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Existence and uniqueness of positive and nondecreasing solutions for a class of fractional boundary value problems involving the p-Laplacian operator

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Abstract

In this article, we investigate the existence of a solution arising from the following fractional q-difference boundary value problem by using the p-Laplacian operator: $D_q^\gamma(\phi_p(D_q^\delta y(t))) + f(t, y(t)) = 0$ $(0 < t < 1; 0 < \gamma < 1; 3 < \delta < 4)$, $y(0) = (D_q y)(0) = (D_q^2 y)(0) = 0$, $a_1(D_q y)(1) + a_2(D_q^2 y)(1) = 0$, $a_1 + |a_2| \neq 0$, $D_{0+}^\gamma y(t)|_{t=0} = 0$. We make use of such a fractional q-difference boundary value problem in order to show the existence and uniqueness of positive and nondecreasing solutions by means of a familiar fixed point theorem.

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1 Introduction, definitions, and preliminaries

Recently, many mathematicians, physicists and engineers have extensively studied various families of fractional differential equations and their applications. The development of the theory of fractional calculus stems from the applications in many widespread disciplines such as engineering, economics and other fields. Jackson [1] introduced the q-difference calculus (or the so-called *quantum calculus*), which is an old subject. New developments in this theory were made. These include (for example) the q-analogs of the fractional integral and the fractional derivative operators, the q-analogs of the Laplace, Fourier, and other integral transforms, and so on (see, for details, [2–13], and [14]; see also a very recent work [15] dealing with q-calculus).

Throughout our present investigation, we make use of the following notations:

$$\mathbb{N} := \{1, 2, 3, \ldots\} \quad \text{and} \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Moreover, as usual, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$ denotes the set of positive real numbers, $\mathbb{Z}_-$ denotes the set of negative integers, and $\mathbb{C}$ denotes the set of complex numbers.

Al-Salam [16] and Agarwal [2] investigated several properties and results for some fractional q-integrals and fractional q-derivatives which are based on the q-analog of the ordinary integral:

$$\int_a^x f(t)\,dt.$$

Atici and Eloe [3] constructed interesting links between the fractional q-calculus in the existing literature and the fractional q-calculus on a time scale given by

$$T_{t_0} = \left\{t : t = t_0 q^n \ (n \in \mathbb{N}_0; t_0 \in \mathbb{R}; 0 < q < 1)\right\}.$$

They also derived some properties of a q-Laplace transform, which are used to solve fractional q-difference equations. Benchohra *et al.* [17] investigated the existence of solutions for fractional-order functional equations by means of the Banach fixed point theorem and its nonlinear alternative of Leray-Schauder type. El-Sayed *et al.* [18] studied the stability, existence, uniqueness, and numerical solution of the fractional-order logistic equation. The work of El-Shahed [19] was concerned with the existence and non-existence of positive solutions for some nonlinear fractional boundary value problems. Ferreira (see [20] and [21]) investigated the existence of nontrivial solutions to some nonlinear q-fractional boundary value problems by applying a fixed point theorem in cones. For more information on the positive solutions (or nontrivial solutions) for a class of boundary value problems with the fractional differential equations (or q-fractional differential equations), we refer the reader to such earlier works as (for example) [5, 10, 22–31], and [32].

We now review briefly some concepts of the quantum calculus.

For $q \in (0,1)$, the q-integer $[\lambda]_q$ is defined by

$$[\lambda]_q = \frac{1-q^\lambda}{1-q} \quad (\lambda \in \mathbb{R}).$$

Clearly, we have

$$\lim_{q\to 1^-} [\lambda]_q = \lambda,$$

so we say that $[\lambda]_q$ is a q-analog of the number λ. The q-analog of the binomial formula $(a-b)^n$ is given by

$$(a-b)^0 = 1 \quad \text{and} \quad (a-b)^n = \prod_{k=0}^{n-1}\left(a - bq^k\right) \quad (a, b \in \mathbb{R}; n \in \mathbb{N}_0).$$

More generally, we have

$$(a-b)^{(\delta)} = a^\delta \prod_{n=0}^{\infty}\left(\frac{a-bq^n}{a-bq^{\delta+n}}\right) \quad (\delta \in \mathbb{R}). \tag{1.1}$$

Clearly, if we set $b = 0$ in Eq. (1.1), it reduces immediately to

$$a^{(\delta)} = a^\delta \quad (\delta \in \mathbb{R}).$$

The q-gamma function is defined as follows:

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}} \quad \left(x \in \mathbb{R} \setminus \left\{\{0\} \cup \mathbb{Z}_-\right\}\right)$$

and satisfies the formula:

$$\Gamma_q(x+1) = [x]\Gamma_q(x).$$

The q-derivative of a function $f(x)$ is given by

$$(D_q f)(x) = \frac{f(x) - f(qx)}{(1-q)x} \quad \text{and} \quad \lim_{q\to 1^-}(D_q f)(x) = f'(x) = \frac{d}{dx}\{f(x)\}.$$

For the q-derivatives of higher order, we have

$$\bigl(D_q^0 f\bigr)(x) = f(x) \quad \text{and} \quad \bigl(D_q^n f\bigr)(x) = D_q\bigl(D_q^{n-1} f\bigr)(x) \quad (n \in \mathbb{N}).$$

Suppose now that $0 < a < b$. Then the definite q-integral is defined as follows:

$$(I_q f)(x) = \int_0^x f(t)\,d_q t = x(1-q)\sum_{n=0}^{\infty} f\bigl(xq^n\bigr)q^n \quad \bigl(x \in [0,b]\bigr)$$

and

$$\int_a^b f(t)\,d_q t = \int_0^b f(t)\,d_q t - \int_0^a f(t)\,d_q t.$$

The operator I_q^n can be defined by

$$\bigl(I_q^0 f\bigr)(x) = f(x) \quad \text{and} \quad \bigl(I_q^n f\bigr)(x) = I_q\bigl(I_q^{n-1} f\bigr)(x) \quad (n \in \mathbb{N}).$$

The *Fundamental Theorem of Calculus* does indeed apply *mutatis mutandis* to the operators I_q and D_q. We thus have

$$(D_q I_q f)(x) = f(x),$$

and if f is continuous at $x = 0$, then

$$(I_q D_q f)(x) = f(x) - f(0).$$

Denoting by ${}_xD_q$ the q-derivative with respect to the variable x, we now recall the following three formulas which will be used in the remainder of this paper:

$$\bigl[a(t-s)\bigr]^{(\delta)} = a^\delta (t-s)^{(\delta)}, \tag{1.2}$$

$${}_xD_q(t-s)^{(\delta)} = [\delta]_q (x-s)^{(\delta-1)}, \tag{1.3}$$

$$\left({}_xD_q \int_0^x f(x,t)\,d_q t\right)(x) = \int_0^x {}_xD_q f(x,t)\,d_q t + f(qx,x). \tag{1.4}$$

Definition 1 (see [21]) Let $\delta \geqq 0$ and f be a function defined on $[0,1]$. The fractional q-integral of the Riemann-Liouville type is given by

$$\bigl(I_q^0 f\bigr)(x) = f(x)$$

and

$$\left(I_q^\delta f\right)(x) = \frac{1}{\Gamma_q(\delta)} \int_0^x (x-qt)^{(\delta-1)} f(t)\, d_q t \quad \left(\delta > 0; x \in [0,1]\right).$$

Definition 2 (see [21] and [13]) The fractional q-derivative of the Riemann-Liouville type of order δ ($\delta \geqq 0$) is defined by

$$\left(D_q^0 f\right)(x) = f(x)$$

and

$$\left(D_q^\delta f\right)(x) = \left(D_q^m I_q^{m-\delta} f\right)(x) \quad (\delta > 0),$$

where m is the smallest integer greater than or equal to δ.

Lemma 1 (see [21]) *Let $\delta \geqq 0$, $\beta \geqq 0$, and f be a function defined on $[0,1]$. Then the following two formulas hold true*:

(1) $(I_q^\beta I_q^\delta f)(x) = (I_q^{\delta+\beta} f)(x)$;

(2) $(D_q^\delta I_q^\delta f)(x) = f(x)$.

Lemma 2 (see [21] and [13]) *Let $\delta > 0$ and p be a positive integer. Then the following equality holds*:

$$\left(I_q^\delta D_q^p f\right)(x) = \left(D_q^p I_q^\delta f\right)(x) - \sum_{k=0}^{p-1} \frac{x^{\delta-p+k}}{\Gamma_q(\delta+k-p+1)} \left(D_q^k f\right)(0).$$

Theorem 1 (see [33, 34], and [35]) (a) *Let $(E, \leqq)$ be a partially ordered set and suppose that there exists a metric d in E such that (E, d) is a complete metric space. Assume that E satisfies the condition that, if $\{x_n\}$ is a nondecreasing sequence in E such that $x_n \to x$, then*

$$x_n \leqq x \quad (n \in \mathbb{N}).$$

Let $T : E \to E$ be a nondecreasing mapping such that

$$d(Tx, Ty) \leqq d(x,y) - \psi\left(d(x,y)\right) \quad (x \geqq y),$$

where

$$\psi : [0, +\infty) \to [0, +\infty)$$

is a continuous and nondecreasing function such that ψ is positive in $(0, \infty)$, $\psi(0) = 0$, and

$$\lim_{t\to\infty} \psi(t) = \infty.$$

If there exists $x_0 \in E$ with $x_0 \leqq T(x_0)$, then T has a fixed point.

(b) *If we assume that $(E, \leqq)$ satisfies the condition that, for $x \in E$ and $y \in E$, there exists $z \in E$ which is comparable to x and y and the hypothesis of* (a), *then it leads to the uniqueness of the fixed point.*

Mena *et al.* [27] investigated the existence and uniqueness of positive and nondecreasing solutions for the following singular fractional boundary value problem:

$$D_{0^+}^{\alpha} u(t) + f\bigl(t, u(t)\bigr) = 0 \quad (0 < t < 1; 2 < \alpha < 3),$$
$$u(0) = u'(1) = u''(0) = 0.$$

Miao and Liang [10], on the other hand, studied the existence and uniqueness of a positive and nondecreasing solution for the following fractional q-difference boundary value problem:

$$D_q^{\gamma}\bigl(\phi_p\bigl(D_q^{\alpha} u(t)\bigr)\bigr) + f\bigl(t, u(t)\bigr) = 0 \quad (0 < t < 1; 2 < \alpha < 3),$$
$$u(0) = (D_q u)(0) = 0, \qquad (D_q u)(1) = 0, \quad \text{and} \quad D_{0+}^{\gamma} u(t)|_{t=0} = 0.$$

Motivated essentially by the aforementioned work by Miao and Liang [10], we introduce and investigate here the following q-difference boundary value problem by using the p-Laplacian operator:

$$D_q^{\gamma}\bigl(\phi_p\bigl(D_q^{\delta} y(t)\bigr)\bigr) + f\bigl(t, y(t)\bigr) = 0 \quad (0 < t < 1; 3 < \delta < 4), \tag{1.5}$$

$$\begin{cases} y(0) = (D_q y)(0) = (D_q^2 y)(0) = 0, \\ a_1(D_q y)(1) + a_2(D_q^2 y)(1) = 0, \quad \text{and} \quad D_{0+}^{\gamma} y(t)|_{t=0} = 0. \end{cases} \tag{1.6}$$

We prove the existence and uniqueness of a positive and nondecreasing solution for the boundary value problem given by Eqs. (1.5) and (1.6) by means of a fixed point theorem involving partially ordered sets.

2 Fractional boundary value problem

Throughout of this paper, we always make use of the usual space $E = C[0,1]$ which is known as the space of continuous functions on $[0,1]$. We note that E is a real Banach space with the norm given by

$$\|u\| = \max_{0 \leqq t \leqq 1} \bigl|u(t)\bigr|.$$

Suppose that $x \in C[0,1]$ and $y \in C[0,1]$. Then we have

$$x \leqq y \quad \Leftrightarrow \quad x(t) \leqq y(t) \quad \bigl(\forall t \in [0,1]\bigr).$$

We know from the recent work [34] that $(C[0,1], \leqq)$ with the familiar metric:

$$d(x, y) = \sup_{0 \leqq t \leqq 1} \bigl\{\bigl|x(t) - y(t)\bigr|\bigr\}$$

satisfies the hypothesis of Theorem 1(a). Moreover, for $x \in C[0,1]$ and $y \in C[0,1]$ such that $\max\{x, y\} \in C[0,1]$, $(C[0,1], \leqq)$ satisfies the condition of Theorem 1(b).

We first demonstrate Lemma 3.

Lemma 3 *If $h \in C[0,1]$, the following boundary value problem*:

$$\big(D_q^\delta y\big)(t) + h(t) = 0 \quad (0 < t < 1; 3 < \delta < 4), \tag{2.1}$$

$$\begin{cases} u(0) = (D_q u)(0) = (D_q^2 u)(0) = 0, \\ a_1(D_q u)(1) + a_2(D_q^2 u)(1) = 0 \quad (|a_1| + |a_2| \neq 0) \end{cases} \tag{2.2}$$

has a unique solution given by

$$u(t) = \int_0^1 G(t, qs)h(s)\, d_q s, \tag{2.3}$$

where

$$\begin{aligned} G(t,s) &= \frac{1}{(a_1 + a_2[\delta - 2]_q)\Gamma_q(\delta)} \\ &\quad \times \begin{cases} (a_1(1-s)^{(\delta-2)} + a_2[\delta-2]_q(1-s)^{(\delta-3)})t^{\delta-1} \\ \quad - (a_1 + a_2[\delta-2]_q)(t-s)^{(\delta-1)} \quad (0 \leqq s \leqq t \leqq 1), \\ (a_1(1-s)^{(\delta-2)} + a_2[\delta-2]_q(1-s)^{(\delta-3)})t^{\delta-1} \quad (0 \leqq t \leqq s \leqq 1). \end{cases} \end{aligned} \tag{2.4}$$

Proof By applying Lemma 1, Lemma 2 (*with* $p = 4$) and Eq. (2.1), we have

$$\big(I_q^\delta D_q^4 I_q^{4-\delta} u\big)(x) = -I_q^\delta f\big(t, u(t)\big)$$

and

$$u(t) = c_1 t^{\delta-1} + c_2 t^{\delta-2} + c_3 t^{\delta-3} + c_4 t^{\delta-4} - \int_0^t \frac{(t-qs)^{(\delta-1)}}{\Gamma_q(\delta)} h(s)\, d_q s. \tag{2.5}$$

From Eq. (2.2), we get $c_4 = 0$. Thus, upon differentiating both sides of Eq. (2.5), if we make use of Eqs. (1.2) and (1.3), we see that

$$(D_q u)(t) = [\delta-1]_q c_1 t^{\delta-2} + [\delta-2]_q c_2 t^{\delta-3} + c_3 t^{\delta-4} - \frac{[\delta-1]_q}{\Gamma_q(\delta)} \int_0^t (t-qs)^{(\delta-2)} h(s)\, d_q s. \tag{2.6}$$

Using the boundary condition (2.2), we have $c_3 = 0$. Moreover, by differentiating both sides of Eq. (2.6), and using Eqs. (1.2) and (1.3), we obtain

$$\begin{aligned} \big(D_q^2 u\big)(t) &= [\delta-1]_q[\delta-2]_q c_1 t^{\delta-3} + [\delta-2]_q[\delta-3]_q c_2 t^{\delta-4} \\ &\quad - \frac{[\delta-1]_q[\delta-2]_q}{\Gamma_q(\delta)} \int_0^t (t-qs)^{(\delta-3)} h(s)\, d_q s. \end{aligned}$$

Similarly, by using the boundary condition (2.2), we have $c_2 = 0$ and

$$c_1 = \frac{a_1 \int_0^1 (1-qs)^{(\delta-2)} h(s)\, d_q s + a_2[\delta-2]_q \int_0^1 (1-qs)^{(\delta-3)} h(s)\, d_q s}{(a_1 + a_2[\delta-2]_q)\Gamma_q(\delta)}.$$

Consequently, we have the following unique solution of the boundary value problem given by Eqs. (2.1) and (2.2):

$$\begin{aligned}
u(t) &= -\int_0^t \frac{(t-qs)^{(\delta-1)}}{\Gamma_q(\delta)} h(s)\,d_qs \\
&\quad + \frac{(a_1\int_0^1 (1-qs)^{(\delta-2)}h(s)\,d_qs + a_2[\delta-2]_q\int_0^1(1-qs)^{(\delta-3)}h(s)\,d_qs)t^{\delta-1}}{(a_1+a_2[\delta-2]_q)\Gamma_q(\delta)} \\
&= \frac{1}{\Gamma_q(\delta)}\int_0^t \biggl(\frac{(a_1(1-qs)^{(\delta-2)} + a_2[\delta-2]_q(1-qs)^{(\delta-3)})t^{\delta-1}}{(a_1+a_2[\delta-2]_q)} - (t-qs)^{(\delta-1)}\biggr) h(s)\,d_qs \\
&\quad + \int_t^1 \frac{(a_1(1-qs)^{(\delta-2)} + a_2[\delta-2]_q(1-qs)^{(\delta-3)})t^{\delta-1}}{(a_1+a_2[\delta-2]_q)\Gamma_q(\delta)} h(s)\,d_qs \\
&= \int_0^1 G(t,qs)h(s)\,d_qs.
\end{aligned}$$

We thus arrive at the desired result asserted by Lemma 3. □

By using the method in [10] *mutatis mutandis*, it can easily be proven that, if $f \in C([0,1]\times[0,+\infty),[0,+\infty))$, then the boundary value problem given by Eqs. (1.5) and (1.6) is equivalent to the following integral equation:

$$u(t) = \int_0^1 G(t,qs)\phi_p^{-1}\biggl(\int_0^s \frac{(s-\tau)^{(\gamma-1)}f(\tau,u(\tau))}{(a_1+a_2[\delta-2]_q)\Gamma_q(\gamma)}\,d_q\tau\biggr)\,d_qs, \tag{2.7}$$

where $G(t,s)$ is defined, as before, by Eq. (2.4).

Lemma 4 *The function $G(t,s)$ given by Eq.* (2.4) *has the following properties*:

(1) *$G(t,s)$ is a continuous function and $G(t,qs) \geqq 0$*;

(2) *$G(t,s)$ is strictly increasing in the first variable t.*

Proof The continuity of $G(t,s)$ can easily be checked. We, therefore, omit the details involved. Next, for $0 \leqq s \leqq t \leqq 1$, we let

$$\begin{aligned}
g_1(t,s) &= \bigl(a_1(1-s)^{\delta-2} + a_2[\delta-2]_q(1-s)^{\delta-3}\bigr)t^{\delta-1} \\
&\quad - \bigl(a_1+a_2[\delta-2]_q\bigr)(t-s)^{\delta-1}
\end{aligned}$$

and, for $0 \leqq t \leqq s \leqq 1$, we suppose that

$$g_2(t,s) = \bigl(a_1(1-s)^{\delta-2} + a_2[\delta-2]_q(1-s)^{\delta-3}\bigr)t^{\delta-1}.$$

Then it is not difficult to see that

$$g_2(t,qs) \geqq 0.$$

Now, for $g_1(0,qs) = 0$, $\delta > 0$, and $a \leqq b \leqq t$, we have

$$(t-a)^{(\delta)} \geqq (t-b)^{(\delta)} \quad (t \neq 0).$$

We thus find that

$$
\begin{aligned}
g_1(t,qs) &= \big(a_1(1-qs)^{\delta-2} + a_2[\delta-2]_q(1-qs)^{\delta-3}\big)t^{\delta-1} \\
&\quad - \big(a_1 + a_2[\delta-2]_q\big)\left(1-q\frac{s}{t}\right)t^{\delta-1} \\
&\geqq \big[\big(a_1(1-qs)^{\delta-2} + a_2[\delta-2]_q(1-qs)^{\delta-3}\big) \\
&\quad - \big(a_1 + a_2[\delta-2]_q\big)(1-qs)^{\delta-1}\big]t^{\delta-1} \\
&\geqq \big[\big(a_1(1-qs)^{\delta-1} + a_2[\delta-2]_q(1-qs)^{\delta-1}\big) \\
&\quad - \big(a_1 + a_2[\delta-2]_q\big)(1-qs)^{\delta-1}\big]t^{\delta-1} = 0.
\end{aligned}
$$

So, clearly, $G(t,qs) \geqq 0$ for all $(t,s) \in [0,1] \times [0,1]$. This completes the proof of Lemma 4(1).

Next, for a fixed $s \in [0,1]$, we see that

$$
\begin{aligned}
{}_tD_q g_1(t,qs) &= [\delta-1]_q\big(a_1(1-qs)^{(\delta-2)} + a_2[\delta-2]_q(1-qs)^{(\delta-3)}\big)t^{\delta-2} \\
&\quad - [\delta-1]_q\big(a_1 + a_2[\delta-2]_q\big)(t-qs)^{\delta-2} \\
&= [\delta-1]_q\big(a_1(1-qs)^{(\delta-2)} + a_2[\delta-2]_q(1-qs)^{(\delta-3)}\big)t^{\delta-2} \\
&\quad - [\delta-1]_q\big(a_1 + a_2[\delta-2]_q\big)\left(1-q\frac{s}{t}\right)^{\delta-2}t^{\delta-2} \\
&\geqq [\delta-1]_q(1-qs)^{(\delta-2)}\big(a_1 + a_2[\delta-2]_q\big)t^{\delta-2} \\
&\quad - [\delta-1]_q(1-qs)^{(\delta-2)}\big(a_1 + a_2[\delta-2]_q\big)t^{\delta-2} \\
&= 0.
\end{aligned}
$$

This implies that $g_1(t,qs)$ is an increasing function of the first argument t. Furthermore, obviously, $g_2(t,qs)$ is an increasing function of the first argument t. Therefore, $G(t,qs)$ is an increasing function of t for a fixed $s \in [0,1]$. This completes the proof of Lemma 4. □

3 Uniqueness of positive solutions

For notational convenience, we write

$$
M := \phi_p^{-1}\left(\frac{1}{\Gamma_q(\gamma)(a_1 + a_2[\delta-2]_q)}\right) \sup_{0\leqq t\leqq 1} \int_0^1 G(t,qs)\,d_qs > 0. \tag{3.1}
$$

The main result of this paper is the assertion in Theorem 2.

Theorem 2 *The boundary value problem given by Eqs.* (1.5) *and* (1.6) *has a unique positive and increasing solution* $u(t)$ *if each of the following two conditions is satisfied*:

(i) *the function* $f : [0,1] \times [0,\infty) \to [0,\infty)$ *is continuous and nondecreasing with respect to the second variable*;

(ii) *there exist* λ *and* M *given by Eq.* (3.1) $(0 < \lambda + 1 < M)$ *such that, for* $u \in [0,\infty)$ *and* $v \in [0,\infty)$ *with* $u \geqq v$ *and* $t \in [0,1]$,

$$
\phi_p\big(\ln(v+2)\big) \leqq f(t,v) \leqq f(t,u) \leqq \phi_p\big(\ln(u+2)(u-v+1)^{\lambda}\big).
$$

Furthermore, if $f(t,0) > 0$ *for* $t \in [0,1]$, *then the solution* $u(t)$ *of the boundary value problem given by Eqs.* (1.5) *and* (1.6) *is strictly increasing on* $[0,\infty)$.

Proof First of all, we set

$$u := u(t) \quad \text{and} \quad v := v(t).$$

We then consider the set K (called a *cone*) given by

$$K = \{u : u \in C[0,1] \text{ and } u(t) \geqq 0\}.$$

Since K is a closed set, K is a complete metric space in accordance with the usual metric

$$d(u,v) = \sup_{t\in[0,1]} \left|u(t) - v(t)\right|.$$

Let us now consider the operator T as follows:

$$Tu(t) = \int_0^1 G(t,qs)\phi_p^{-1}\left(\frac{1}{(a_1 + a_2[\delta - 2]_q)\Gamma_q(\gamma)} \int_0^s (s-\tau)^{(\gamma-1)} f\bigl(\tau,u(\tau)\bigr)\,d_q\tau\right) d_qs.$$

Then, by applying Lemma 4 and the condition (i) of Theorem 2, we see that $T(K) \subset K$.

We now show that all conditions of Theorem 1 are satisfied. Firstly, by the condition (i) of Theorem 2, for $u, v \in K$ and $u \geqq v$, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,qs)\phi_p^{-1}\left(\frac{1}{(a_1 + a_2[\delta - 2]_q)\Gamma_q(\gamma)} \int_0^s (s-\tau)^{(\gamma-1)} f\bigl(\tau,u(\tau)\bigr)\,d_q\tau\right) d_qs \\ &\geqq \int_0^1 G(t,qs)\phi_p^{-1}\left(\frac{1}{(a_1 + a_2[\delta - 2]_q)\Gamma_q(\gamma)} \int_0^s (s-\tau)^{(\gamma-1)} f\bigl(\tau,v(\tau)\bigr)\,d_q\tau\right) d_qs \\ &= Tv(t). \end{aligned}$$

This shows that T is a nondecreasing operator. On the other hand, for $u \geqq v$ and by the condition (ii) of Theorem 2, we have

$$\begin{aligned} d(Tu,Tv) &= \sup_{0\leqq t\leqq 1} \left|(Tu)(t) - (Tv)(t)\right| \\ &= \sup_{0\leqq t\leqq 1} \bigl((Tu)(t) - (Tv)(t)\bigr) \\ &\leqq \sup_{0\leqq t\leqq 1}\Biggl[\int_0^1 G(t,qs)\phi_p^{-1}\left(\frac{1}{(a_1 + a_2[\delta - 2]_q)\Gamma_q(\gamma)}\right. \\ &\quad \left.\times \int_0^s (s-\tau)^{(\gamma-1)} f\bigl(\tau,u(\tau)\bigr)\,d_q\tau\right) d_qs \\ &\quad - \int_0^1 G(t,qs)\phi_p^{-1}\left(\frac{1}{(a_1 + a_2[\delta - 2]_q)\Gamma_q(\gamma)}\right. \\ &\quad \left.\times \int_0^s (s-\tau)^{(\gamma-1)} f\bigl(\tau,v(\tau)\bigr)\,d_q\tau\right) d_qs\Biggr] \end{aligned}$$

$$\begin{aligned}
&\leqq \big(\ln(u+2)(u-v+1)^{\lambda} - \ln(v+2)\big) \\
&\quad \times \sup_{0\leqq t\leqq 1} \int_0^1 G(t,qs)\phi_p^{-1}\left(\frac{1}{(a_1+a_2[\delta-2]_q)\Gamma_q(\gamma)} \int_0^s (s-\tau)^{(\gamma-1)}\,d_q\tau\right) d_qs \\
&\leqq \ln\frac{(u+2)(u-v+1)^{\lambda}}{v+2}\phi_p^{-1}\left(\frac{1}{(a_1+a_2[\delta-2]_q)\Gamma_q(\gamma)}\right)\sup_{0\leqq t\leqq 1}\int_0^1 G(t,qs)\,d_qs \\
&\leqq (\lambda+1)\ln(u-v+1)\phi_p^{-1}\left(\frac{1}{(a_1+a_2[\delta-2]_q)\Gamma_q(\gamma)}\right)\sup_{0\leqq t\leqq 1}\int_0^1 G(t,qs)\,d_qs.
\end{aligned}$$

Since the function $h(x) = \ln(x+1)$ is nondecreasing, from the condition (ii) of Theorem 2, we have

$$\begin{aligned}
d(Tu,Tv) &\leqq (\lambda+1)\ln\big(\|u-v\|+1\big)\phi_p^{-1}\left(\frac{1}{(a_1+a_2[\delta-2]_q)\Gamma_q(\gamma)}\right)\sup_{0\leqq t\leqq 1}\int_0^1 G(t,qs)\,d_qs \\
&= (\lambda+1)\ln\big(\|u-v\|+1\big)M \\
&\leqq \|u-v\| - \big(\|u-v\| - \ln\big(\|u-v\|+1\big)\big).
\end{aligned}$$

We now let $\psi(x) = x - \ln(x+1)$. Then, obviously, the function ψ given by

$$\psi : [0,+\infty) \to [0,+\infty)$$

is continuous, nondecreasing, and positive in $(0,\infty)$. It is also clearly seen that $\psi(x)$ satisfies the following conditions:

$$\psi(0) = 0 \quad \text{and} \quad \lim_{x\to+\infty}\psi(x) = \infty.$$

Thus, for $u \geqq v$, we have

$$d(Tu,Tv) \leqq d(u,v) - \psi\big(d(u,v)\big).$$

As $G(t,qs) \geqq 0$ and $f \geqq 0$, we have

$$(T0)(t) = \int_0^1 G(t,qs)f(s,0)\,d_qs \geqq 0.$$

Consequently, in view of Theorem 1, the boundary value problem given by Eqs. (1.5) and (1.6) has at least one nonnegative solution. Since $(K,\leqq)$ satisfies the condition (ii) of Theorem 2, Theorem 1 implies the uniqueness of the solution. Thus, clearly, the proof of the last assertion of Theorem 2 follows immediately from the proof of a well-known result in [10, Theorem 4.2]. Our proof Theorem 2 is thus completed. □

4 Concluding remarks and observations

Our present study was motivated by several aforementioned recent works. Here, we have successfully addressed the problem involving the existence and uniqueness of positive and nondecreasing solutions of a family of fractional q-difference boundary value problems given by Eqs. (1.5) and (1.6). The proof of our main result asserted by Theorem 2 of the preceding section has made use of some familiar fixed point theorems. We have also indicated the relevant connections of the results derived in this investigation with those in earlier works on the subject.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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