W_n\bigl(t,x(t)\bigr) \ge L_1, \quad t \in [t_1,\infty)_{\mathbb{T}}.$$

If $L_1 > 0$, then we get from Lemma 2.1 that $\lim_{t\to\infty} x(t) = \infty$, which is a contradiction to the boundedness of $x(t)$. Therefore $L_1 = 0$, *i.e.*,

$$\lim_{t\to\infty} W_n\bigl(t,x(t)\bigr) = 0. \tag{2.3}$$

Because of $W_n(t,x(t)) > 0$, we know $W_{n-1}(t,x(t))$ is strictly increasing on $[t_1,\infty)_{\mathbb{T}}$, this implies that exactly one of the following is true:

(a_1) $W_{n-1}(t,x(t)) < 0$ for $t \geq t_1$;
(b_1) there exists $t_3 \geq t_1$ such that $W_{n-1}(t,x(t)) \geq W_{n-1}(t_3,x(t_3)) > 0$ for $t \geq t_3$.

If (b_1) holds, by Lemma 2.1, we obtain

$$\lim_{t\to\infty} W_{n-2}\bigl(t,x(t)\bigr) = \lim_{t\to\infty} W_{n-3}\bigl(t,x(t)\bigr) = \cdots = \lim_{t\to\infty} x(t) = \infty,$$

which contradicts the fact that $x(t)$ is bounded. Then $W_{n-1}(t,x(t)) < 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$. Thus, we conclude

$$\lim_{t\to\infty} W_{n-1}\bigl(t,x(t)\bigr) = L_2 \leq 0 \quad \text{and} \quad W_{n-1}\bigl(t,x(t)\bigr) \leq L_2, \quad t \in [t_1,\infty)_{\mathbb{T}}.$$

If $L_2 < 0$, then by Lemma 2.1 we get $\lim_{t\to\infty} x(t) = -\infty$, which is a contradiction to the boundedness of $x(t)$. Therefore $L_2 = 0$, *i.e.*,

$$\lim_{t\to\infty} W_{n-1}\bigl(t,x(t)\bigr) = 0. \tag{2.4}$$

Since $W_{n-2}^{\Delta}(t,x(t)) = \frac{S_{n-1}(t,x(t))}{r_{n-1}(t)} < 0$, we see $W_{n-2}(t,x(t))$ is strictly decreasing, which implies that exactly one of the following is true:

(a_2) $W_{n-2}(t,x(t)) > 0$ for $t \geq t_1$;
(b_2) there exists $t_4 \geq t_1$ such that $W_{n-2}(t,x(t)) \leq W_{n-2}(t_4,x(t_4)) < 0$ for $t \geq t_4$.

If (b_2) holds, by Lemma 2.1, we obtain

$$\lim_{t\to\infty} W_{n-3}\bigl(t,x(t)\bigr) = \lim_{t\to\infty} Wn-4\bigl(t,x(t)\bigr) = \cdots = \lim_{t\to\infty} x(t) = -\infty,$$

which contradicts the fact that $x(t)$ is bounded. Then $W_{n-2}(t,x(t)) > 0$ for all $t \in [t_1,\infty)_{\mathbb{T}}$.

Therefore we can repeat the above argument and show that Lemma 2.2 holds. The proof is completed. □

Lemma 2.3 [21] *Suppose that $X \subset BC_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ is bounded and uniformly Cauchy. Further, suppose that X is equi-continuous on $[t_0,t_1]_{\mathbb{T}}$ for any $t_1 \in [t_0,\infty)_{\mathbb{T}}$. Then X is relatively compact.*

Lemma 2.4 [21] *Suppose that X is a Banach space and Ω is a bounded, convex and closed subset of X. Further, suppose that there exist two operators U and $V : \Omega \to X$ such that*
(i) *$Ux + Vy \in \Omega$ for all $x,y \in \Omega$;*
(ii) *U is a contraction mapping;*
(iii) *V is completely continuous.*
Then $U + V$ has a fixed point in Ω.

3 Main results

Now, we state and prove our main results.

Theorem 3.1 *Assume that* (H_1) *and* (H_2) *hold. Then Eq.* (1.1) *has a nonoscillatory bounded solution $x(t)$ with $\liminf_{t\to\infty} |x(t)| > 0$ if and only if there exists some constant $M \neq 0$ such* *that*

$$\int_{t_0}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\int_{u_{n-1}}^{\infty}A(u_n)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1 < \infty, \tag{3.1}$$

where $A(u_n)=[\frac{1}{r_n(u_n)}\int_{u_n}^{\infty}f(s,|M|)\Delta s]^{\frac{1}{\gamma}}$.

Proof Necessity. Assume that Eq. (1.1) has a nonoscillatory bounded solution $x(t)$ on $[t_0,\infty)_{\mathbb{T}}$ with $\liminf_{t\to\infty}|x(t)|>0$. Without loss of generality, we assume that there is a constant $M>0$ and some $t_1\geq t_0$ such that $x(t)>M$ and $x(\delta(t))>M$ for $t\geq t_1$. By assumption that $x(t)$ is bounded and condition (H_2), we see that $z(t)$ is bounded and $W_n^{\Delta}(t,z(t))=-f(t,x(\delta(t)))<0$. Thus, by Lemma 2.2 we know that there exists $t_2\geq t_1$ such that

$$(-1)^{n-k}W_k\bigl(t,z(t)\bigr)>0 \quad \text{for } t\geq t_2 \text{ and } 0\leq k\leq n. \tag{3.2}$$

Integrating Eq. (1.3) from t ($\geq t_2$) to ∞, we have

$$\int_t^{\infty}W_n^{\Delta}\bigl(s,z(s)\bigr)\Delta s=-\int_t^{\infty}f\bigl(s,x\bigl(\delta(s)\bigr)\bigr)\Delta s.$$

That is,

$$\begin{aligned}r_n(t)\bigl[W_{n-1}^{\Delta}\bigl(t,z(t)\bigr)\bigr]^{\gamma} &= W_n\bigl(t,z(t)\bigr)=\int_t^{\infty}f\bigl(s,x\bigl(\delta(s)\bigr)\bigr)\Delta s\\ &\geq\int_t^{\infty}f(s,M)\Delta s \quad (t\geq t_2).\end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have that for $t\geq t_2$,

$$\begin{aligned}&\int_t^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\int_{u_{n-1}}^{\infty}A(u_n)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1\\ &\quad\leq\int_t^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\\ &\qquad\times\int_{u_{n-1}}^{\infty}W_{n-1}^{\Delta}\bigl(u_n,z(u_n)\bigr)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1\\ &\quad=\int_t^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}(-1)^{n-(n-1)}\\ &\qquad\times W_{n-1}\bigl(u_{n-1},z(u_{n-1})\bigr)\Delta u_{n-1}\cdots\Delta u_2\Delta u_1\\ &\quad=\int_t^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}(-1)^{n-(n-1)}W_{n-2}^{\Delta}\bigl(u_{n-1},z(u_{n-1})\bigr)\Delta u_{n-1}\cdots\Delta u_2\Delta u_1\\ &\quad=\int_t^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-3}}^{\infty}\frac{1}{r_{n-2}(u_{n-2})}(-1)^{n-(n-2)}\\ &\qquad\times W_{n-2}\bigl(u_{n-2},z(u_{n-2})\bigr)\Delta u_{n-2}\cdots\Delta u_2\Delta u_1\\ &\qquad\cdots\end{aligned}$$

$$
\begin{aligned}
&= \int_t^\infty \frac{1}{r_1(u_1)}(-1)^{n-1} W_1\bigl(u_1, z(u_1)\bigr)\Delta u_1 \\
&= \int_t^\infty (-1)^{n-1} z^\Delta(u_1)\Delta u_1 \\
&= (-1)^{n-1} z(u_1)\big|_t^\infty < \infty.
\end{aligned}
$$

By using the boundedness of $z(t)$, we see that (3.1) holds.

Sufficiency. Suppose that there exists some constant $M > 0$ such that

$$\int_{t_0}^\infty \frac{1}{r_1(u_1)} \int_{u_1}^\infty \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^\infty \frac{1}{r_{n-1}(u_{n-1})} \int_{u_{n-1}}^\infty A(u_n)\Delta u_n \Delta u_{n-1} \cdots \Delta u_2 \Delta u_1 < \infty.$$

Then we may choose $t_1 \geq t_0$ such that

$$
\begin{aligned}
&\int_t^\infty \frac{1}{r_1(u_1)} \int_{u_1}^\infty \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^\infty \frac{1}{r_{n-1}(u_{n-1})} \int_{u_{n-1}}^\infty A(u_n)\Delta u_n \cdots \Delta u_2 \Delta u_1 \\
&\quad < \frac{(1-\beta_1)(1-\beta_1-\alpha_1)}{2} M
\end{aligned}
$$

and $\min\{\delta(t), \tau(t)\} \geq t_0$ for $t \geq t_1$. Let

$$\Omega = \bigl\{x \in BC_{\mathrm{rd}}\bigl([t_0, \infty)_{\mathbb{T}}, \mathbb{R}\bigr) : \beta_1(1-\beta_1-\alpha_1)M \leq x(t) \leq M \text{ for } t \geq t_0\bigr\}.$$

It is easy to verify that Ω is a bounded, convex and closed subset of $BC_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$.

Now we define two operators U and $V : \Omega \to BC_{\mathrm{rd}}([t_0, \infty)_{\mathbb{T}}, \mathbb{R})$ as follows:

$$(Ux)(t) = q\bigl(t^*\bigr)x\bigl(\tau\bigl(t^*\bigr)\bigr)$$

and

$$
\begin{aligned}
(Vx)(t) = {} & \frac{(1-\beta_1)(1+\alpha_1+\beta_1)}{2} M \\
& + (-1)^n \int_{t^*}^\infty \frac{1}{r_1(u_1)} \int_{u_1}^\infty \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^\infty \frac{1}{r_{n-1}(u_{n-1})} \\
& \times \int_{u_{n-1}}^\infty A(u_n, x)\Delta u_n \Delta u_{n-1} \cdots \Delta u_2 \Delta u_1,
\end{aligned}
$$

where $t^* = \max\{t, t_1\}$, $A(u_n, x) = [\frac{1}{r_n(u_n)} \int_{u_n}^\infty f(s, x(\delta(s)))\Delta s]^{\frac{1}{\gamma}}$. Now we show that U and V satisfy the conditions in Lemma 2.4.

(1) We will prove that $Ux + Vy \in \Omega$ for any $x, y \in \Omega$. In fact, for any $x, y \in \Omega$ and $t \geq t_0$, $x(t), y(t) \in [\beta_1(1-\beta_1-\alpha_1)M, M]$ and

$$
\begin{aligned}
(Ux)(t) + (Vy)(t) = {} & \frac{(1-\beta_1)(1+\alpha_1+\beta_1)}{2} M + q\bigl(t^*\bigr)x\bigl(\tau\bigl(t^*\bigr)\bigr) \\
& + (-1)^n \int_{t^*}^\infty \frac{1}{r_1(u_1)} \int_{u_1}^\infty \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^\infty \frac{1}{r_{n-1}(u_{n-1})} \\
& \times \int_{u_{n-1}}^\infty A(u_n, x)\Delta u_n \Delta u_{n-1} \cdots \Delta u_2 \Delta u_1
\end{aligned}
$$

$$\begin{aligned}
&\leq \frac{(1-\beta_1)(1+\alpha_1+\beta_1)}{2}M + \beta_1 M + \frac{(1-\beta_1)(1-\beta_1-\alpha_1)}{2}M \\
&= M
\end{aligned}$$

and

$$\begin{aligned}
(Ux)(t) + (Vy)(t) &= \frac{(1-\beta_1)(1+\alpha_1+\beta_1)}{2}M + q(t^*)x(\tau(t^*)) \\
&\quad + (-1)^n \int_{t^*}^{\infty} \frac{1}{r_1(u_1)} \int_{u_1}^{\infty} \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^{\infty} \frac{1}{r_{n-1}(u_{n-1})} \\
&\quad \times \int_{u_{n-1}}^{\infty} A(u_n, x)\Delta u_n \Delta u_{n-1} \cdots \Delta u_2 \Delta u_1 \\
&\geq \frac{(1-\beta_1)(1+\alpha_1+\beta_1)}{2}M - \alpha_1 M - \frac{(1-\beta_1)(1-\beta_1-\alpha_1)}{2}M \\
&= \beta_1(1-\beta_1-\alpha_1)M,
\end{aligned}$$

which implies that $Ux + Vy \in \Omega$ for any $x, y \in \Omega$.

(2) We will show that U is a contraction mapping. Indeed, for any $x, y \in \Omega$ and $t \geq t_0$, we have

$$\begin{aligned}
\left|(Ux)(t) - (Uy)(t)\right| &= \left|q(t^*)x(\tau(t^*)) - q(t^*)y(\tau(t^*))\right| \\
&\leq \max\{\alpha_1, \beta_1\}\|x-y\|.
\end{aligned}$$

Therefore, we conclude

$$\|Ux - Uy\| \leq \max\{\alpha_1, \beta_1\}\|x-y\|,$$

then U is a contraction mapping.

(3) We will show that V is a completely continuous mapping.

(i) By the proof of (1), we see that $\beta_1(1-\beta_1-\alpha_1)M \leq (Vx)(t) \leq M$ for $t \in [t_0, \infty)_{\mathbb{T}}$. That is, $V\Omega \subset \Omega$.

(ii) We consider the continuity of V. Let $x_n \in \Omega$ and $\|x_n - x\| \to 0$ as $n \to \infty$, then $x \in \Omega$ and $|x_n(t) - x(t)| \to 0$ as $n \to \infty$ for any $t \in [t_0, \infty)_{\mathbb{T}}$. Consequently, for any $s \in [t_1, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned}
\lim_{n\to\infty} \Bigg| &\frac{1}{r_1(u_1)} \int_{u_1}^{\infty} \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^{\infty} \frac{1}{r_{n-1}(u_{n-1})} \\
&\times \int_{u_{n-1}}^{\infty} \left[A(u_n, x_n) - A(u_n, x)\right]\Delta u_n \Delta u_{n-1} \cdots \Delta u_2 \Bigg| = 0.
\end{aligned}$$

Since

$$\begin{aligned}
&\left|\frac{1}{r_1(u_1)} \int_{u_1}^{\infty} \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^{\infty} \frac{1}{r_{n-1}(u_{n-1})} \int_{u_{n-1}}^{\infty} \left[A(u_n, x_n) - A(u_n, x)\right]\Delta u_n \Delta u_{n-1} \cdots \Delta u_2\right| \\
&\quad \leq 2\frac{1}{r_1(u_1)} \int_{u_1}^{\infty} \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^{\infty} \frac{1}{r_{n-1}(u_{n-1})} \int_{u_{n-1}}^{\infty} A(u_n)\Delta u_n \Delta u_{n-1} \cdots \Delta u_2
\end{aligned}$$

and for any $t \in [t_0, \infty)_{\mathbb{T}}$,

$$\begin{aligned}&\left|(Vx_n)(t) - (Vx)(t)\right| \\ &\quad\le \int_{t_1}^{\infty} \frac{1}{r_1(u_1)} \int_{u_1}^{\infty} \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^{\infty} \frac{1}{r_{n-1}(u_{n-1})} \\ &\qquad\times \int_{u_{n-1}}^{\infty} \left|A(u_n, x_n) - A(u_n, x)\right| \Delta u_n \Delta u_{n-1} \cdots \Delta u_2 \Delta u_1,\end{aligned}$$

we have

$$\begin{aligned}&\| Vx_n - Vx\| \\ &\quad\le \int_{t_1}^{\infty} \frac{1}{r_1(u_1)} \int_{u_1}^{\infty} \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^{\infty} \frac{1}{r_{n-1}(u_{n-1})} \\ &\qquad\times \int_{u_{n-1}}^{\infty} \left|A(u_n, x_n) - A(u_n, x)\right| \Delta u_n \Delta u_{n-1} \cdots \Delta u_2 \Delta u_1.\end{aligned}$$

By Chapter 5 in [4], we see that the Lebesgue dominated convergence theorem satisfies the integral on time scales. Then

$$\lim_{n\to\infty} \| Vx_n - Vx\| = 0,$$

which implies that V is continuous on Ω.

(iii) We show that $V\Omega$ is uniformly Cauchy. In fact, for any $\varepsilon > 0$, let us take $t_2 > t_1$ so that

$$\int_{t_2}^{\infty} \frac{1}{r_1(u_1)} \int_{u_1}^{\infty} \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^{\infty} \frac{1}{r_{n-1}(u_{n-1})} \int_{u_{n-1}}^{\infty} A(u_n) \Delta u_n \Delta u_{n-1} \cdots \Delta u_2 \Delta u_1 < \varepsilon.$$

Then, for any $x \in \Omega$ and $u, v \in [t_2, \infty)_{\mathbb{T}}$, we have

$$\left|(Vx)(u) - (Vx)(v)\right| < 2\varepsilon,$$

which implies that $V\Omega$ is uniformly Cauchy.

(iv) We show that $V\Omega$ is equi-continuous on $[t_0, t_2]_{\mathbb{T}}$ for any $t_2 \in [t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we assume $t_2 \ge t_1$. Note $L \le r_1(t)$. For any $\varepsilon > 0$, choose $\delta = L\varepsilon/(1 + \int_{t_0}^{\infty} \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^{\infty} \frac{1}{r_{n-1}(u_{n-1})} \int_{u_{n-1}}^{\infty} A(u_n) \Delta u_n \Delta u_{n-1} \cdots \Delta u_2)$, then when $u, v \in [t_0, t_2]$ with $|u - v| < \delta$, for any $x \in \Omega$, we have

$$\begin{aligned}&\left|(Vx)(u) - (Vx)(v)\right| \\ &\quad= \biggl|\int_{u^*}^{\infty} \frac{1}{r_1(u_1)} \int_{u_1}^{\infty} \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^{\infty} \frac{1}{r_{n-1}(u_{n-1})} \int_{u_{n-1}}^{\infty} A(u_n) \Delta u_n \Delta u_{n-1} \cdots \Delta u_2 \Delta u_1 \\ &\qquad- \int_{v^*}^{\infty} \frac{1}{r_1(u_1)} \int_{u_1}^{\infty} \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^{\infty} \frac{1}{r_{n-1}(u_{n-1})} \int_{u_{n-1}}^{\infty} A(u_n) \Delta u_n \Delta u_{n-1} \cdots \Delta u_2 \Delta u_1\biggr| \\ &\quad= \biggl|\int_{u^*}^{v^*} \frac{1}{r_1(u_1)} \int_{u_1}^{\infty} \frac{1}{r_2(u_2)} \cdots \int_{u_{n-2}}^{\infty} \frac{1}{r_{n-1}(u_{n-1})} \int_{u_{n-1}}^{\infty} A(u_n) \Delta u_n \Delta u_{n-1} \cdots \Delta u_2 \Delta u_1\biggr|\end{aligned}$$

$$\begin{aligned}&\leq \left|\frac{1}{L}\int_{u^*}^{v^*}\int_{t_0}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\int_{u_{n-1}}^{\infty}A(u_n)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1\right|\\&\leq \frac{1}{L}\left|u^*-v^*\right|\int_{t_0}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\int_{u_{n-1}}^{\infty}A(u_n)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\\&< \varepsilon,\end{aligned}$$

which implies that $V\Omega$ is equi-continuous on $[t_0,t_2]_{\mathbb{T}}$ for any $t_2\in[t_0,\infty)_{\mathbb{T}}$.

We see from Lemma 2.3 that V is a completely continuous mapping. By Lemma 2.4 it follows that there exists $x\in\Omega$ such that $(U+V)x=x$, which is the desired bounded solution of Eq. (1.1) with $\liminf_{t\to\infty}|x(t)|>0$. The proof is completed. □

Theorem 3.2 *Assume that* (H_1) *and* (H_3) *hold, and that* τ *has the inverse* $\tau^{-1}\in C(\mathbb{T},\mathbb{T})$. *Then Eq.* (1.1) *has a nonoscillatory bounded solution* $x(t)$ *with* $\liminf_{t\to\infty}|x(t)|>0$ *if and only if there exists some constant* $M\neq 0$ *such that* (3.1) *holds.*

Proof The proof of necessity is similar to that of Theorem 3.1.

Sufficiency. Suppose that there exists some constant $M>0$ such that

$$\int_{t_0}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\int_{u_{n-1}}^{\infty}A(u_n)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1<\infty.$$

Then we may choose $t_1\geq t_0$ such that

$$\begin{aligned}&\int_{t}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\int_{u_{n-1}}^{\infty}A(u_n)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1\\&\quad<\frac{\beta_2(\beta_2-1)}{2(\beta_2+1)}M\end{aligned}$$

and $\min\{\delta(\tau^{-1}(t)),\tau^{-1}(t)\}\geq t_0$ for $t\geq t_1$. Let

$$\Omega=\left\{x\in BC_{\mathrm{rd}}\big([t_0,\infty)_{\mathbb{T}},\mathbb{R}\big):\frac{(\beta_2-1)}{\alpha_2(\beta_2+1)}M\leq x(t)\leq M\text{ for }t\geq t_0\right\}.$$

It is easy to verify that Ω is a bounded, convex and closed subset of $BC_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$.

Now we define two operators U and $V:\Omega\to BC_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ as follows:

$$(Ux)(t)=\frac{x(\tau^{-1}(t^*))}{q(\tau^{-1}(t^*))}+\frac{(\beta_2+3)\beta_2M}{-2(\beta_2+1)q(\tau^{-1}(t^*))}$$

and

$$\begin{aligned}(Vx)(t)={}&\frac{1}{q(\tau^{-1}(t^*))}(-1)^n\int_{\tau^{-1}(t^*)}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\\&\times\int_{u_{n-1}}^{\infty}A(u_n,x)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1,\end{aligned}$$

where $t^*=\max\{t,t_1\}$ for any $t\in[t_0,\infty)_{\mathbb{T}}$. In order to prove the theorem, we will show that U and V satisfy the conditions in Lemma 2.4.

We first show that $Ux + Vy \in \Omega$ for any $x, y \in \Omega$. In fact, for any $x, y \in \Omega$ and $t \geq t_0$, $x(t), y(t) \in [\frac{\beta_2 - 1}{\alpha_2(\beta_2+1)}M, M]$ and

$$\begin{aligned}(Ux)(t) + (Vy)(t) &= \frac{x(\tau^{-1}(t^*))}{q(\tau^{-1}(t^*))} + \frac{1}{-q(\tau^{-1}(t^*))}\bigg[\frac{(\beta_2+3)\beta_2}{2(\beta_2+1)}M \\ &\quad + (-1)^{n+1}\int_{\tau^{-1}(t^*)}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})} \\ &\quad \times \int_{u_{n-1}}^{\infty}A(u_n, x)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1\bigg] \\ &\leq \frac{1}{\beta_2}\bigg[\frac{(\beta_2+3)\beta_2}{2(\beta_2+1)}M + \frac{\beta_2(\beta_2-1)}{2(\beta_2+1)}M\bigg] \\ &= M\end{aligned}$$

and

$$\begin{aligned}(Ux)(t) + (Vy)(t) &= \frac{x(\tau^{-1}(t^*))}{q(\tau^{-1}(t^*))} + \frac{1}{-q(\tau^{-1}(t^*))}\bigg[\frac{(\beta_2+3)\beta_2}{2(\beta_2+1)}M \\ &\quad + (-1)^{n+1}\int_{\tau^{-1}(t^*)}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})} \\ &\quad \times \int_{u_{n-1}}^{\infty}A(u_n, x)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1\bigg] \\ &\geq \frac{1}{\alpha_2}\bigg[\frac{(\beta_2+3)\beta_2}{2(\beta_2+1)}M - M - \frac{\beta_2(\beta_2-1)}{2(\beta_2+1)}M\bigg] \\ &= \frac{\beta_2 - 1}{\alpha_2(\beta_2+1)}M,\end{aligned}$$

and $|(Vx)(t)| \leq \frac{\beta_2(\beta_2-1)}{2(\beta_2+1)}M$, which means that $Ux + Vy \in \Omega$ for any $x, y \in \Omega$ and $V\Omega$ is uniformly bounded.

Now we show that $V\Omega$ is equi-continuous on $[t_0, t_2]_{\mathbb{T}}$ for any $t_2 \in [t_0, \infty)_{\mathbb{T}}$. Without loss of generality, we assume $t_2 \geq t_1$. Since $1/q(\tau^{-1}(t))$, $\tau^{-1}(t)$ are continuous on $[t_0, t_2]_{\mathbb{T}}$, so they are uniformly continuous on $[t_0, t_2]_{\mathbb{T}}$. For any $\varepsilon > 0$, choose $\delta > 0$ such that when $u, v \in [t_0, t_2]_{\mathbb{T}}$ with $|u - v| < \delta$, we have

$$\left|\frac{1}{q(\tau^{-1}(u))} - \frac{1}{q(\tau^{-1}(v))}\right| < \frac{\varepsilon}{1 + A}$$

and

$$\left|\tau^{-1}(u) - \tau^{-1}(v)\right| < \frac{L\varepsilon}{1 + B},$$

where

$$\begin{aligned}A &= \int_{t_0}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\int_{u_{n-1}}^{\infty}A(u_n, x)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1, \\ B &= \int_{t_0}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\int_{u_{n-1}}^{\infty}A(u_n, x)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2.\end{aligned}$$

Then, we obtain that for any $x \in \Omega$,

$$\begin{aligned}
&\bigl|(Vx)(u)-(Vx)(v)\bigr| \\
&= \biggl|\frac{1}{q(\tau^{-1}(u^*))}\int_{\tau^{-1}(u^*)}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-1}}^{\infty}A(u_n,x)\Delta u_n\cdots\Delta u_2\Delta u_1 \\
&\quad -\frac{1}{q(\tau^{-1}(v^*))}\int_{\tau^{-1}(v^*)}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-1}}^{\infty}A(u_n,x)\Delta u_n\cdots\Delta u_2\Delta u_1\biggr| \\
&\le \biggl|\biggl[\frac{1}{q(\tau^{-1}(u^*))}-\frac{1}{q(\tau^{-1}(v^*))}\biggr] \\
&\quad \times\int_{\tau^{-1}(u^*)}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-1}}^{\infty}A(u_n,x)\Delta u_n\cdots\Delta u_2\Delta u_1\biggr| \\
&\quad +\biggl|\frac{1}{q(\tau^{-1}(v^*))}\int_{\tau^{-1}(u^*)}^{\tau^{-1}(v^*)}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-1}}^{\infty}A(u_n,x)\Delta u_n\cdots\Delta u_2\Delta u_1\biggr| \\
&\le \biggl|\biggl[\frac{1}{q(\tau^{-1}(u^*))}-\frac{1}{q(\tau^{-1}(v^*))}\biggr] \\
&\quad \times\int_{t_0}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-1}}^{\infty}A(u_n,x)\Delta u_n\cdots\Delta u_2\Delta u_1\biggr| \\
&\quad +\biggl|\frac{1}{q(\tau^{-1}(v^*))}\frac{1}{L}\int_{\tau^{-1}(u^*)}^{\tau^{-1}(v^*)}\int_{t_0}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-1}}^{\infty}A(u_n,x)\Delta u_n\cdots\Delta u_2\Delta u_1\biggr| \\
&\le \varepsilon+\frac{1}{L}\bigl|\tau^{-1}\bigl(u^*\bigr)-\tau^{-1}\bigl(v^*\bigr)\bigr|\int_{t_0}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-1}}^{\infty}A(u_n,x)\Delta u_n\cdots\Delta u_2 \\
&< 2\varepsilon,
\end{aligned}$$

which implies that $V\Omega$ is equi-continuous on $[t_0,t_2]_{\mathbb{T}}$ for any $t_2 \in [t_0,\infty)_{\mathbb{T}}$. The rest of the proof is similar to that of Theorem 3.1. The proof is completed. □

Theorem 3.3 *Assume that* (H_1) *and* (H_4) *hold, and that* τ *has the inverse* $\tau^{-1} \in C(\mathbb{T},\mathbb{T})$. *Then Eq.* (1.1) *has a nonoscillatory bounded solution* $x(t)$ *with* $\liminf_{t\to\infty}|x(t)| > 0$ *if and only if there exists some constant* $M \ne 0$ *such that* (3.1) *holds.*

Proof The proof of necessity is similar to that of Theorem 3.1.

Sufficiency. Suppose that there exists some constant $M > 0$ such that

$$\int_{t_0}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\int_{u_{n-1}}^{\infty}A(u_n)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1 < \infty.$$

Then we may choose $t_1 \ge t_0$ such that

$$\int_{\tau^{-1}(t)}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\int_{u_{n-1}}^{\infty}A(u_n)\Delta u_n\cdots\Delta u_2\Delta u_1 < \frac{\alpha_3-1}{3}M,$$

and $\min\{\delta(\tau^{-1}(t)),\tau^{-1}(t)\} \ge t_0$ for $t \ge t_1$. Let

$$\Omega=\biggl\{x\in BC_{\mathrm{rd}}\bigl([t_0,\infty)_{\mathbb{T}},\mathbb{R}\bigr):\frac{\alpha_3-1}{3(\beta_3-1)}M\le x(t)\le M \text{ for } t\ge t_0\biggr\}.$$

It is easy to verify that Ω is a bounded, convex and closed subset of $BC_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$.

Now we define two operators U and $V:\Omega\to BC_{\mathrm{rd}}([t_0,\infty)_{\mathbb{T}},\mathbb{R})$ as follows:

$$(Ux)(t)=\frac{x(\tau^{-1}(t^*))}{q(\tau^{-1}(t^*))}+\frac{2(\alpha_3-1)M}{3q(\tau^{-1}(t^*))}$$

and

$$\begin{aligned}(Vx)(t)={}&\frac{1}{q(\tau^{-1}(t^*))}(-1)^n\int_{\tau^{-1}(t^*)}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\\&\times\int_{u_{n-1}}^{\infty}A(u_n,x)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1,\end{aligned}$$

where $t^*=\max\{t,t_1\}$ for any $t\in[t_0,\infty)_{\mathbb{T}}$. Now we show that U and V satisfy the conditions in Lemma 2.4.

We will show that $Ux+Vy\in\Omega$ for any $x,y\in\Omega$. In fact, for any $x,y\in\Omega$ and $t\geq t_0$, $x(t),y(t)\in[(\alpha_3-1)M/3(\beta_3-1),M]$ and

$$\begin{aligned}(Ux)(t)+(Vy)(t)={}&\frac{x(\tau^{-1}(t^*))}{q(\tau^{-1}(t^*))}+\frac{1}{q(\tau^{-1}(t^*))}\biggl[\frac{2\alpha_3}{3}M\\&+(-1)^{n+1}\int_{\tau^{-1}(t^*)}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\\&\times\int_{u_{n-1}}^{\infty}A(u_n,x)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1\biggr]\\\leq{}&\frac{1}{\alpha_3}\biggl[\frac{2(\alpha_3-1)}{3}M+M+\frac{\alpha_3-1}{3}M\biggr]\\={}&M\end{aligned}$$

and

$$\begin{aligned}(Ux)(t)+(Vy)(t)={}&\frac{x(\tau^{-1}(t^*))}{q(\tau^{-1}(t^*))}+\frac{1}{q(\tau^{-1}(t^*))}\biggl[\frac{2(\alpha_3-1)}{3}M\\&+(-1)^{n+1}\int_{\tau^{-1}(t^*)}^{\infty}\frac{1}{r_1(u_1)}\int_{u_1}^{\infty}\frac{1}{r_2(u_2)}\cdots\int_{u_{n-2}}^{\infty}\frac{1}{r_{n-1}(u_{n-1})}\\&\times\int_{u_{n-1}}^{\infty}A(u_n,x)\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1\biggr]\\\geq{}&\frac{1}{\beta_3}\biggl[\frac{2(\alpha_3-1)}{3}M+\frac{\alpha_3-1}{3(\beta_3-1)}M-\frac{\alpha_3-1}{3}M\biggr]\\={}&\frac{\alpha_3-1}{3(\beta_3-1)}M,\end{aligned}$$

and $|(Vx)(t)|\leq(\alpha_3-1)M/3$, which implies that $Ux+Vy\in\Omega$ for any $x,y\in\Omega$ and $V\Omega$ is uniformly bounded. The rest of the proof is similar to that of Theorem 3.2. The proof is completed. □

4 Example

In this section, we give an example to illustrate our main results.

Lemma 4.1 [23, 24] *Assume that* $s,t\in\mathbb{T}$ *and* $g\in C_{\mathrm{rd}}(\mathbb{T}\times\mathbb{T},\mathbb{R})$, *then*

$$\int_s^t\left[\int_\eta^t g(\eta,\zeta)\Delta\zeta\right]\Delta\eta=\int_s^t\left[\int_s^{\sigma(\zeta)} g(\eta,\zeta)\Delta\eta\right]\Delta\zeta.$$

Example 4.1 Let $\mathbb{T}=\{q^n:n\in\mathbb{Z}\}\cup\{0\}$ with $q>1$. Consider the following higher order dynamic equation:

$$\begin{aligned}&\left\{t^\gamma\left[\left(t\left(\cdots\left(t\left(x(t)-q_k(t)x(qt)\right)^\Delta\right)^\Delta\cdots\right)^\Delta\right)^\Delta\right]^\gamma\right\}^\Delta\\&\quad+\frac{q^{n\gamma+1}-1}{(q-1)q^{n\gamma+1}t^{n\gamma+2}}x^{2r+1}\left(q^{3+r}t\right)=0,\end{aligned}\tag{4.1}$$

where $t\in[q,\infty)_\mathbb{T}$, γ is the quotient of odd positive integers and r is a positive integer, $q_k(t)=-2[(-1)^kk^2+(-1)^{\log_q t}]/5$ ($k\in\{1,2,3\}$), $r_n(t)=t^\gamma$, $r_k(t)=t$ ($1\le k\le n-1$), $\tau(t)=qt$, $\delta(t)=q^{3+r}t$ and $f(t,u)=\frac{q^{n\gamma+1}-1}{(q-1)q^{n\gamma+1}t^{n\gamma+2}}u^{2r+1}$.

It is easy to verify that $q_k(t)$ satisfies the condition (H_{k+1}). On the other hand, we have

$$\begin{aligned}&\int_q^\infty\left[\frac{1}{r_n(t)}\right]^{\frac{1}{\gamma}}\Delta t=\int_q^\infty\left[\frac{1}{t^\gamma}\right]^{\frac{1}{\gamma}}\Delta t=\infty,\\&\int_q^\infty\frac{1}{r_k(t)}\Delta t=\int_q^\infty\frac{1}{t}\Delta t=\infty,\quad 1\le k\le n-1.\end{aligned}$$

If $s>t\ge q$ and $s=q^nt$, then we have the following inequality:

$$\begin{aligned}&\int_t^{\sigma(s)}\frac{1}{\tau}\Delta\tau\\&\quad=\int_t^{qt}\frac{1}{\tau}\Delta\tau+\int_{qt}^{q^2t}\frac{1}{\tau}\Delta\tau+\cdots+\int_{q^nt}^{q^{n+1}t}\frac{1}{\tau}\Delta\tau\\&\quad=\frac{qt-t}{t}+\frac{q^2t-qt}{qt}+\cdots+\frac{q^{n+1}t-q^nt}{q^nt}\\&\quad=(n+1)(q-1)\\&\quad\le q^{n+1}\\&\quad\le q^nt\\&\quad=s.\end{aligned}$$

Combining Lemma 4.1 with the above inequality, we see that for any $M>0$,

$$\begin{aligned}&\int_q^\infty\frac{1}{u_1}\int_{u_1}^\infty\frac{1}{u_2}\cdots\int_{u_{n-2}}^\infty\frac{1}{u_{n-1}}\int_{u_{n-1}}^\infty\left[\frac{1}{u_n^\gamma}\int_{u_n}^\infty f(s,M)\Delta s\right]^{\frac{1}{\gamma}}\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1\\&\quad=M^{\frac{2r+1}{\gamma}}\int_q^\infty\frac{1}{u_1}\int_{u_1}^\infty\frac{1}{u_2}\cdots\int_{u_{n-2}}^\infty\frac{1}{u_{n-1}}\\&\qquad\times\int_{u_{n-1}}^\infty\left[\frac{1}{u_n^\gamma}\int_{u_n}^\infty\left(\frac{-1}{s^{n\gamma+1}}\right)^\Delta\Delta s\right]^{\frac{1}{\gamma}}\Delta u_n\Delta u_{n-1}\cdots\Delta u_2\Delta u_1\end{aligned}$$

$$
\begin{aligned}
&= M^{\frac{2r+1}{\gamma}} \int_q^\infty \frac{1}{u_1} \int_{u_1}^\infty \frac{1}{u_2} \cdots \int_{u_{n-2}}^\infty \frac{1}{u_{n-1}} \int_{u_{n-1}}^\infty \frac{1}{u_n^{n+1+\frac{1}{\gamma}}} \Delta u_n \Delta u_{n-1} \cdots \Delta u_2 \Delta u_1 \\
&= M^{\frac{2r+1}{\gamma}} \int_q^\infty \frac{1}{u_1} \int_{u_1}^\infty \frac{1}{u_2} \cdots \int_{u_{n-3}}^\infty \frac{1}{u_{n-2}} \\
&\quad \times \left[\int_{u_{n-2}}^\infty \int_{u_{n-1}}^\infty \frac{1}{u_{n-1}} \frac{1}{u_n^{n+1+\frac{1}{\gamma}}} \Delta u_n \Delta u_{n-1} \right] \Delta u_{n-2} \cdots \Delta u_2 \Delta u_1 \\
&= M^{\frac{2r+1}{\gamma}} \int_q^\infty \frac{1}{u_1} \int_{u_1}^\infty \frac{1}{u_2} \cdots \int_{u_{n-3}}^\infty \frac{1}{u_{n-2}} \\
&\quad \times \left[\int_{u_{n-2}}^\infty \frac{1}{u_n^{n+1+\frac{1}{\gamma}}} \int_{u_{n-2}}^{\sigma(u_n)} \frac{1}{u_{n-1}} \Delta u_{n-1} \Delta u_n \right] \Delta u_{n-2} \cdots \Delta u_2 \Delta u_1 \\
&\le M^{\frac{2r+1}{\gamma}} \int_q^\infty \frac{1}{u_1} \int_{u_1}^\infty \frac{1}{u_2} \cdots \int_{u_{n-3}}^\infty \frac{1}{u_{n-2}} \left[\int_{u_{n-2}}^\infty \frac{1}{u_n^{n+\frac{1}{\gamma}}} \Delta u_n \right] \Delta u_{n-2} \cdots \Delta u_2 \Delta u_1 \\
&= M^{\frac{2r+1}{\gamma}} \int_q^\infty \frac{1}{u_1} \int_{u_1}^\infty \frac{1}{u_2} \cdots \left[\int_{u_{n-3}}^\infty \int_{u_{n-2}}^\infty \frac{1}{u_{n-2}} \frac{1}{u_n^{n+\frac{1}{\gamma}}} \Delta u_n \Delta u_{n-2} \right] \cdots \Delta u_2 \Delta u_1 \\
&= M^{\frac{2r+1}{\gamma}} \int_q^\infty \frac{1}{u_1} \int_{u_1}^\infty \frac{1}{u_2} \cdots \left[\int_{u_{n-3}}^\infty \frac{1}{u_n^{n+\frac{1}{\gamma}}} \int_{u_{n-3}}^{\sigma(u_n)} \frac{1}{u_{n-2}} \Delta u_{n-2} \Delta u_n \right] \cdots \Delta u_2 \Delta u_1 \\
&\le M^{\frac{2r+1}{\gamma}} \int_q^\infty \frac{1}{u_1} \int_{u_1}^\infty \frac{1}{u_2} \cdots \left[\int_{u_{n-3}}^\infty \frac{1}{u_n^{n-1+\frac{1}{\gamma}}} \Delta u_n \right] \cdots \Delta u_2 \Delta u_1 \\
&\quad \cdots \\
&= M^{\frac{2r+1}{\gamma}} \int_q^\infty \frac{1}{u_1} \left[\int_{u_1}^\infty \int_{u_2}^\infty \frac{1}{u_2} \frac{1}{u_n^{4+\frac{1}{\gamma}}} \Delta u_n \Delta u_2 \right] \Delta u_1 \\
&= M^{\frac{2r+1}{\gamma}} \int_q^\infty \frac{1}{u_1} \left[\int_{u_1}^\infty \int_{u_1}^{\sigma(u_n)} \frac{1}{u_2} \frac{1}{u_n^{4+\frac{1}{\gamma}}} \Delta u_2 \Delta u_n \right] \Delta u_1 \\
&\le M^{\frac{2r+1}{\gamma}} \int_q^\infty \int_{u_1}^\infty \frac{1}{u_1} \frac{1}{u_n^{3+\frac{1}{\gamma}}} \Delta u_n \Delta u_1 \\
&\le M^{\frac{2r+1}{\gamma}} \int_q^\infty \frac{1}{u_n^{2+\frac{1}{\gamma}}} \Delta u_n \\
&= M^{\frac{2r+1}{\gamma}} \frac{q-1}{q^{1+\frac{1}{\gamma}} - 1} \\
&< \infty.
\end{aligned}
$$

Thus conditions (H_1) and (3.1) hold. By Theorem 3.1, Theorem 3.2 and Theorem 3.3, we see that Eq. (4.1) has a nonoscillatory bounded solution $x(t)$ with $\liminf_{t\to\infty} |x(t)| > 0$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Guangxi Colleges and Universities Key Laboratory of Mathematics and Its Applications, Nanning, 530004, China.
[2]College of Information and Statistics, Guangxi University of Finance and Economics, Nanning, 530003, China.

Acknowledgements
This project is supported by NNSF of China (11461003) and NSF of Guangxi (2012GXNSFDA276040).

Received: 17 January 2015 Accepted: 25 August 2015 Published online: 07 September 2015

References

1. Hilger, S: Analysis on measure chains - a unified approach to continuous and discrete calculus. Results Math. **18**, 18-56 (1990)
2. Kac, V, Chueng, P: Quantum Calculus. Universitext (2002)
3. Bohner, M, Peterson, A: Dynamic Equations on Time Scales: An Introduction with Applications. Birkhäuser, Boston (2001)
4. Bohner, M, Peterson, A: Advances in Dynamic Equations on Time Scales. Birkhäuser, Boston (2003)
5. Bohner, M, Saker, SH: Oscillation of second order nonlinear dynamic equations on time scales. Rocky Mt. J. Math. **34**, 1239-1254 (2004)
6. Erbe, L: Oscillation results for second-order linear equations on a time scale. J. Differ. Equ. Appl. **8**, 1061-1071 (2002)
7. Hassan, TS: Oscillation criteria for half-linear dynamic equations on time scales. J. Math. Anal. Appl. **345**, 176-185 (2008)
8. Agarwal, RP, Bohner, M, Saker, SH: Oscillation of second order delay dynamic equations. Can. Appl. Math. Q. **13**, 1-17 (2005)
9. Bohner, M, Karpuz, B, Öcalan, Ö: Iterated oscillation criteria for delay dynamic equations of first order. Adv. Differ. Equ. **2008**, Article ID 458687 (2008)
10. Erbe, L, Peterson, A, Saker, SH: Oscillation criteria for second-order nonlinear delay dynamic equations. J. Math. Anal. Appl. **333**, 505-522 (2007)
11. Han, Z, Shi, B, Sun, S: Oscillation criteria for second-order delay dynamic equations on time scales. Adv. Differ. Equ. **2007**, Article ID 70730 (2007)
12. Han, Z, Sun, S, Shi, B: Oscillation criteria for a class of second-order Emden-Fowler delay dynamic equations on time scales. J. Math. Anal. Appl. **334**, 847-858 (2007)
13. Şahiner, Y: Oscillation of second-order delay differential equations on time scales. Nonlinear Anal., Theory Methods Appl. **63**, e1073-e1080 (2005)
14. Şahiner, Y: Oscillation of second-order neutral delay and mixed-type dynamic equations on time scales. Adv. Differ. Equ. **2006**, Article ID 65626 (2006)
15. Zhang, B, Zhu, S: Oscillation of second-order nonlinear delay dynamic equations on time scales. Comput. Math. Appl. **49**, 599-609 (2005)
16. Grace, SR, Agarwal, RP, Kaymakcalan, B, Sae-jie, W: On the oscillation of certain second order nonlinear dynamic equations. Math. Comput. Model. **50**, 273-286 (2009)
17. Erbe, L, Peterson, A, Saker, SH: Asymptotic behavior of solutions of a third-order nonlinear dynamic equation on time scales. J. Comput. Appl. Math. **181**, 92-102 (2005)
18. Erbe, L, Peterson, A, Saker, SH: Hille and Nehari type criteria for third order dynamic equations. J. Math. Anal. Appl. **329**, 112-131 (2007)
19. Erbe, L, Peterson, A, Saker, SH: Oscillation and asymptotic behavior a third-order nonlinear dynamic equation. Can. Appl. Math. Q. **14**, 129-147 (2006)
20. Karpuz, B, Öcalan, Ö, Rath, RN: Necessary and sufficient conditions on the oscillatory and asymptotic behaviour of solutions to neutral delay dynamic equation. Electron. J. Differ. Equ. **2009**, 64 (2009)
21. Zhu, Z, Wang, Q: Existence of nonoscillatory solutions to neutral dynamic equations on time scales. J. Math. Anal. Appl. **335**, 751-762 (2007)
22. Karpuz, B, Öcalan, Ö, Rath, RN: Necessary and sufficient conditions on the asymptotic behaviour of solutions of forced neutral delay dynamic equations. Nonlinear Anal. **71**, 3063-3071 (2009)
23. Karpuz, B: Asymptotic behaviour of bounded solutions of a class of higher-order neutral dynamic equations. Appl. Math. Comput. **215**, 2174-2183 (2009)
24. Karpuz, B: Unbounded oscillation of higher-order nonlinear delay dynamic equations of neutral type with oscillating coefficients. Electron. J. Qual. Theory Differ. Equ. **2009**, 34 (2009)
25. Wu, X, Sun, T, Xi, H, Cheng, C: Kamenev-type oscillation criteria for higher-order nonlinear dynamic equations on time scales. Adv. Differ. Equ. **2013**, 248 (2013)
26. Sun, T, Xi, H, Peng, X, Yu, W: Nonoscillatory solutions for higher-order neutral dynamic equations on time scales. Abstr. Appl. Anal. **2010**, Article ID 428963 (2010)
27. Zhang, Z, Dong, W, Li, Q, Liang, H: Positive solutions for higher order nonlinear neutral dynamic equations on time scales. Appl. Math. Model. **33**, 2455-2463 (2009)
28. Agarwal, RP, Bohner, M: Basic calculus on time scales and some of its applications. Results Math. **35**, 3-22 (1999)
29. Chen, D: Oscillation and asymptotic behavior for *n*th-order nonlinear neutral delay dynamic equations on time scales. Acta Appl. Math. **109**, 703-719 (2010)