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Properties of multivalent functions associated with the integral operator defined by the hypergeometric function

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Abstract
In this paper, we introduce a new class of multivalent functions by using a generalized integral operator defined by the hypergeometric function. Some properties such as inclusion, radius problem and integral preserving are considered.

MSC: 30C45; 30C50

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1 Introduction and preliminaries

Let A_p denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}), \tag{1.1}$$

which are analytic in the open unit disc E. Also $A_1 = A$, the usual class of analytic functions defined in the open unit disc $E = \{z : |z| < 1\}$. A function $f \in A_p$ is a p-valent starlike function of order ρ if and only if

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > \rho, \quad 0 \le \rho < p, z \in E.$$

This class of functions is denoted by $S_p^*(\rho)$. It is noted that $S_p^*(0) = S_p^*$. Let $f(z)$ and $g(z)$ be analytic in E, we say $f(z)$ is subordinate to $g(z)$, written $f \prec g$ or $f(z) \prec g(z)$ if there exists a Schwarz function $w(z)$, $w(0) = 0$ and $|w(z)| < 1$ in E, then $f(z) = g(w(z))$. In particular, if g is univalent in E, then we have the following equivalence

$$f(z) \prec g(z) \quad \Longleftrightarrow \quad f(0) = g(0) \quad \text{and} \quad f(E) \subset g(E).$$

For any two analytic functions $f(z)$ and $g(z)$ with

$$f(z) = \sum_{n=0}^{\infty} b_n z^{n+1} \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} c_n z^{n+1}, \quad z \in E,$$

the convolution (the Hadamard product) is given by

$$(f * g)(z) = \sum_{n=0}^{\infty} b_n c_n z^{n+1}, \quad z \in E.$$

A function $f \in A$ is said to be in the class, denoted by $SD(k,\delta)$ $(0 \leq \delta < 1)$, if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > k\left|\frac{zf'(z)}{f(z)} - 1\right| + \delta, \quad k \geq 0, z \in E. \tag{1.2}$$

Similarly, a function $f \in A$ is said to be in the class, denoted by $CD(k,\delta)$ of k-uniformly convex of order δ $(0 \leq \delta < 1)$, if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > k\left|\frac{zf''(z)}{f'(z)}\right| + \delta, \quad k \geq 0, z \in E. \tag{1.3}$$

Geometric interpretation The functions $f \in SD(k,\delta)$ and $f \in CD(k,\delta)$ if and only if $\frac{zf'(z)}{f(z)}$ and $\frac{zf''(z)}{f(z)} + 1$, respectively, take all the values in the conic domain $\Omega_{k,\delta}$ defined by

$$\Omega_{k,\delta} = \left\{u + iv : u > k\sqrt{(u-1)^2 + v^2} + \delta\right\}$$

with $p(z) = \frac{zf'(z)}{f(z)}$ or $p(z) = \frac{zf''(z)}{f'(z)} + 1$ and considering the functions which map E onto the conic domain $\Omega_{k,\delta}$ such that $1 \in \Omega_{k,\delta}$. One may rewrite the conditions (1.2) or (1.3) in the form

$$p(z) \prec q_{k,\delta}(z).$$

The function $q_{k,\delta}(z)$ plays the role of extremal for these classes and is given by

$$q_{k,\delta}(z) = \begin{cases} \frac{1+(1-2\delta)z}{1-z}, & k = 0, \\ 1 + \frac{2\delta\gamma}{\pi^2}(\log\frac{1+\sqrt{z}}{1-\sqrt{z}})^2, & k = 1, \\ 1 + \frac{2\delta}{1-k^2}\sinh^2[(\frac{2}{\pi}\arccos k)\arctan h\sqrt{z}], & 0 < k < 1, \\ 1 + \frac{\delta}{k^2-1}\sin(\frac{\pi}{2R(t)}\int_0^{\frac{u(z)}{\sqrt{t}}}\frac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\,dx) + \frac{\delta}{k^2-1}, & k > 1. \end{cases} \tag{1.4}$$

For $f(z)$ in A_p, the operator $D^{\mu+p-1} : A_p \longrightarrow A_p$ is defined by

$$D^{\mu+p-1}f(z) = \frac{z^p}{(1-z)^{\mu+p}} * f(z) \quad (\mu > -p),$$

or equivalently

$$D^{\mu+p-1}f(z) = \frac{z^p(z^{\mu-1}f(z))^{\mu+p-1}}{(\mu+p-1)!}, \tag{1.5}$$

where μ is any integer greater than $-p$. If $f(z)$ is given by (1.1), then it follows that

$$D^{\mu+p-1}f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{(\mu+n-1)!}{(n-p)!(\mu+p-1)!} a_n z^n.$$

The symbol $D^{\mu+p-1}$ when $p = 1$, was introduced by Ruscheweyh [1] and $D^{\mu+p-1}$ is called the $(\mu + p - 1)$th order Ruscheweyh derivative. We now introduce a function $(z^p_2F_1(a,b,c;z))^{-1}$ given by

$$\left(z^p{}_2F_1(a,b,c;z)\right) * \left(z^p{}_2F_1(a,b,c;z)\right)^{-1} = \frac{z^p}{(1-z)^{\mu+p}} \quad (\mu > -p),$$

and the following linear operator

$$I_{\mu,p}(a,b,c)f(z) = \left(z^p{}_2F_1(a,b,c;z)\right)^{-1} * f(z), \tag{1.6}$$

where a, b, c are real or complex numbers other than $0,-1,-2,\ldots$, $\mu > -p$, $z \in E$ and $f(z) \in A_p$. This operator was recently introduced in [2]. In particular, for $p = 1$, this operator is studied by Noor [3]. For $b = 1$, this operator reduces to the well-known Cho-Kwon-Srivastava operator $I_{\mu,p}(a,c)$, which was studied by Cho *et al.* [4], and for $\mu = 1$, $b = c$, $a = n + p$, see [5]. For $a = n + p$, $b = c = 1$, this operator was investigated by Liu [6] and Liu and Noor [7].

Simple computations yield

$$I_{\mu,p}(a,b,c)f(z) = z^p + \sum_{n=p+1}^{\infty} \frac{(c)_n(\mu+p)_n}{(a)_n(b)_n} a_n z^n.$$

From (1.6), we note that

$$I_{\mu,1}(a,b,c)f(z) = I_{\mu}(a,b,c)f(z) \quad \text{(see [3])},$$
$$I_{0,p}(a,p,a)f(z) = f(z), I_{1,p}(a,p,a)f(z) = \frac{zf'(z)}{p}.$$

Also, it can be easily seen that

$$z\left(I_{\mu,p}(a,b,c)f(z)\right)' = (\mu+p)I_{\mu+1,p}(a,b,c)f(z) - \mu I_{\mu,p}(a,b,c)f(z), \tag{1.7}$$

and

$$z\left(I_{\mu,p}(a+1,b,c)f(z)\right)' = aI_{\mu,p}(a,b,c)f(z) - (a-p)I_{\mu,p}(a,b,c)f(z).$$

We define the following class of multivalent analytic functions by using the operator $I_{\mu,p}(a,b,c)f(z)$ above.

Definition 1.1 Let $f \in A_p$ for $p \in \mathbb{N}$. Then $f \in UB^{\alpha}_{\mu,p}(a,b,c;\gamma,k,\delta)$ for $a,b,c \in \mathbb{R}\backslash\mathbb{Z}^-_0$, $\mu > -p$, $\alpha > 0$, $k \geq 0$, $0 \leq \delta < 1$ and $\gamma > 0$ if and only if

$$(1-\gamma)\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha} + \gamma \frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)}\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha-1}$$
$$\prec q_{k,\delta}(z), \tag{1.8}$$

where $g \in A_p$ is such that

$$q(z) = \frac{I_{\mu+1,p}(a,b,c)g(z)}{I_{\mu,p}(a,b,c)g(z)} \in P(\rho), \quad \rho = \frac{k+\delta}{k+1}, z \in E. \tag{1.9}$$

Furthermore, for different choices of parameters being involved, we obtain many other well-known subclasses of the class A_p and A as special cases.

(i) $a = c$, $b = 1$, $k = 0$, $\mu = m \in \mathbb{N}_0$, we have $B^{\alpha}_{m,p}(\gamma,\delta)$ studied in [8].

(ii) $a = c = b = p = \gamma = 1$, $k = \mu = 0$, $g(z) = z$, the class $UB^{\alpha}_{\mu,p}(a,b,c;\gamma,k,\delta)$ reduces to the class

$$B^{\alpha}(\delta) = \left\{ f \in A(1) : \frac{zf'(z)}{f(z)} \left(\frac{f(z)}{z} \right)^{\alpha} \in P(\delta) \right\}$$

studied in [9].

(iii) $a = c = b = p = \gamma = 1$, $k = \mu = 0$, $g(z) = z$, the $UB^{\alpha}_{\mu,p}(a,b,c;\gamma,k,\delta)$ reduces to $B(\alpha)$ is the class of Bazilevich functions investigated by Singh [10].

(iv) $a = c = b = p = \alpha = 1$, $\gamma = 0$, $k = \mu = 0$, $g(z) = z$, the class $UB^{\alpha}_{\mu,p}(a,b,c;\gamma,k,\delta)$ reduces to the class

$$B_{\delta} = \left\{ f \in A(1) : \frac{f(z)}{z} \in P(\delta) \right\},$$

the class studied by Chen [11].

Let $f \in A_p$. and $F_{\eta,p} : A_p. \to A_p$. be defined by

$$F_{\eta,p}(z) = \frac{(\eta+p)}{z^{\eta}} \int_0^z t^{\eta-1} f(t)\,dt, \quad \eta > -p. \tag{1.10}$$

We need the following lemmas which will be used in our main results.

Lemma 1.2 [12] *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$, and let $\psi : D \subset \mathbb{C}^2 \to \mathbb{C}$ be a complex-valued function satisfying the conditions*:

(i) *$\psi(u,v)$ is continuous in a domain $D \subset \mathbb{C}^2$,*

(ii) *$(1,0) \in D$ and $\psi(1,0) > 0$,*

(iii) *$\operatorname{Re}\psi(iu_2, v_1) \leq 0$, whenever $(iu_2, v_1) \in D$ and $v_1 \leq -\frac{1}{2}(1+u_2^2)$.*

If $h(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is analytic in E such that $(h, zh') \in D$ and $\operatorname{Re}\psi(h(z), zh'(z)) > 0$ for $z \in E$, then $\operatorname{Re} h(z) > 0$.

Lemma 1.3 [13] *Let h be convex in the unit disc E, and let $A \geq 0$. Suppose that $B(z)$ is analytic in E with $\operatorname{Re} B(z) \geq A$. If g is analytic in E and $g(0) = h(0)$. Then*

$$Az^2 g''(z) + B(z)zg'(z) + g(z) \prec h(z) \quad \textit{implies that} \quad g(z) \prec h(z).$$

Lemma 1.4 [14] *Let F be analytic and convex in E. If $f, g \in A_p$ and $f, g \prec F$. Then*

$$\sigma f + (1-\sigma)g \prec F, \quad 0 \leq \sigma \leq 1.$$

Lemma 1.5 [15] *Let h be convex in E with $h(0) = a$ and $\beta \in \mathbb{C}$ such that* $\operatorname{Re}\beta \geq 0$. *If $p \in H[a,n]$ and*

$$p(z) + \frac{zp'(z)}{\beta} \prec h(z),$$

then $p(z) \prec q(z) \prec h(z)$, where

$$q(z) = \frac{\beta}{nz^{\beta/n}} \int_0^z h(t)t^{\beta/n-1}\,dt$$

and $q(z)$ is the best dominant.

2 Main results

Theorem 2.1 *Let $f \in UB^{\alpha}_{\mu,p}(a,b,c;\gamma,k,\delta)$ for $a,b,c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $\mu > -p$, $p \in \mathbb{N}$, $\alpha > 0$, $k \geq 0$, $0 \leq \delta < 1$ and $\gamma > 0$. Then $f \in UB^{\alpha}_{\mu,p}(a,b,c;0,k,\delta)$.*

Proof Consider

$$h(z) = \left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha}, \tag{2.1}$$

where h is analytic in E with $h(0) = 1$, and $g \in A_p$ satisfies condition (1.9). Differentiating (2.1) logarithmically and using (1.7), we have

$$\frac{zh'(z)}{h(z)} = \alpha(\mu + p)\left\{\left(\frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)f(z)} - \frac{I_{\mu+1,p}(a,b,c)g(z)}{I_{\mu,p}(a,b,c)g(z)}\right)\right\}.$$

Using (2.1) and simplifying, we obtain

$$\begin{aligned} h(z) + \frac{\gamma zh'(z)}{\alpha(\mu + p)q(z)} &= (1-\gamma)\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha} \\ &\quad + \gamma\frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)}\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha-1}. \end{aligned}$$

Since $f \in UB^{\alpha}_{\mu,p}(a,b,c;\gamma,k,\delta)$, therefore, we can write

$$h(z) + \frac{\gamma zh'(z)}{\alpha(\mu + p)q(z)} \prec q_{k,\delta}(z), \quad z \in E.$$

Now using Lemma 1.3 for $A = 0$ and $B(z) = \frac{\gamma}{\alpha(\mu+p)q(z)}$ with $\operatorname{Re} q(z) > 0$, we have $\operatorname{Re} B(z) \geq 0$, therefore, $h(z) \prec q_{k,\delta}(z)$. Hence $f \in UB^{\alpha}_{\mu,p}(a,b,c;0,k,\delta)$. □

Theorem 2.2 *Let $f \in UB^{\alpha}_{\mu,p}(a,b,c;\gamma,0,\delta)$ for $a,b,c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $\mu > -p$, $p \in \mathbb{N}$. Then $f \in UB^{\alpha}_{\mu,p}(a,b,c;0,0,\delta_1)$, where*

$$\delta_1 = \frac{2\alpha\delta(p + \mu)|q(z)|^2 + \gamma\rho}{2\alpha(p + \mu)|q(z)|^2 + \gamma\rho}.$$

Proof Consider

$$h(z) = \frac{1}{(1-\delta_1)}\left\{\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha} - \delta_1\right\}, \tag{2.2}$$

where h is analytic in E with $h(0) = 1$, and $g \in A_p$ satisfies condition (1.9). Differentiating (2.2), we have

$$\begin{aligned}\frac{(1-\delta_1)}{\alpha}h'(z) &= \left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha-1} \\ &\quad\times\left\{\frac{(I_{\mu,p}(a,b,c)f(z))'}{I_{\mu,p}(a,b,c)g(z)} - \frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\frac{(I_{\mu,p}(a,b,c)g(z))'}{I_{\mu,p}(a,b,c)g(z)}\right\}.\end{aligned}$$

Using (1.7) and simplifying, we obtain

$$\begin{aligned}&(1-\gamma)\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha} + \gamma\frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)}\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha-1} \\ &\quad= (1-\delta_1)h(z) + \delta_1 + \frac{\gamma(1-\delta_1)zh'(z)}{\alpha(p+\mu)q(z)}.\end{aligned}$$

Since $f \in UB^{\alpha}_{\mu,p}(a,b,c;\gamma,0,\delta)$, therefore we have

$$(1-\delta_1)h(z) + \delta_1 + \frac{\gamma(1-\delta_1)zh'(z)}{\alpha(p+\mu)q(z)} \prec \frac{1+(1-2\delta)z}{1-z}, \quad 0 \le \delta < 1, z \in E.$$

This implies that

$$\frac{1}{1-\delta}\left\{(1-\delta_1)h(z) + \delta_1 - \delta + \frac{\gamma(1-\delta_1)zh'(z)}{\alpha(p+\mu)q(z)}\right\} \in q_{0,0}(E) = P.$$

To obtain our desired result, we show that $h \in P$, for $z \in E$. Let $u = u_1 + iu_2$, $v = v_1 + iv_2$, and let $\Psi : D \subset \mathbb{C}^2 \to \mathbb{C}$ be a complex-valued function such that $u = h(z)$, $v = zh'(z)$. Then

$$\Psi(u,v) = (1-\delta_1)u + \delta_1 - \delta + \frac{\gamma(1-\delta_1)v}{\alpha(p+\mu)q(z)}.$$

The first two conditions of Lemma 1.2 are easily verified. To verify the third condition, we consider

$$\begin{aligned}&\operatorname{Re}\Psi(iu_2, v_1) \\ &\quad= \operatorname{Re}\left\{(1-\delta_1)iu_2 + \delta_1 - \delta + \frac{\gamma(1-\delta_1)v_1}{\alpha(p+\mu)q(z)}\right\} \\ &\quad= \delta_1 - \delta + \operatorname{Re}\frac{\gamma(1-\delta_1)v_1}{\alpha(p+\mu)q(z)} \\ &\quad\le \delta_1 - \delta - \operatorname{Re}\frac{\gamma(1-\delta_1)(1+u_2^2)\overline{q(z)}}{2\alpha(p+\mu)|q(z)|^2} \\ &\quad\le \delta_1 - \delta - \frac{\gamma(1-\delta_1)(1+u_2^2)\rho}{2\alpha(p+\mu)|q(z)|^2} = \frac{A+Bu^2}{C},\end{aligned}$$

where $A = 2\alpha(p+\mu)(\delta_1 - \delta)|q(z)|^2 - \gamma\rho(1-\delta_1)$, $B = -\gamma\rho(1-\delta_1) \leq 0$ if $0 \leq \delta_1 < 1$ and $C = 2\alpha(p+\mu)|q(z)|^2 > 0$. From the relation $\delta_1 = \frac{2\alpha\delta(p+\mu)|q(z)|^2+\gamma\rho}{2\alpha(p+\mu)|q(z)|^2+\gamma\rho}$, we have $A \leq 0$. This implies that $\operatorname{Re}\Psi(iu_2, v_1) \leq 0$. Using Lemma 1.2, we have $h \in P$ for $z \in E$. This completes the proof. □

Theorem 2.3 *Let $a,b,c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $\mu > -p$, $p \in \mathbb{N}$, $\alpha > 0$, $k \geq 0$ and $0 \leq \delta < 1$. Then*

$$UB^\alpha_{\mu,p}(a,b,c;\gamma_2,k,\delta) \subset UB^\alpha_{\mu,p}(a,b,c;\gamma_1,k,\delta), \quad 0 \leq \gamma_1 < \gamma_2, z \in E.$$

Proof Since $f \in UB^\alpha_{\mu,p}(a,b,c;\gamma_2,k,\delta)$, therefore, we have

$$\begin{aligned}&(1-\gamma_2)\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^\alpha + \gamma_2 \frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)}\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha-1}\\ &\quad = h_1(z) \prec q_{k,\delta}(z),\end{aligned} \tag{2.3}$$

where $g \in A_p$ satisfies condition (1.9). From Theorem 2.1, we write

$$\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^\alpha = h_2(z) \prec q_{k,\delta}(z), \quad z \in E. \tag{2.4}$$

Now, for $\gamma_1 \geq 0$, we obtain

$$\begin{aligned}&(1-\gamma_1)\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^\alpha + \gamma_1 \frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)}\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha-1}\\ &\quad = \left(1-\frac{\gamma_1}{\gamma_2}\right)\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^\alpha\\ &\qquad + \frac{\gamma_1}{\gamma_2}\left\{(1-\gamma_2)\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^\alpha + \gamma_2 \frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)}\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha-1}\right\}\\ &\quad = \frac{\gamma_1}{\gamma_2}h_1(z) + \left(1-\frac{\gamma_1}{\gamma_2}\right)h_2(z).\end{aligned}$$

Using the convexity of the class of the function $q_{k,\delta}(z)$ and Lemma 1.4, we write

$$\frac{\gamma_1}{\gamma_2}h_1(z) + \left(1-\frac{\gamma_1}{\gamma_2}\right)h_2(z) \prec q_{k,\delta}(z), \quad z \in E,$$

where h_1 and h_2 are given by (2.3) and (2.4), respectively. This implies that $f \in UB^\alpha_{\mu,p}(a,b,c;\gamma_1,k,\delta)$. Hence the proof of the theorem is completed. □

Theorem 2.4 *Let $f \in UB^1_{\mu,p}(a,b,c;\gamma,k,\delta)$, $a,b,c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $\mu > -p$, $p \in \mathbb{N}$, $\alpha > 0$, $k \geq 0$, $\gamma \geq 1$ and $0 \leq \delta < 1$. Then $f \in UB^1_{\mu+1,p}(a,b,c;1,k,\delta)$.*

Proof Since $f \in UB^1_{\mu,p}(a,b,c;\gamma,k,\delta)$, therefore, we have

$$(1-\gamma)\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right) + \gamma \frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)} \prec q_{k,\delta}(z).$$

Now, consider

$$\gamma \frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)} = (1-\gamma)\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right) + \gamma \frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)} + (\gamma-1)\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right).$$

This implies that

$$\frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)} = \frac{1}{\gamma}\left\{(1-\gamma)\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right) + \gamma \frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)}\right\} + \left(1-\frac{1}{\gamma}\right)\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right).$$

Using Theorem 2.1, Lemma 1.4 and the convexity of $q_{k,\delta}(z)$, we have the required result. □

Now, using the operators $I_{\mu,p}(a,b,c)$ and $F_{\eta,p}$ defined by (1.7) and (1.10), respectively, we have

$$z\big(I_p^{\mu}(a,b,c)F_{\eta,p}(f)(z)\big)' = (p+\eta)I_p^{\mu}(a,b,c)f(z) - \eta I_p^{\mu}(a,b,c)F_{\eta,p}(f)(z), \quad \eta > -p. \tag{2.5}$$

Theorem 2.5 *Let $f \in A_p$ and $F_{\eta,p}$ be given by* (1.10). *If*

$$(1-\gamma)\frac{I_{\mu,p}(a,b,c)F_{\eta,p}(f)(z)}{z^p} + \gamma \frac{I_{\mu,p}(a,b,c)(f(z))}{z^p} \prec q_{k,\delta}(z), \quad z \in E, \tag{2.6}$$

with $a,b,c \in \mathbb{R}\backslash\mathbb{Z}_0^-$, $\mu,\eta > -p$, $p \in \mathbb{N}$, $\gamma > 0$, then

$$\frac{I_{\mu,p}(a,b,c)F_{\eta,p}(f(z))}{z^p} \prec h(z) \prec q_{k,\delta}(z), \quad z \in E,$$

where

$$h(z) = \frac{p+\eta}{\gamma z^{(p+\eta)/\gamma}} \int_0^z q_{k,\delta}(z) t^{(p+\eta)/\gamma - 1}\,dt.$$

Proof Let

$$\frac{I_{\mu,p}(a,b,c)F_{\eta,p}(f(z))}{z^p} = h_1(z), \quad z \in E,$$

where h_1 is analytic in E with $h_1(0) = 1$. Then

$$z\big(I_{\mu,p}(a,b,c)F_{\eta,p}\big(f(z)\big)\big)' = pz^p h_1(z) + z^{p+1}h_1'(z).$$

Using (2.5), we have

$$\gamma(p+\eta)\frac{I_{\mu,p}(a,b,c)f(z)}{z^p} - \gamma\eta \frac{I_{\mu,p}(a,b,c)F_{\eta,p}(f)(z)}{z^p} = p\gamma h_1(z) + \gamma z h_1'(z).$$

Thus,

$$(1-\gamma)\frac{I_{\mu,p}(a,b,c)F_{\eta,p}(f(z))}{z^p}+\gamma\frac{I_{\mu,p}(a,b,c)(f(z))}{z^p}=h_1(z)+\gamma\frac{zh_1'(z)}{p+\eta}. \tag{2.7}$$

From (2.7), it follows that

$$h_1(z)+\gamma\frac{zh_1'(z)}{p+\eta}\prec q_{k,\delta}(z),\quad z\in E.$$

Using Lemma 1.5, for $\beta_1=\frac{p+\eta}{\gamma}$, $n=1$ and $a=1$, we obtain $h_1(z)\prec h(z)\prec q_{k,\delta}(z)$. That is, $\frac{I_{\mu,p}(a,b,c)F_{\eta,p}(f(z))}{z^p}\prec q_{k,\delta}(z)$. □

Theorem 2.6 *Let $f\in UB^{\alpha}_{\mu,p}(a,b,c;0,0,\delta)$ for $a,b,c\in\mathbb{R}\backslash\mathbb{Z}_0^-$, $\mu>-p$, $p\in\mathbb{N}$, $\alpha,\gamma>0$, $0\le\delta<1$. Then $f\in UB^{\alpha}_{\mu,p}(a,b,c;\gamma,0,\delta)$, for $|z|<r_0$, where*

$$r_0=\frac{\alpha(p}{+}\mu)+\gamma-\sqrt{\gamma^2+2\alpha\gamma(p+\mu)}\alpha(p+\mu). \tag{2.8}$$

Proof Let $f\in UB^{\alpha}_{\mu,p}(a,b,c;0,0,\delta)$. Then we have

$$\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha}=(1-\delta)h(z)+\delta, \tag{2.9}$$

where $g\in A_p$ satisfies the condition

$$q(z)=\frac{I_{\mu+1,p}(a,b,c)g(z)}{I_{\mu,p}(a,b,c)g(z)}\in P,\quad z\in E$$

and $h\in P$. Differentiating (2.9) and then using (1.7), we obtain

$$\gamma\frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)}\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha-1}-\gamma\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha}=\frac{\gamma(p-\delta)zh'(z)}{\alpha(p+\mu)q(z)}.$$

This implies that

$$\begin{aligned}&\frac{1}{1-\delta}\left\{(1-\gamma)\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha}+\gamma\frac{I_{\mu+1,p}(a,b,c)f(z)}{I_{\mu+1,p}(a,b,c)g(z)}\left(\frac{I_{\mu,p}(a,b,c)f(z)}{I_{\mu,p}(a,b,c)g(z)}\right)^{\alpha-1}-\delta\right\}\\&\quad=h(z)+\frac{\gamma zh'(z)}{\alpha(p+\mu)q(z)},\quad z\in E.\end{aligned} \tag{2.10}$$

Now, using the well-known distortion result for class P, we have

$$\left|zh'(z)\right|\le\frac{2r\operatorname{Re}h(z)}{1-r^2}\quad\text{and}\quad\operatorname{Re}h(z)\ge\frac{1-r}{1+r},\quad |z|<r<1, z\in E.$$

Thus, due to the applications of these inequalities, we have

$$\begin{aligned}\operatorname{Re}\left(h(z)+\frac{\gamma zh'(z)}{\alpha(p+\mu)q(z)}\right) &\geq \operatorname{Re}h(z)-\frac{\gamma|zh'(z)|}{\alpha(p+\mu)|q(z)|}\\ &\geq \operatorname{Re}h(z)\left(1-\frac{2\gamma r}{\alpha(p+\mu)(1-r)^2}\right)\\ &= \operatorname{Re}h(z)\left(\frac{\alpha(p+\mu)(1-r)^2-2\gamma r}{\alpha(p+\mu)(1-r)^2}\right).\end{aligned}$$

For $|z|<r_0$, where r_0 is given in (2.8), the inequality above is positive. Sharpness of the result follows by taking $h(z)=\frac{1+z}{1-z}$. Hence from (2.10), we have the required result. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
MR and SNM jointly discussed and presented the ideas of this article. MR made the text file and corresponded it to the journal. Both authors read and approved the final manuscript.

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