

## CONSTRUCTING IRREDUCIBLE POLYNOMIALS WITH PRESCRIBED LEVEL CURVES OVER FINITE FIELDS

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(Received 14 January 2001 and in revised form 28 March 2001)

ABSTRACT. We use Eisenstein's irreducibility criterion to prove that there exists an absolutely irreducible polynomial  $P(X, Y) \in GF(q)[X, Y]$  with coefficients in the finite field  $GF(q)$  with  $q$  elements, with prescribed level curves  $X_c := \{(x, y) \in GF(q)^2 \mid P(x, y) = c\}$ .

2000 Mathematics Subject Classification. 11T06.

**1. Introduction.** Let  $GF(q)$  be the finite field with  $q$  elements. Assume that for any  $c \in GF(q)$ , a subset  $X_c$  (possibly empty) of the finite affine plane  $GF(q)^2$  is given, such that  $X_c \cap X_d \neq \emptyset$  for any  $c \neq d$  and

$$GF(q)^2 = \bigcup_{c \in GF(q)} X_c. \quad (1.1)$$

In this paper, we use Eisenstein's irreducibility criterion to build absolutely irreducible polynomials

$$P(X, Y) \in GF(q)[X, Y] \quad (1.2)$$

such that for any  $c \in GF(q)$  the level curve  $\{(x, y) \in GF(q)^2 \mid P(x, y) = c\}$  coincides with  $X_c$ . Note that  $P(X, Y) \in GF(q)[X, Y]$  is called absolutely irreducible if it is irreducible over the algebraic closure of  $GF(q)$ .

If we define a function  $f : GF(q)^2 \rightarrow GF(q)$  taking a constant value  $c$  on the set  $X_c$  for any  $c \in GF(q)$ , it is easy to see that this is equivalent to the fact that there exists an absolutely irreducible polynomial which interpolates the function  $f$ .

It is of course well known that there exists a polynomial that interpolates the function  $f$  (see [3, Section 7.5] for a general discussion on this topic). Thus, our result can be viewed as a stronger version of this basic fact, going back to Weber [4].

The basic facts about bivariate polynomial interpolation over finite fields that we will need are summarized in the following theorem.

**THEOREM 1.1.** *Any function  $f : GF(q)^2 \rightarrow GF(q)$  can be interpolated by some polynomial in two variables. Moreover, there exists a unique polynomial  $F(X, Y) \in GF(q)[X, Y]$  of degree less than  $q$  in both  $X$  and  $Y$  that interpolates the function  $f$ , that is, satisfying  $F(a, b) = f(a, b)$  for any  $(a, b) \in GF(q)^2$ . Also, any two interpolating polynomials for  $f$  are congruent modulo the ideal of  $GF(q)[X, Y]$  generated by  $X^q - X$  and  $Y^q - Y$ .*

Our main result is the following theorem.

**THEOREM 1.2.** *Let  $f : GF(q)^2 \rightarrow GF(q)$  be a function. Then there exists an absolutely irreducible polynomial  $P(X, Y) \in GF(q)[X, Y]$  that interpolates the function  $f$ .*

**2. Proof of the main result.** Let  $f : GF(q)^2 \rightarrow GF(q)$  be an arbitrary function. By [Theorem 1.1](#), there exists a unique interpolating polynomial  $H(X, Y) \in GF(q)[X, Y]$  for  $f$ , of degree at most  $q - 1$  in both  $X$  and  $Y$ . We order  $H(X, Y)$  in terms of the powers of  $Y$

$$H(X, Y) = c_0(X) + c_1(X)Y + \dots + c_{q-1}(X)Y^{q-1}, \tag{2.1}$$

where  $c_0(X), c_1(X), \dots, c_{q-1}(X) \in GF(q)[X, Y]$  are of degree at most  $q - 1$ .

Clearly, if we add  $Y^q - Y$  to  $H(X, Y)$ , we still get an interpolating polynomial for  $f$ , say  $K(X, Y)$ , that is, monic in  $Y$ . Thus, it will be perfectly legitimate to start with an interpolating polynomial of the form

$$K(X, Y) = Y^q + d_{q-1}(X)Y^{q-1} + \dots + d_1(X)Y + d_0(X), \tag{2.2}$$

where  $d_0(X), d_1(X), \dots, d_{q-1}(X) \in GF(q)[X, Y]$  are of degree at most  $q - 1$ .

It is well known (see [\[3, Corollary 2.11\]](#)) that there are irreducible polynomials of any degree over a finite field  $GF(q)$ . Fix such an irreducible polynomial  $h(X) \in GF(q)[X]$  of degree 2. Clearly  $h(X)$  has two roots in the algebraic closure of  $GF(q)$ , each of them generating the quadratic extension of  $GF(q)$ . Let  $\alpha$  be a root of  $h(X)$  in  $\overline{GF(q)}$ , the algebraic closure of  $GF(q)$ .

Our construction is based on replacing each polynomial coefficient  $d_i(X)$  of [\(2.2\)](#) with a polynomial of the form

$$e_i(X) = d_i(X) + (X^q - X)u_i(X), \tag{2.3}$$

where  $u_i(X) \in GF(q)[X]$ , such that each  $e_i(X)$  is divisible by  $h(X)$  for  $i = 0, \dots, q - 1$ , while  $e_0(X)$  is not divisible by  $h(X)^2$ . Clearly, the polynomial  $F(X, Y)$  we get by performing these replacements

$$F(X, Y) = Y^q + e_{q-1}(X)Y^{q-1} + \dots + e_1(X)Y + e_0(X) \tag{2.4}$$

will still be an interpolating polynomial for  $f$ , by [Theorem 1.1](#). We will then see that  $F(X, Y)$  follows to be absolutely irreducible.

We prove that for some choice of  $u_i(X) \in GF(q)[X]$  in [\(2.3\)](#),  $e_i(X)$  is divisible by  $h(X)$ , that is,

$$d_i(X) + (X^q - X)u_i(X) \equiv 0 \pmod{h(X)} \tag{2.5}$$

is solvable. Indeed, from the way we defined  $h(X)$ ,  $X^q - X$  is relatively prime to  $h(X)$ . Thus, [\(2.5\)](#) is a linear congruence modulo  $h(X)$  in the Euclidean ring  $GF(q)[X]$  in which the coefficient  $X^q - X$  of the unknown  $u_i(X)$  is relatively prime to the modulus  $h(X)$ . This being the case, a solution  $u_i(X)$  of [\(2.5\)](#) exists, and is uniquely determined up to a multiple of  $h(X)$ . It follows that we can select a solution  $u_i(X)$  of [\(2.5\)](#) which is

a polynomial of degree one. This will completely take care of the cases  $i = 1, \dots, q - 1$ . For the special case  $i = 0$  we are looking for a solution  $u_0(X)$  of (2.5) satisfying the additional requirement

$$d_0(X) + (X^q - X)u_0(X) \not\equiv 0 \pmod{h(X)^2}. \tag{2.6}$$

This can be done as follows. If the solution  $u_0(X)$  of the  $i = 0$  case of (2.5) already satisfies (2.6) there is nothing to prove. Otherwise, if  $u_0(X)$  satisfies

$$d_0(X) + (X^q - X)u_0(X) \equiv 0 \pmod{h(X)^2}, \tag{2.7}$$

just replace  $u_0(X)$  with  $u_0(X) + h(X)$ . This last polynomial will satisfy both (2.5) and (2.6).

The last step in our proof will consist in showing that the polynomial  $F(X, Y)$  constructed above is absolutely irreducible.

The key ingredient of this last step is *Eisenstein's irreducibility criterion* (see [2, Theorem 6.15]), to the effect that if  $P(X) = \gamma_n X^n + \gamma_{n-1} X^{n-1} + \dots + \gamma_1 X + \gamma_0$  is a polynomial with coefficients in some unique factorization domain  $\mathbb{R}$ , if we can find some irreducible element  $p \in \mathbb{R}$  which divides  $\gamma_0, \dots, \gamma_{n-1}$ , does not divide  $\gamma_n$ , while  $p^2$  does not divide  $\gamma_0$ , then  $P(X)$  is an irreducible element of  $R[X]$ .

We view  $F(X, Y)$  as a (monic) polynomial in  $Y$  with coefficients in the unique factorization domain  $\overline{GF(q)}[X]$ , that is,  $F(X, Y) \in (\overline{GF(q)}[X])[Y]$ .

Pick up the irreducible

$$p(X) := X - \alpha \in \overline{GF(q)}[X]. \tag{2.8}$$

Since  $\alpha$  is a root of  $h(X)$ , by the way we constructed  $F(X, Y)$  it follows that  $p(X)$  divides the polynomial coefficients  $e_0(X), e_1(X), \dots, e_{q-1}(X) \in GF(q)[X]$  and  $p(X)^2$  does not divide the free coefficient  $e_0(X)$ . Also, the coefficient of the highest power of  $Y$  in (2.4) is 1. Thus, we can apply now Eisenstein's criterion to conclude that  $F(X, Y)$  is an irreducible element of the polynomial ring  $(\overline{GF(q)}[X])[Y] \cong \overline{GF(q)}[X, Y]$ . In other words, the interpolating polynomial  $F(X, Y)$  for  $f$  is absolutely irreducible. This concludes the proof of our main theorem. □

By our construction, the degrees of the polynomial coefficients  $e_1(X), \dots, e_{q-1}(X)$  of  $F(X, Y)$  are at most  $q + 1$ , the degree of  $e_0(X)$  is at most  $q + 2$ , while  $F(X, Y)$  is monic of degree  $q$  in  $Y$ .

**Theorem 1.2** may be seen as a useful tool in the theory of curves over finite fields, since it allows a fairly elementary and efficient construction of equations of absolutely irreducible plane curves over  $GF(q)$  with a given set  $Z \subset GF(q)^2$  of  $GF(q)$ -rational points (we may, for example, apply our construction to the special case in which the level curves are  $X_0 = Z, X_1 = GF(q)^2 \setminus Z$ , and  $X_c = \emptyset$  for any  $c \in GF(q) \setminus \{0, 1\}$ ). Finally, our interpolation result (with a construction based on a different method, though less direct) still holds true for the case of more than two variables (the proof of this will appear in [1]).

**REFERENCES**

- [1] M. Caragiu, *Multivariate interpolation by absolutely irreducible polynomials over finite fields*, to appear in Rev. Roumaine Math. Pures Appl.
- [2] T. W. Hungerford, *Algebra*, Graduate Texts in Mathematics, vol. 73, Springer-Verlag, New York, 1980, reprint of the 1974 original. [MR 82a:00006](#). [Zbl 442.00002](#).
- [3] R. Lidl and H. Niederreiter, *Finite Fields*, 2nd ed., Encyclopedia of Mathematics and its Applications, vol. 20, Cambridge University Press, Cambridge, 1997. [MR 97i:11115](#). [Zbl 866.11069](#).
- [4] H. Weber, *Lehrbuch der Algebra*, vol. 2, Vieweg & Sohn, Braunschweig, 1899.

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