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ON SEPARATION AXIOMS IN INTUITIONISTIC TOPOLOGICAL SPACES

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ABSTRACT. The purpose of this paper is to investigate several types of separation axioms in intuitionistic topological spaces, developed by Çoker (2000). After giving some characterizations of T_1 and T_2 separation axioms in intuitionistic topological spaces, we give interrelations between several types of separation axioms and some counterexamples.

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1. Introduction. After the introduction of the concept of a fuzzy set by Zadeh [15], Atanassov [1, 2] has introduced the concept of intuitionistic fuzzy set. Later Çoker et al. [4, 5, 8] have defined intuitionistic fuzzy topological spaces, intuitionistic sets, and intuitionistic topological spaces in [6, 9, 12].

2. Preliminaries. First we present the fundamental definitions (see Çoker [4]).

Definition 2.1 (see [4]). Let X be a nonempty fixed set. An intuitionistic fuzzy set (IS for short) A is an object having the form $A = \langle X, A_1, A_2\rangle$, where A_1 and A_2 are subsets of X satisfying $A_1 \cap A_2 = \varnothing$. The set A_1 is called the set of members of A, while A_2 is called the set of nonmembers of A.

Definition 2.2 (see [4]). Let X be a nonempty set and let the IS's A and B be in the form $A = \langle X, A_1, A_2\rangle$, $B = \langle X, B_1, B_2\rangle$, respectively. Furthermore, let $\{A_i : i \in J\}$ be an arbitrary family of IS's in X, where $A_i = \langle X, A_i^{(1)}, A_i^{(2)}\rangle$. Then

(a) $A \subseteq B$ if and only if $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$;
(b) $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$;
(c) $\bar{A} = \langle X, A_2, A_1\rangle$;
(d) $\cup A_i = \langle X, \cup A_i^{(1)}, \cap A_i^{(2)}\rangle$;
(e) $\cap A_i = \langle X, \cap A_i^{(1)}, \cup A_i^{(2)}\rangle$;
(f) $[\ ]A = \langle X, A_1, A_1^c\rangle$;
(g) $\langle\,\rangle A = \langle X, A_2^c, A_2\rangle$;
(h) $\underset{\sim}{\varnothing} = \langle X, \varnothing, X\rangle$; $\underset{\sim}{X} = \langle X, X, \varnothing\rangle$.

Let X be a nonempty set, $p \in X$ a fixed element in X, and let $A = \langle X, A_1, A_2\rangle$ be an IS. The IS $\underset{\sim}{p}$ defined by $\underset{\sim}{p} = \langle X, \{p\}, \{p\}^c\rangle$ is called an intuitionistic point (IP for short) in X. The IS $\underset{\approx}{p} = \langle \varnothing, \{p\}^c\rangle$ is called a vanishing intuitionistic point (VIP for short) in X. The IS $\underset{\sim}{p}$ is said to be contained in A ($\underset{\sim}{p} \in A$ for short) if and only if $p \in A_1$, and similarly, $\underset{\approx}{p}$ is said to be contained in A ($\underset{\approx}{p} \in A$ for short) if and only if $p \notin A_2$. For a given IS A in X, we may write

$$A = (\cup\{\underset{\sim}{p} : \underset{\sim}{p} \in A\}) \cup (\cup\{\underset{\approx}{p} : \underset{\approx}{p} \in A\}), \tag{2.1}$$

(cf. [9]) and whenever A is not a proper IS (i.e., if A is not of the form $A = \langle X, A_1, A_2\rangle$, where $A_1 \cup A_2 \neq X$), then $A = \cup\{\underset{\sim}{p} : \underset{\sim}{p} \in A\}$ follows. In general, any IS A in X can be written in the form $A = \underset{\sim}{A} \cup \underset{\approx}{A}$, where $\underset{\sim}{A} = \cup\{\underset{\sim}{p} : \underset{\sim}{p} \in A\}$ and $\underset{\approx}{A} = \cup\{\underset{\approx}{p} : \underset{\approx}{p} \in A\}$. Furthermore it is easy to show that, if $A = \langle X, A_1, A_2\rangle$, then $\underset{\sim}{A} = \langle X, A_1, A_1^c\rangle$ and $\underset{\approx}{A} = \langle X, \varnothing, A_2\rangle$ (cf. [4, 7]).

Definition 2.3 (see [4]). Let X and Y be two nonempty sets and $f : X \to Y$ a function, $B = \langle Y, B_1, B_2\rangle$ an IS in Y and $A = \langle X, A_1, A_2\rangle$ an IS in X. Then the preimage of B under f, denoted by $f^{-1}(B)$, is the IS in X defined by $f^{-1}(B) = \langle X, f^{-1}(B_1), f^{-1}(B_2)\rangle$, and the image of A under f, denoted by $f(A)$, is the IS in Y defined by $f(A) = \langle Y, f(A_1), f_-(A_2)\rangle$ where $f_-(A_2) = (f(A_2^c))^c$.

You may find the fundamental properties of preimages and images in [4].

Definition 2.4 (see [6]). An intuitionistic topology (IT for short) on a nonempty set X is a family τ of IS's in X containing $\underset{\sim}{\varnothing}$, $\underset{\sim}{X}$ and closed under finite infima and arbitrary suprema. In this case the pair (X,τ) is called an intuitionistic topological space (ITS for short) and any IS in τ is known as an intuitionistic open set (IOS for short) in X. The complement $\bar{A}$ of an IOS A in an ITS (X,τ) is called an intuitionistic closed set (ICS for short) in X.

Let (X,τ) be an ITS on X. Then, we can also construct several other ITS's on X in the following way: $\tau_{0,1} = \{[\ ]G : G \in \tau\}$ and $\tau_{0,2} = \{\langle\ \rangle G : G \in \tau\}$. Furthermore,

$$\tau_1 = \{G_1 : G = \langle X, G_1, G_2\rangle \in \tau\}, \qquad \tau_2 = \{G_2^c : G = \langle X, G_1, G_2\rangle \in \tau\} \tag{2.2}$$

are topological spaces in X (cf. [6]).

Definition 2.5. Let A and B be two IS's on X and Y, respectively. Then the product intuitionistic set (PIS for short) of A and B on $X \times Y$ is defined by $U \times V = \langle (X,Y), A_1 \times B_1, (A_2^c \times B_2^c)^c\rangle$, where $A = \langle X, A_1, A_2\rangle$ and $B = \langle Y, B_1, B_2\rangle$.

If (X,τ) and (Y,Φ) are ITS's, then the product topology $\tau \times \Phi$ on $X \times Y$ is the IT generated by the base $\mathcal{B} = \{A \times B : A \in \tau, B \in \Phi\}$. This is so, because, if $A \times B, C \times D \in \mathcal{B}$, then $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$. Let $A \in \tau$, $B \in \Phi$, and $A = \langle X, A_1, A_2\rangle$, $B = \langle Y, B_1, B_2\rangle$. Then we have $\pi_1^{-1}(A) = \langle (x,y), A_1 \times Y, A_2 \times Y\rangle = A \times \underset{\sim}{Y}$, $\pi_2^{-1}(B) = \langle (X,Y), X \times B_1, X \times B_2\rangle = \underset{\sim}{X} \times B$, and

$$\begin{aligned}\pi_1^{-1}(A) \cap \pi_2^{-1}(B) &= (A \times \underset{\sim}{Y}) \cap (\underset{\sim}{X} \times B)\\ &= \langle (X,Y), (A_1 \times Y) \cap (X \times B_1), (A_2 \times Y) \cup (X \times B_2)\rangle\\ &= \langle (X,Y), A_1 \times B_1, (A_2 \times Y) \cup (X \times B_2)\rangle\\ &= \langle (X,Y), A_1 \times B_1, (A_2^c \times B_2^c)^c\rangle = A \times B.\end{aligned} \tag{2.3}$$

The definition of "neighborhoods" of IP's and VIP's can be found in Coşkun and Çoker [9] and "continuous function" between ITS's can be found in Çoker [6].

Lemma 2.6. *The projections* $\pi_1 : X\times Y \to X$, $\pi_2 : X\times Y \to Y$, $\pi_1(x,y) = x$, $\pi_2(x,y) = y$ *are continuous.*

Proof. Let $A \in \tau$, then $\pi_1^{-1}(A) = \langle (x,y), \pi_1^{-1}(A_1), \pi_1^{-1}(A_2)\rangle$. Thus we have $\pi_1^{-1}(A) = \langle (x,y), A_1\times Y, A_2 \times Y\rangle = A \times \underset{\sim}{Y}$, that is, π_1 is continuous.

In other words, the product topology $\tau\times\Phi$ on $X\times Y$ is indeed the initial topology on $X\times Y$ with respect to the projections $\pi_1 : X\times Y \to X$ and $\pi_2 : X\times Y \to Y$. Here the subbase $\{\pi_1^{-1}(A), \pi_2^{-1}(B) : A\in\tau,\ B\in\Phi\}$ generates this product topology and the base $\mathcal{B}$ is given by

$$\mathcal{B} = \{\pi_1^{-1}(A)\cap\pi_2^{-1}(B) : A\in\tau,\ B\in\Phi\} = \{A\times B : A\in\tau,\ B\in\Phi\}. \tag{2.4}$$

□

Definition 2.7. Given the nonempty set X, we define the diagonal Δ_x as the following IS in $X\times X$:

$$\Delta_x = \langle (x_1,x_2), \{(x_1,x_2) : x_1 = x_2\}, \{(x_1,x_2) : x_1 \neq x_2\}\rangle. \tag{2.5}$$

Notice that, if X and Y are two nonempty sets and $(p,q)\in X\times Y$ a fixed element in $X\times Y$, then $(p,q)_\sim$ is contained in $U\times V((p,q)_\sim \in U\times V$ for short) if and only if $(p,q)\in U_1\times V_1$, and $(p,q)_\approx$ is contained in $U\times V((p,q)_\approx \in U\times V$ for short) if and only if $(p,q)\notin (U_2^c\times V_2^c)^c$, or equivalently $(p,q)\in U_2^c\times V_2^c$.

Definition 2.8. Let X, Y be two nonempty sets and $f : X\to Y$ a function. The graph of f, denoted by $\mathrm{GR}(f)$, is defined as the following IS in $X\times Y$:

$$\mathrm{GR}(f) = \langle (x,y), \{(x,f(x)) : x\in X\}, \{(x,f(x)) : x\in X\}^c\rangle. \tag{2.6}$$

3. Separation axioms in intuitionistic topological spaces. In this section, we present T_1 and T_2 separation axioms in ITS's. The separation axioms T_1 and T_2 presented here have certain similarities to those in Bayhan and Çoker [3].

Definition 3.1. Let (X,τ) be an ITS, (X,τ) is said to be

(a) $T_1(i) \Leftrightarrow \forall x,y\in X\ (x\neq y)\ \exists U,V\in\tau$ such that $\underset{\sim}{x}\in U$, $\underset{\sim}{y}\notin U$, and $\underset{\sim}{y}\in V$, $\underset{\sim}{x}\notin V$ (cf. [3, 14]);

(b) $T_1(ii) \Leftrightarrow \forall x,y\in X\ (x\neq y)\ \exists U,V\in\tau$ such that $\underset{\approx}{x}\in U$, $\underset{\approx}{y}\notin U$, and $\underset{\approx}{y}\in V$, $\underset{\approx}{x}\notin \underset{\sim}{x}\in V$ (cf. [3, 14]);

(c) $T_1(iii) \Leftrightarrow \forall x,y\in X\ (x\neq y)\ \exists U,V\in\tau$ such that $\underset{\sim}{x}\in U\subseteq \underset{\sim}{\bar{y}}$ and $\underset{\sim}{y}\in V\subseteq\underset{\sim}{\bar{x}}$ (cf. [3]);

(d) $T_1(iv) \Leftrightarrow \forall x,y\in X\ (x\neq y)\ \exists U,V\in\tau$ such that $\underset{\approx}{x}\in U\subseteq \underset{\approx}{\bar{y}}$ and $\underset{\approx}{y}\in V\subseteq\underset{\approx}{\bar{x}}$ (cf. [3]);

(e) $T_1(v) \Leftrightarrow \forall x,y\in X\ (x\neq y)\ \exists U,V\in\tau$ such that $\underset{\sim}{y}\notin U$ and $\underset{\sim}{x}\notin V$ (cf. [3]);

(f) $T_1(vi) \Leftrightarrow \forall x,y\in X\ (x\neq y)\ \exists U,V\in\tau$ such that $\underset{\approx}{y}\notin U$ and $\underset{\approx}{x}\notin V$ (cf. [3]);

(g) $T_1(vii) \Leftrightarrow \forall x\in X$, $\underset{\sim}{x}$ is τ-closed;

(h) $T_1(viii) \Leftrightarrow \forall x\in X$, $\underset{\approx}{x}$ is τ-closed.

Theorem 3.2. *Let (X,τ) be an ITS, then the following implications are valid:*

$$
\begin{array}{ccccc}
T_1(v) & \longleftarrow & & & T_1(vi) \\
\uparrow & & & & \uparrow \\
T_1(i) & \longleftarrow & T_1(i)+T_1(ii) & \longrightarrow & T_1(ii) \\
 & & \updownarrow & & \downarrow \\
T_1(vii) & \longleftarrow & T_1(iii) & & T_1(iv)
\end{array}
\tag{3.1}
$$

Proof. The proof is obvious. □

Counterexample 3.3. Let $X=\{a,b,c\}$ and define the IT $\tau=\{\underset{\sim}{\emptyset},\underset{\sim}{X},A,B,C,D,E,F,G\}$, where $A=\langle X,\{a,c\},\emptyset\rangle$, $B=\langle X,\{b\},\emptyset\rangle$, $C=\langle X,\{a\},\emptyset\rangle$, $D=\langle X,\{c\},\emptyset\rangle$, $E=\langle X,\{a,b\},\emptyset\rangle$, $F=\langle X,\{b,c\},\emptyset\rangle$, $G=\langle X,\emptyset,\emptyset\rangle$. Then (X,τ) is $T_1(i)$, but not $T_1(ii)$.

Counterexample 3.4. Let $X=\{a,b\}$ and define the IT $\tau=\{\underset{\sim}{\emptyset},\underset{\sim}{X},A,B\}$ on X, where $A=\langle X,\emptyset,\{a\}\rangle$, $B=\langle X,\emptyset,\{b\}\rangle$. Then it is clear that (X,τ) is $T_1(v)$, but not $T_1(i)$.

Counterexample 3.5. Let $X=\{a,b,c\}$ and define the IT $\tau=\{\underset{\sim}{\emptyset},\underset{\sim}{X},A,B,C,D,E,F\}$ on X, where $A=\langle X,\emptyset,\{a,b\}\rangle$, $B=\langle X,\{c\},\{a,b\}\rangle$, $C=\langle X,\emptyset,\{b,c\}\rangle$, $D=\langle X,\{c\},\{b\}\rangle$, $E=\langle X,\{a,c\},\{b\}\rangle$, $F=\langle X,\emptyset,\{b\}\rangle$. Then (X,τ) is $T_1(vi)$, but not $T_1(ii)$.

Counterexample 3.6. Let $X=\{a,b,c\}$ and define the IS's $A=\langle X,\{a\},\{c\}\rangle$, $B=\langle X,\{b\},\{a\}\rangle$, $C=\langle X,\{a\},\{b,c\}\rangle$, $D=\langle X,\emptyset,\{b\}\rangle$, $E=\langle X,\{a,b\},\emptyset\rangle$, $F=\langle X,\emptyset,\{a,c\}\rangle$, $G=\langle X,\emptyset,\{b,c\}\rangle$, $H=\langle X,\{a\},\emptyset\rangle$, $K=\langle X,\{a\},\{b\}\rangle$. Let τ denote the IT on X generated by the subbase $S=\{A,B,C,D,E,F,G,H,K\}$. Then (X,τ) is clearly $T_1(iv)$, but not $T_1(iii)$.

Counterexample 3.7. Let $X=\{a,b,c,d\}$ and consider the family $\tau=\{\underset{\sim}{\emptyset},\underset{\sim}{X},A,B,C,D,E,F,G\}$, where $A=\langle X,\{a\},\emptyset\rangle$, $B=\langle X,\{b\},\{\emptyset\}\rangle$, $C=\langle X,\{c\},\emptyset\rangle$, $D=\langle X,\{a,b\},\emptyset\rangle$, $E=\langle X,\{b,c\},\emptyset\rangle$, $F=\langle X,\{a,b,c\},\emptyset\rangle$, $G=\langle X,\emptyset,\emptyset\rangle$. Then the ITS (X,τ) is $T_1(v)$, but not $T_1(vi)$.

Counterexample 3.8. Let $X=\{a,b,c\}$ and consider the family $\tau=\{\underset{\sim}{\emptyset},\underset{\sim}{X},A,B,C,D,E,F,G,H,K\}$, where $A=\langle X,\{a\},\{c\}\rangle$, $B=\langle X,\{b\},\emptyset\rangle$, $C=\langle X,\{c\},\emptyset\rangle$, $D=\langle X,\{a,b\},\emptyset\rangle$, $E=\langle X,\{a,c\},\emptyset\rangle$, $F=\langle X,\{b,c\},\emptyset\rangle$, $G=\langle X,\emptyset,\{c\}\rangle$, $H=\langle X,\emptyset,\emptyset\rangle$, $K=\langle X,\{a\},\emptyset\rangle$. Then the ITS (X,τ) on X is $T_1(i)$, but not $T_1(iii)$.

Counterexample 3.9. Let $X=\{a,b,c\}$ and consider the family $\tau=\{\underset{\sim}{\emptyset},\underset{\sim}{X},A,B,C,D,E,F,G\}$, where $A=\langle X,\{a,c\},\emptyset\rangle$, $B=\langle X,\{b,c\},\emptyset\rangle$, $C=\langle X,\{b\},\emptyset\rangle$, $D=\langle X,\{a,b\},\emptyset\rangle$, $E=\langle X,\{c\},\emptyset\rangle$, $F=\langle X,\{a\},\emptyset\rangle$, $G=\langle X,\emptyset,\emptyset\rangle$. Then the ITS (X,τ) on X is $T_1(iv)$, but not $T_1(ii)$.

Counterexample 3.10 (see [6]). Let $X = \mathbb{N}^+$ and consider the IS's A_n given below:

$$
\begin{aligned}
A_1 &= \langle X, \{2,3,4,\dots\}, \varnothing\rangle, \\
A_2 &= \langle X, \{3,4,5,\dots\}, \{1\}\rangle, \\
A_3 &= \langle X, \{4,5,6,\dots\}, \{1,2\}\rangle, \\
A_n &= \langle X, \{n+1, n+2, n+3,\dots\}, \{1,2,3,\dots,n-1\}\rangle \quad (n \geq 2).
\end{aligned}
\tag{3.2}
$$

Then $\tau = \{\underset{\sim}{\varnothing}, \underset{\sim}{X}\} \cup \{A_n : n = 1,2,3,\dots\}$ is an IT on X. Clearly (X,τ) is $T_1(vi)$, but not $T_1(ii)$.

Proposition 3.11. *Let (X,τ) be an ITS. Then*
(a) (X,τ) *is* $T_1(i)$ *if and only if* (X,τ_1) *is* T_1.
(b) (X,τ) *is* $T_1(ii)$ *if and only if* (X,τ_2) *is* T_1.
(c) (X,τ) *is* $T_1(i)$ *if and only if* $(X,\tau_{0,1})$ *is* $T_1(i)$.
(d) (X,τ) *is* $T_1(ii)$ *if and only if* $(X,\tau_{0,2})$ *is* $T_1(ii)$.

Definition 3.12. Let (X,τ) be an ITS. (X,τ) is said to be

(a) $T_2(i) \Leftrightarrow \forall x,y \in X\ (x \neq y)\ \exists U,V \in \tau$ such that $\underset{\sim}{x} \in U$, $\underset{\sim}{y} \in V$, and $U \cap V = \underset{\sim}{\varnothing}$ (cf. [3, 13]);

(b) $T_2(ii) \Leftrightarrow \forall x,y \in X\ (x \neq y)\ \exists U,V \in \tau$ such that $\underset{\approx}{x} \in U$, $\underset{\approx}{y} \in V$, and $U \cap V = \underset{\sim}{\varnothing}$ (cf. [3, 13]);

(c) $T_2(iii) \Leftrightarrow \forall x,y \in X\ (x \neq y)\ \exists U,V \in \tau$ such that $\underset{\sim}{x} \in U$, $\underset{\sim}{y} \in V$, and $U \subseteq \bar{V}$ (cf. [3, 10]);

(d) $T_2(iv) \Leftrightarrow \forall x,y \in X\ (x \neq y)\ \exists U,V \in \tau$ such that $\underset{\approx}{x} \in U$, $\underset{\approx}{y} \in V$, and $U \subseteq \bar{V}$ (cf. [3, 10]);

(e) $T_2(v) \Leftrightarrow \forall x,y \in X\ (x \neq y)\ \exists U,V \in \tau$ such that $\underset{\sim}{x} \in U \subseteq \bar{\underset{\sim}{y}}$, $\underset{\sim}{y} \in V \subseteq \bar{\underset{\sim}{x}}$, and $U \cap V = \underset{\sim}{\varnothing}$ (cf. [3, 11]);

(f) $T_2(vi) \Leftrightarrow \forall x,y \in X\ (x \neq y)\ \exists U,V \in \tau$ such that $\underset{\approx}{x} \in U \subseteq \bar{\underset{\sim}{y}}$, $\underset{\sim}{y} \in V \subseteq \bar{\underset{\sim}{x}}$, and $U \cap V = \underset{\sim}{\varnothing}$ (cf. [3, 11]);

(g) $T_2(vii) \Leftrightarrow \Delta_X$ is an ICS in the product ITS $(X \times X, \tau_{X\times X})$.

Theorem 3.13. *Let (X,τ) be an ITS. Then the following implications are valid:*

$$
\begin{array}{ccccc}
 & & T_2(v) & \longrightarrow & T_2(vi) \\
 & & \downarrow & & \downarrow \\
T_2(vii) & \longleftarrow & T_2(i) & \longrightarrow & T_2(ii) \\
 & & \downarrow & & \downarrow \\
 & & T_2(iii) & \longrightarrow & T_2(iv)
\end{array}
\tag{3.3}
$$

Proof. We prove only the case $T_2(i) \Rightarrow T_2(vii)$. We must see that $\bar{\Delta}_X$ is an IOS in $(X \times X, \tau_{X\times X})$. Let $(x,y)_\sim \in \bar{\Delta}_X$. This means that $(x,y) \in \{(x,y) : x \neq y\}$, that is, $x \neq y$. Since (X,τ) is $T_2(i)$, there exist $U,V \in \tau$ such that $\underset{\sim}{x} \in U$, $\underset{\sim}{y} \in V$, and $U \cap V = \underset{\sim}{\varnothing}$. Now in this case we have $(x,y)_\sim \in U \times V \subseteq \bar{\Delta}_X$. Indeed, from $x \in U_1$ and $y \in V_1$ we get

$(x,y) \in U_1 \times V_1$, that is, $(x,y)_{\sim} \in U \times V$. We also know that $U \times V \subseteq \bar{\Delta}_X \Leftrightarrow U_1 \times V_1 \subseteq \{(x,y) : x \neq y\}$ and $(U_2^c \times V_2^c)^c \supseteq \{(x,y) : x = y\}$. If $(y_1,y_2) \in U_1 \times V_1$, then $y_1 \in U_1$, $y_2 \in V_1 \Rightarrow y_1 \neq y_2 \Rightarrow (y_1,y_2) \in \{(x,y) : x \neq y\}$ follows. Thus the first inclusion is true. For the second, $(y_1,y_2) \in U_2^c \times V_2^c \Rightarrow y_1 \in U_2^c$ and $y_2 \in V_2^c \Rightarrow y_1 \neq y_2$, that is, we have $U_2^c \times V_2^c \subseteq \{(x,y) : x \neq y\}$. Thus we see that $(y_1,y_2) \in \{(x,y) : x = y\}$. The second inclusion is true, too. Now since

$$\bar{\Delta}_X = \bigcup_{(y_1,y_2)_{\sim} \in \bar{\Delta}_X} (y_1,y_2)_{\sim}, \tag{3.4}$$

it follows from the fact that $\bar{\Delta}_X$ is not a proper IS, that $\bar{\Delta}_X$ is an IOS in $(X \times X)$, that is, (X,τ) is $T_2(vii)$. □

Counterexample 3.14. Let $X = \{a,b\}$ and consider the family $\tau = \{\underset{\sim}{\varnothing}, \underset{\sim}{X}, A, B\}$ on X, where $A = \langle X, \varnothing, \{b\}\rangle$, $B = \langle X, \varnothing, \{a\}\rangle$. Then the ITS (X,τ) on X is $T_2(ii)$, but not $T_2(i)$.

Counterexample 3.15. Let $X = \{a,b,c\}$ and define the IS's $A = \langle X, \varnothing, \{b,c\}\rangle$, $B = \langle X, \{b\}, \{a\}\rangle$, $C = \langle X, \{a\}, \{c\}\rangle$, and $D = \langle X, \varnothing, \{a,b\}\rangle$. Let τ denote the IT on X generated by the subbase $S = \{A,B,C,D\}$. Then (X,τ) is $T_2(iv)$, but not $T_2(iii)$

Counterexample 3.16. Let $X = \{a,b,c\}$ and consider the family $\tau = \{\underset{\sim}{\varnothing}, \underset{\sim}{X}, A, B, C, D, E, F, G, H, K, L, M\}$ on X, where $A = \langle X, \varnothing, \{b\}\rangle$, $B = \langle X, \varnothing, \{a,c\}\rangle$, $C = \langle X, \{a\}, \{b,c\}\rangle$, $D = \langle X, \varnothing, \{a\}\rangle$, $E = \langle X, \varnothing, \{a,b\}\rangle$, $F = \langle X, \varnothing, \{c\}\rangle$, $G = \langle X, \{a\}, \{c\}\rangle$, $H = \langle X, \{a\}, \varnothing\rangle$, $K = \langle X, \{a\}, \{b\}\rangle$, $L = \langle X, \varnothing, \{b,c\}\rangle$, and $M = \langle X, \varnothing, \varnothing\rangle$. Then the ITS (X,τ) on X is $T_2(vi)$, but not $T_2(v)$.

Counterexample 3.17. Let $X = \{a,b,c,d\}$ and define the IS's $A = \langle X, \{a\}, \{b\}\rangle$, $B = \langle X, \{b\}, \{a,d\}\rangle$, $C = \langle X, \{b\}, \{c\}\rangle$, $D = \langle X, \{c\}, \{a,b\}\rangle$, $E = \langle X, \{a\}, \{d\}\rangle$, $F = \langle X, \{d\}, \{a\}\rangle$, $G = \langle X, \{b\}, \{d\}\rangle$, $H = \langle X, \{d\}, \{b\}\rangle$, $K = \langle X, \{c\}, \{d\}\rangle$, $L = \langle X, \{d\}, \{c\}\rangle$, $M = \langle X, \{a\}, \{c\}\rangle$, and $N = \langle X, \{c\}, \{a\}\rangle$. Let τ denote the IT on X generated by the subbase $S = \{A,B,C,D,E,F,G,H,K,L,M,N\}$. Then (X,τ) is $T_2(iii)$, but not $T_2(i)$.

Counterexample 3.18. Let $X = \{a,b\}$ and consider the family $\tau = \{\underset{\sim}{\varnothing}, \underset{\sim}{X}, A, B\}$ on X, where $A = \langle X, \{b\}, \varnothing\rangle$, $B = \langle X, \varnothing, \{b\}\rangle$. Then the ITS (X,τ) on X is $T_2(iv)$, but not $T_2(ii)$.

Counterexample 3.19. We consider the IT on X as in Counterexample 3.15. (X,τ) is $T_2(iv)$, but not $T_2(i)$.

Counterexample 3.20. We consider the ITS on X as in Counterexample 3.14. (X,τ) is $T_2(ii)$, but not $T_2(v)$.

Proposition 3.21. *Let (X,τ) be an ITS. Then*
(a) (X,τ) is $T_2(i) \Rightarrow (X,\tau_1)$ is T_2.
(b) (X,τ) is $T_2(ii) \Rightarrow (X,\tau_2)$ is T_2.

Proposition 3.22. *Let (X,τ) be an ITS. Then*
(a) (X,τ) is $T_2(i) \Rightarrow (X,\tau_{0,1})$ is $T_2(i)$.
(b) (X,τ) is $T_2(ii) \Rightarrow (X,\tau_{0,2})$ is $T_2(ii)$.

Theorem 3.23. *Let (X,τ) be an ITS. Then the following implications are valid:*
(a) $T_2(i) \Rightarrow T_1(iii)$.
(b) $T_2(ii) \Rightarrow T_1(ii)$.
(c) $T_2(iii) \Rightarrow T_1(iii)$.
(d) $T_2(iv) \Rightarrow T_1(iv)$.
(e) $T_2(v) \Rightarrow T_1(iii)$.
(f) $T_2(vi) \Rightarrow T_1(vi)$.

Proof. The proof is obvious. □

Proposition 3.24. *Let (X,τ) be $T_2(i)$. Then every intuitionistic point $\underset{\sim}{x}$ is the intersection of all the intuitionistic closed neighborhoods of $\underset{\sim}{x}$.*

Proof. Let (X,τ) be $T_2(i)$ and $x \in X$. We denote the intersection of IC neighborhoods of $\underset{\sim}{x}$ by the IS $C = \langle X, C_1, C_2\rangle$. We assume the contrary and suppose that there exists a distinct IP $\underset{\sim}{y}$ in C, that is, $y \in C_1$.

Case 1. $\{x\} \subsetneq C_1$, then there exists $y \in C_1$ such that $x \neq y$. Since (X,τ) is $T_2(i)$, there exist IOS's U and V such that $\underset{\sim}{x} \in U$, $\underset{\sim}{y} \in V$, and $U \cap V = \underset{\sim}{\varnothing}$ which implies that $U \subseteq \bar{V}$. Hence we have $\underset{\sim}{x} \in U \subseteq \bar{V}$. Thus $\bar{V}$ is a closed neighborhood of $\underset{\sim}{x}$. From our assumption, we get $\underset{\sim}{y} \in \bar{V}$. But it is a contradiction, since $V_1 \cap V_2 = \varnothing$. Thus our assumption is false. This means that C consists only of the IP $\underset{\sim}{x}$.

Case 2. $\{x\} \subsetneq C_2^c$ and $\{x\} = C_1$, then there exists $y \in C_2^c$ such that $y \neq x$. Since (X,τ) is $T_2(i)$, there exist IOS's $U, V \in \tau$ such that $\underset{\sim}{x} \in U$, $\underset{\sim}{y} \in V$, and $U \cap V = \underset{\sim}{\varnothing}$ and the same result as in the previous assumption holds in this case, too. □

Proposition 3.25. *Let (X,τ) be an ITS, (Y,Φ) a $T_2(i)$ ITS and $f : (X,\tau) \to (Y,\Phi)$ a continuous function. Then the graph of f is an ICS in $X \times Y$.*

Proof. We must show that $\overline{\mathrm{GR}(f)}$ is an IOS in $X \times Y$. Let $(x,y)_\sim \in \overline{\mathrm{GR}(f)}$. Then $(x,y) \in \{(x,f(x)) : x \in X\}^c$ which implies that $y \neq f(x)$. Since (Y,Φ) is $T_2(i)$, there exist $U,V \in \Phi$ such that $\underset{\sim}{y} \in U$, $f(\underset{\sim}{x}) \in V$, and $U \cap V = \underset{\sim}{\varnothing}$. From the assumption that f is continuous, we see that $f^{-1}(V) = \langle X, f^{-1}(V_1), f^{-1}(V_2)\rangle$ is an open neighborhood of $\underset{\sim}{x}$. Also $f^{-1}(V) \times U$ is an open neighborhood of $(x,y)_\sim$. It can be shown easily that $f^{-1}(V) \times U \subseteq \overline{\mathrm{GR}(f)}$. Since $\overline{\mathrm{GR}(f)}$ is not a proper IS in $X \times Y$, our assumption holds, that is, $\overline{\mathrm{GR}(f)}$ is an IOS in $X \times Y$. □

Proposition 3.26. *Let (X,τ) be an ITS, (Y,Φ) a $T_2(i)$ ITS and $f : (X,\tau) \to (Y,\Phi)$ a continuous function. Then the IS $C = \langle (x_1,x_2), \{(x_1,x_2) : f(x_1) = f(x_2)\}, \{(x_1,x_2) : f(x_1) \neq f(x_2)\}\rangle$ in $X \times Y$ is an ICS in $X \times Y$.*

Proof. A similar argument as in the proof of Proposition 3.25 can be followed. □

Proposition 3.27. *Let (X,τ) and (Y,Φ) be two ITS's. Then*
(a) If (X,τ) and (Y,Φ) are $T_1(i)$, then so is $(X \times Y, \tau \times \Phi)$.
(b) If (X,τ) and (Y,Φ) are $T_1(ii)$, then so is $(X \times Y, \tau \times \Phi)$.

Proof. (a) Let (X,τ) and (Y,Φ) be $T_1(i)$. Let $(x_1,y_1),(x_2,y_2)\in X\times Y$ and $(x_1,y_1)\neq(x_2,y_2)$. Now suppose that $x_1\neq x_2$. Since (X,τ) is $T_1(i)$ then there exist $U,V\in\tau$ such that $\underset{\sim}{x_1}\in U$, $\underset{\sim}{x_2}\notin U$, and $\underset{\sim}{x_2}\in V$, $\underset{\sim}{x_1}\notin V$. Then we have IOS's $U\times\underset{\sim}{Y}=\langle(X,Y),U_1\times Y,(U_2^c\times\varnothing^c)^c\rangle$ and $V\times\underset{\sim}{Y}=\langle(X,Y),V_1\times Y,(V_2^c\times\varnothing^c)^c\rangle$ in $\tau\times\Phi$ having the properties $(x_1,y_1)_\sim\in U\times\underset{\sim}{Y}$, $(x_2,y_2)_\sim\notin U\times\underset{\sim}{Y}$, and $(x_2,y_2)_\sim\in V\times\underset{\sim}{Y}$, $(x_1,y_1)_\sim\notin V\times\underset{\sim}{Y}$. We can prove the case $y_1\neq y_2$ similarly. Thus we conclude that $(X\times Y,\tau\times\Phi)$ is $T_1(i)$.

(b) Similar to the previous one. □

Proposition 3.28. *Let (X,τ) and (y,Φ) be two ITS's. Then*

(a) If (X,τ) and (Y,Φ) are $T_2(i)$, then so is $(X\times Y,\tau\times\Phi)$.

(b) If (X,τ) and (Y,Φ) are $T_2(ii)$, then so is $(X\times Y,\tau\times\Phi)$.

(c) If (X,τ) and (Y,Φ) are $T_2(iii)$, then so is $(X\times Y,\tau\times\Phi)$.

(d) If (X,τ) and (Y,Φ) are $T_2(vii)$, then so is $(X\times Y,\tau\times\Phi)$.

Proof. (a) Let (X,τ), (Y,Φ) be $T_2(i)$. Let (x_1,y_1), $(x_2,y_2)\in X\times Y$, and $(x_1,y_1)\neq(x_2,y_2)$ and suppose that $x_1\neq x_2$. Since (X,τ) is $T_2(i)$ then there exist $U,V\in\tau$ such that $\underset{\sim}{x_1}\in U$, $\underset{\sim}{x_2}\in V$, and $U\cap V=\underset{\sim}{\varnothing}$. Then we can form the IOS's $U\times\underset{\sim}{Y}=\langle(X,Y),U_1\times Y,(U_2^c\times\varnothing^c)^c\rangle$ and $V\times\underset{\sim}{Y}=\langle(X,Y),V_1\times Y,(V_2^c\times\varnothing^c)^c\rangle$ in $\tau\times\Phi$ which contains $(x_1,y_1)_\sim$ and $(x_2,y_2)_\sim$, respectively. Now we must see that $(U\times\underset{\sim}{Y})\cap(V\times\underset{\sim}{Y})=\underset{\sim}{\varnothing}$. Indeed,

$$\begin{aligned}(U\times\underset{\sim}{Y})\cap(V\times\underset{\sim}{Y})&=\langle(X,Y),(U_1\times Y)\cap(V_1\times Y),(U_2^c\times\varnothing^c)^c\cup(V_2^c\times\varnothing^c)^c\rangle\\&=\langle(X,Y),(U_1\cap V_1)\times(Y\cap Y),[(U_2^c\times Y)\cap(V_2^c\times Y)]^c\rangle\\&=\Big\langle(X,Y),\varnothing\times Y,[(U_2^c)\cap(V_2^c)\times(Y\cap Y)]^c\Big\rangle\\&=\langle(X,Y),\varnothing,X\times Y\rangle=\underset{\sim}{\varnothing}.\end{aligned}\tag{3.5}$$

Thus $(X\times Y,\tau\times\Phi)$ is $T_2(i)$.

(b) Similar to previous one.

(c) Assume that (X,τ) and (Y,Φ) are $T_2(iii)$. Let (x_1,y_1), $(x_2,y_2)\in X\times Y$ and $(x_1,y_1)\neq(x_2,y_2)$. Suppose that $x_1\neq x_2$. Since (X,τ) is $T_2(iii)$, then there exist $U,V\in\tau$ such that $\underset{\sim}{x_1}\in U$, $\underset{\sim}{x_2}\in V$, and $U\subseteq\bar{V}$. Then we have IOS's $U\times\underset{\sim}{Y}=\langle(X,Y),U_1\times Y,(U_2^c\times\varnothing^c)^c\rangle$ and $V\times\underset{\sim}{Y}=\langle(X,Y),V_1\times Y,(V_2^c\times\varnothing^c)^c\rangle$ in $\tau\times\Phi$ containing $(x_1,y_1)_\sim$ and $(x_2,y_2)_\sim$, respectively. Now, it is easy to see that $U\times\underset{\sim}{Y}\subseteq\overline{V\times\underset{\sim}{Y}}$ holds, which is identical to $U_1\times Y\subseteq(V_2^c\times Y)^c$ and $V_1\times Y\subseteq(U_2^c\times Y)^c$. A similar argument holds if $y_1\neq y_2$. Thus we conclude that $(X\times Y,\tau\times\Phi)$ is $T_2(iii)$.

(d) We are to show that $\Delta_{X\times Y}$ is an ICS, that is, $\bar{\Delta}_{X\times Y}$ is an IOS. Since $\bar{\Delta}_{X\times Y}$ is not a proper IS in $X\times Y$, it is sufficient to show that for every $((p_1,q_1),(p_2,q_2))_\sim\in\bar{\Delta}_{X\times Y}$, there exists an IOS S in $(X\times Y)\times(X\times Y)$ such that $((p_1,q_1),(p_2,q_2))_\sim\in S\subseteq\bar{\Delta}_{X\times Y}$. Since $((p_1,q_1),(p_2,q_2))_\sim\in\bar{\Delta}_{X\times Y}$, we get $((p_1,q_1)\neq(p_2,q_2))_\sim$, that is, $p_1\neq p_2$ or $q_1\neq q_2$. Here come three possible cases:

(1) $p_1\neq p_2$, $q_1=q_2$;

(2) $p_1=p_2$, $q_1\neq q_2$;

(3) $p_1\neq p_2$, $q_1\neq q_2$.

Here we show only case (3). Other cases can be proved similarly. Let $p_1\neq p_2$, $q_1\neq q_2$. Since $(p_1,p_2)_\sim\in\bar{\Delta}_X$, $(q_1,q_2)_\sim\in\bar{\Delta}_Y$ and $\bar{\Delta}_X$, $\bar{\Delta}_Y$ are IOS's, $\exists U_1,U_2\in\tau$ and V_1,

$V_2 \in \Phi$ such that $(p_1,p_2)_\sim \in U_1 \times U_2 \subseteq \bar{\Delta}_X$ and $(q_1,q_2)_\sim \in V_1 \times V_2 \subseteq \bar{\Delta}_Y$. We prove that $((p_1,q_1),(p_2,q_2))_\sim \in (U_1 \times V_1) \times (U_2 \times V_2) \subseteq \bar{\Delta}_{X\times Y}$. This can be shown in two steps.

Step 1. The expression $((p_1,q_1),(p_2,q_2))_\sim \in (U_1 \times V_1) \times (U_2 \times V_2)$ is equivalent to $((p_1,q_1),(p_2,q_2)) \in (U_1 \times V_1)^{(1)} \times (U_2 \times V_2)^{(1)} \Leftrightarrow ((p_1,q_1),(p_2,q_2)) \in (U_1^{(1)} \times V_1^{(1)}) \times (U_2^{(1)} \times V_2^{(1)})$. This means that $(p_1,q_1) \in U_1^{(1)} \times V_1^{(1)}$ and $(p_2,q_2) \in U_2^{(1)} \times V_2^{(1)}$ which are true, since $p_1 \in U_1^{(1)}$, $p_2 \in U_2^{(1)}$, $q_1 \in V_1^{(1)}$, $q_2 \in V_2^{(1)}$.

Step 2. We show the inclusion $(U_1 \times V_1) \times (U_2 \times V_2) \subseteq \bar{\Delta}_{X\times Y}$. For this purpose we must first show that $(U_1 \times V_1)^{(1)} \times (U_2 \times V_2)^{(1)} \subseteq \{((u_1,v_1),(u_2,v_2)) : (u_1,v_1) \neq (u_2,v_2)\}$ or equivalently, $(U_1^{(1)} \times V_1^{(1)}) \times (U_2^{(1)} \times V_2^{(1)}) \subseteq \{((u_1,v_1),(u_2,v_2)) : (u_1,v_1) \neq (u_2,v_2)\}$. This is true since $U_1 \times U_2 \subseteq \bar{\Delta}_X$ and $V_1 \times V_2 \subseteq \bar{\Delta}_Y$, we have $U_1^{(1)} \times U_2^{(1)} \subseteq \{(u_1,u_2) : u_1 \neq u_2\}$ and $V_1^{(1)} \times V_2^{(1)} \subseteq \{(v_1,v_2) : v_1 \neq v_2\}$, respectively. Thus the first inclusion is true. The second inclusion can be proved similarly. Hence $\bar{\Delta}_{X\times Y}$ is an IOS, that is, $\bar{\Delta}_{X\times Y}$ is an ICS, which means that $(X,Y,\tau\times\Phi)$ is $T_2(vii)$. □

Remark 3.29. Let (X,τ) and (Y,Φ) be $T_2(iv)$. Then $(X\times Y,\tau\times\Phi)$ may not be $T_2(iv)$.

Here come the reverse implications.

Proposition 3.30. *Let (X,τ) and (Y,Φ) be two ITS's. Then*
(a) If $(X\times Y,\tau\times\Phi)$ is $T_2(i)$, then so are (X,τ) and (Y,Φ).
(b) If $(X\times Y,\tau\times\Phi)$ is $T_2(ii)$, then so are (X,τ) and (Y,Φ).
(c) If $(X\times Y,\tau\times\Phi)$ is $T_2(iii)$, then so are (X,τ) and (Y,Φ).

Proof. The proofs of (a) and (b) are easy. (c) Let $(X\times Y,\tau\times\Phi)$ be $T_2(iii)$, and $x_1 \neq x_2$ $(x_1,x_2 \in X)$. We take a fixed $y \in Y$. Then, since $(x_1,y) \neq (x_2,y)$ and $X\times Y$ is $T_2(iii)$, there exist $U\times Z$ and $V\times T$ where $U,V \in \tau$ and $Z,T \in \Phi$ such that $(x_1,y)_\sim \in U\times Z$, $(x_2,y)_\sim \in V\times T$, and $U\times Z \subseteq \overline{V\times T}$. Thus we get $(x_1,y) \in U_1\times Z_1$, $(x_2,y) \in V_1\times T_1$, and $U_1\times Z_1 \subseteq (V_2^c\times T_2^c)^c$, $V_1\times T_1 \subseteq (U_2^c\times Z_2^c)^c$; in other words $x_1 \in U_1$, $y \in Z_1$, $x_2 \in V_1$, $y \in T_1$, and $(U_1\times Z_1)\cap(V_2^c\times T_2^c) = \varnothing, (V_1\times T_1)\cap(U_2^c\times Z_2^c) = \varnothing$. From the last intersection we get $(U_1^c\times V_2^c)\times(Z_1\cap T_2^c) = \varnothing$ and $(V_1\cap U_2^c)\times(T_1\cap Z_2^c) = \varnothing$, respectively. $y \in Z_1$ and $y \in T_1$ implies that $Z_1\cap T_2^c \neq \varnothing$ and $U_1\cap V_2^c = \varnothing$ from which $U_1 \subseteq V_2$ follows. Similarly $y \in T_1\cap Z_2^c$ and $V_1\cap U_2^c = \varnothing$ meaning that $V_1 \subseteq U_2$. Thus $\underset{\sim}{x_1} \in U$, $\underset{\sim}{x_2} \in V$, and $U \subseteq \bar{V}$, that is, (X,τ) is $T_2(iii)$. Similarly (Y,Φ) is $T_2(iii)$, too. □

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