COMBINATORIAL INTEGERS $\binom{m,n}{j}$ AND SCHUBERT CALCULUS IN THE INTEGRAL COHOMOLOGY RING OF INFINITE SMOOTH FLAG MANIFOLDS

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We discuss the calculation of integral cohomology ring of LG/T and ΩG. First we describe the root system and Weyl group of LG, then we give some homotopy equivalences on the loop groups and homogeneous spaces, and calculate the cohomology ring structures of LG/T and ΩG for affine group $\widehat{A}_2$. We introduce combinatorial integers $\binom{m,n}{j}$ which play a crucial role in our calculations and give some interesting identities among these integers. Last we calculate generators for ideals and rank of each module of graded integral cohomology algebra in the local coefficient ring $\mathbb{Z}[1/2]$.

1. Introduction

Kumar described the Schubert classes which are the dual to the closures of the Bruhat cells in the flag varieties of the Kac-Moody groups associated to the infinite dimensional Kac-Moody algebras [17]. These classes are indexed by affine Weyl groups and can be chosen as elements of integral cohomologies of the homogeneous space $\widehat{L}_{\text{pol}}G_{\mathbb{C}}/\widehat{B}$ for any compact simply connected semisimple Lie group G. Later, S. Kumar, and B. Kostant described explicit cup product formulas of these classes in the cohomology algebras by using the relation between the invariant-theoretic relative Lie algebra cohomology theory (using the representation module of the nilpotent part) with the purely nil-Hecke rings [16]. These explicit product formulas involve some BGG-type operators A^i and reflections. In the published work [20] of the first author, using some homotopy equivalences, cohomology ring structures of LG/T have been determined where LG is the smooth loop space on G. He has calculated the products and explicit ring structure of LSU_2/T using these ideas. He found that it has a quotient of the divided power algebra. In this work, we list explicit presentation of affine Weyl group of the loop group LSU_3. We calculate generators for ideals and the rank of the modules of graded cohomology algebra of LSU_3/T and ΩSU_3 in the coefficient ring $Z[1/2]$.

Some comments about the structure of this work are in order. It is written for a reader with a first course in algebraic topology and some understanding of the structure of

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compact semisimple Lie groups and their representations, plus symbolic computation and some mathematical maturity. Some good general references are Bredon [2] for topology and geometry, Pressley-Segal [22] for loop groups and their representations, Kac [14] for Kac-Moody algebra theory, Hiller [10] for reflection and Coxeter groups, and Humphreys [11] for Lie algebras and representations.

The organization of this work is as follows.

In Section 2, we describe the root system and Weyl group of LG and we give the group presentation of the affine Weyl group of $\widehat{A}_2$. We classify all elements of affine Weyl group $\widetilde{W}$ and $\widehat{W} = \widetilde{W}/W$ for $\widehat{A}_2$.

In Section 3 some homotopy equivalences between loop groups and homogeneous spaces are given.

Section 4 includes all details about Schubert calculus and cohomology of the flag space G/B for Kac-Moody group G. In this section, we give some facts and results about Kac-Moody Lie algebras and associated groups and the construction of dual Schubert cocycles on the flag spaces by using the relative Lie algebra cohomology tools. The rest of the section includes cup product formula.

In Section 5, we introduce combinatorial integers $\binom{m,n}{j}$ and give some interesting properties of them.

In Section 6, we discuss the calculation of cohomology ring of LG/T. Last using cup product formula we explicitly calculate the cohomology structures of LG/T and ΩG for $\widehat{A}_2$.

2. The root system, Weyl group, and Cartan matrix of the loop group LG

We know from compact simply connected semisimple Lie theory that the complexified Lie algebra $\mathbf{g}_{\mathbb{C}}$ of the compact Lie group G has a decomposition under the adjoint action of the maximal torus T of G. Then, from [11], we have the following.

THEOREM 2.1. *There is a decomposition*

$$\mathbf{g}_{\mathbb{C}} = \mathbf{t}_{\mathbb{C}} \bigoplus_{\alpha} \mathbf{g}_{\alpha}, \tag{2.1}$$

where $\mathbf{g}_0 = \mathbf{t}_{\mathbb{C}}$ *is the complexified Lie algebra of* T *and*

$$\mathbf{g}_{\alpha} = \{\xi \in \mathbf{g}_{\mathbb{C}} : t \cdot \xi = \alpha(t)\xi \ \forall t \in T\}. \tag{2.2}$$

The homomorphisms $\alpha : T \to \mathbb{T}$ for which $\mathbf{g}_{\alpha} \neq 0$ are called the *roots* of G. They form a finite subset of the lattice $\check{T} = \operatorname{Hom}(T, \mathbb{T})$. By analogy, the complexified Lie algebra $L\mathbf{g}_{\mathbb{C}}$ of the loop group LG has a decomposition

$$L\mathbf{g}_{\mathbb{C}} = \bigoplus_{k \in \mathbb{Z}} \mathbf{g}_{\mathbb{C}} \cdot z^k, \tag{2.3}$$

where $\mathbf{g}_{\mathbb{C}}$ is the complexified Lie algebra of G. This is the decomposition into eigenspaces of the rotation action of the circle group $\mathbb{T}$ on the loops. The rotation action commutes with the adjoint action of the constant loops G, and from [22], we have the following.

Theorem 2.2. *There is a decomposition of $L\mathbf{g}_{\mathbb{C}}$ under the action of the maximal torus T of G,*

$$L\mathbf{g}_{\mathbb{C}} = \bigoplus_{k\in\mathbb{Z}} \mathbf{g}_0 \cdot z^k \oplus \bigoplus_{(k,\alpha)} \mathbf{g}_\alpha \cdot z^k. \tag{2.4}$$

The pieces in this decomposition are indexed by homomorphisms

$$(k,\alpha) : \mathbb{T} \times T \longrightarrow \mathbb{T}. \tag{2.5}$$

The homomorphisms $(k,\alpha) \in \mathbb{Z} \times \check{T}$ which occur in the decomposition are called the *roots* of LG.

Definition 2.3. The set of roots is called the *root system* of LG and is denoted by $\widehat{\Delta}$.

Let δ be $(0,1)$. Then

$$\widehat{\Delta} = \bigcup_{k\in\mathbb{Z}} (\Delta \cup \{0\} + k\delta) = \Delta \cup \{0\} + \mathbb{Z}\delta, \tag{2.6}$$

where Δ is the root system of G. The root system $\widehat{\Delta}$ is the union of real roots and imaginary roots:

$$\widehat{\Delta} = \widehat{\Delta}_{\text{re}} \cup \widehat{\Delta}_{\text{im}}, \tag{2.7}$$

where

$$\begin{gathered} \widehat{\Delta}_{\text{re}} = \{(\alpha,n) : \alpha \in \Delta,\ n \in \mathbb{Z}\}, \\ \widehat{\Delta}_{\text{im}} = \{(0,r) : r \in \mathbb{Z}\}. \end{gathered} \tag{2.8}$$

Definition 2.4. Let the rank of G be l. Then, the set of simple roots of LG is

$$\{(\alpha_i,0) : \alpha_i \in \Sigma \text{ for } 1 \le i \le l\} \cup \{(-\alpha_{l+1},1)\}, \tag{2.9}$$

where α_{l+1} is the highest weight of the adjoint representation of G.

The root system $\widehat{\Delta}$ can be divided into three parts as the positive and the negative and 0:

$$\widehat{\Delta} = \widehat{\Delta}^+ \cup \{0\} \cup \widehat{\Delta}^-, \tag{2.10}$$

where

$$\widehat{\Delta}^+ = \widehat{\Delta}^+_{\text{re}} \cup \widehat{\Delta}^+_{\text{im}}, \qquad \widehat{\Delta}^- = \widehat{\Delta}^-_{\text{re}} \cup \widehat{\Delta}^-_{\text{im}}, \tag{2.11}$$

where

$$\begin{gathered} \widehat{\Delta}^+_{\text{re}} = \{(\alpha,n) \in \widehat{\Delta}_{\text{re}} : n > 0\} \cup \{(\alpha,0) : \alpha \in \Delta^+\}, \\ \widehat{\Delta}^+_{\text{im}} = \{n\delta : n > 0\}, \\ \widehat{\Delta}^-_{\text{re}} = -\widehat{\Delta}^+_{\text{re}}, \qquad \widehat{\Delta}^-_{\text{im}} = -\widehat{\Delta}^+_{\text{im}}. \end{gathered} \tag{2.12}$$

In the case of LSU_n, for $n \geq 3$, the root system $\widehat{\Delta}$ of the loop group LSU_n has basic elements $\mathbf{a}_0 = (-\alpha_0, 1)$ and $\mathbf{a}_i = (\alpha_i, 0)$, $1 \leq i \leq n-1$ where α_i is the simple root of SU_n and $\alpha_0 = \sum_{i=1}^{n-1} \alpha_i$. All roots of LSU_n can be written as a sum of the simple roots $\mathbf{a}_i$.

Theorem 2.5 (see [14]). *The set of roots of LSU_n, for $n \geq 3$, is*

$$\widehat{\Delta} = \left\{ k \sum_{r=0}^{i-1} \mathbf{a}_r + l \sum_{r=i}^{j-1} \mathbf{a}_r + k \sum_{r=j}^{n-1} \mathbf{a}_r : |k-l| = 1,\ k \in \mathbb{Z},\ 0 \leq i \leq j \leq n \right\}. \tag{2.13}$$

Corollary 2.6. *The set of positive roots of LSU_n, for $n \geq 3$, is*

$$\widehat{\Delta}^+ = \left\{ k \sum_{r=0}^{i-1} \mathbf{a}_r + l \sum_{r=i}^{j-1} \mathbf{a}_r + k \sum_{r=j}^{n-1} \mathbf{a}_r : |k-l| = 1,\ k \in \mathbb{Z}^+,\ 0 \leq i \leq j \leq n \right\}. \tag{2.14}$$

Corollary 2.7. *The simple roots of LSU_3 are $a_0 = (-\alpha_1 - \alpha_2, 1)$, $a_1 = (\alpha_1, 0)$, $a_2 = (\alpha_2, 0)$, where α_1 and α_2 are the simple roots of compact Lie group SU_3.*

The set of all positive real roots of LSU_3 is

$$\{(\alpha_1, m), (\alpha_2, m), (\alpha_1 + \alpha_2, m), (-\alpha_1, s), (-\alpha_2, s), (-\alpha_1 - \alpha_2, s) : m \geq 0,\ s > 0\}. \tag{2.15}$$

Now, we will discuss the Weyl group of the loop group LG. In order to define this group, we need a larger group structure. We define the semidirect product $\mathbb{T} \ltimes LG$ of $\mathbb{T}$ and LG in which $\mathbb{T}$ acts on LG by the rotation. From [22], we have the following.

Theorem 2.8. *$\mathbb{T} \times T$ is a maximal abelian subgroup of $\mathbb{T} \ltimes LG$.*

Theorem 2.9. *The complexified Lie algebra of $\mathbb{T} \ltimes LG$ has a decomposition*

$$(\mathbb{C} \oplus \mathbf{t}_{\mathbb{C}}) \oplus \left(\bigoplus_{k \neq 0} \mathbf{t}_{\mathbb{C}} \cdot z^k \oplus \bigoplus_{(k,\alpha)} \mathbf{g}_\alpha \cdot z^k \right), \tag{2.16}$$

according to the characters of $\mathbb{T} \times T$.

We know that the roots of G are permuted by the Weyl group W. This is the group of automorphisms of the maximal torus T which arise from conjugation in G, that is, $W = N(T)/T$, where

$$N(T) = \{n \in G : nTn^{-1} = T\} \tag{2.17}$$

is the normalizer of T in G. Exactly in the same way, the infinite set of roots of LG is permuted by the Weyl group $\widetilde{W} = N(\mathbb{T} \times T)/(\mathbb{T} \times T)$, where $N(\mathbb{T} \times T)$ is the normalizer in $\mathbb{T} \ltimes LG$. The group $\widetilde{W}$ is called the *affine Weyl group*.

Proposition 2.10. *The affine Weyl group $\widetilde{W}$ is the semidirect product of the coweight lattice $T^\vee = \mathrm{Hom}(\mathbb{T}, T)$ by the Weyl group W of G.*

We know that the Weyl group W of G acts on the Lie algebra of the maximal torus T. It is a finite group of isometries of the Lie algebra $\mathbf{t}$ of the maximal torus T. It preserves the coweight lattice $T^\vee$. For each simple root α, the Weyl group W contains an element r_α of order two represented by $\exp((\pi/2)(e_\alpha + e_{-\alpha}))$ in $N(T)$. Since the roots α can be considered as the linear functionals on the Lie algebra $\mathbf{t}$ of the maximal torus T, the action of r_α on $\mathbf{t}$ is given by

$$r_\alpha(\xi) = \xi - \alpha(\xi)h_\alpha \quad \text{for } \xi \in \mathbf{t}, \tag{2.18}$$

where h_α is the coroot in $\mathbf{t}$ corresponding to simple root α. Also, we can give the action of r_α on the roots by

$$r_\alpha(\beta) = \beta - \alpha(h_\beta)\alpha \quad \text{for } \alpha, \beta \in \mathbf{t}^*, \tag{2.19}$$

where $\mathbf{t}^*$ is the dual vector space of $\mathbf{t}$. The element r_α is the reflection in the hyperplane H_α of $\mathbf{t}$ whose equation is $\alpha(\xi) = 0$. These reflections r_α generate the Weyl group W. For $G = SU_n$, we have from [12] the following.

Theorem 2.11. *The Weyl group of SU_n is the symmetric group S_n.*

Now, we want to describe the Weyl group structure of LG. By analogy with $\mathbb{R}$ for real form, the roots of the loop group LG can be considered as linear forms on the Lie algebra $\mathbb{R} \times \mathbf{t}$ of the maximal abelian group $\mathbb{T} \times T$. The Weyl group $\widetilde{W}$ acts linearly on $\mathbb{R} \times \mathbf{t}$, the action of W is an obvious reflection in the affine hyperplane $1 \times \mathbf{t}$ and the action of $\lambda \in T^\vee$ is given by

$$\lambda \cdot (x, \xi) = (x, \xi + x\lambda). \tag{2.20}$$

Thus, the Weyl group $\widetilde{W}$ preserves the hyperplane $1 \times \mathbf{h}$, and $\lambda \in \check{T}$ acts on it by translation by the vector $\lambda \in T^\vee \subset \mathbf{t}$. If $\alpha \neq 0$, the affine hyperplane $H_{\alpha,k}$ can be defined as follows. For each root (α, k),

$$H_{\alpha,k} = \{\xi \in \mathbf{t} : \alpha(\xi) = -k\}. \tag{2.21}$$

We know that the Weyl group W of G is generated by the reflections r_α in the hyperplanes H_α for the simple roots α. A corresponding statement holds for the affine Weyl group $\widetilde{W}$.

Proposition 2.12. *Let G be a simply connected semisimple compact Lie group. Then the Weyl group $\widetilde{W}$ of the loop group LG is generated by the reflections in the hyperplanes $H_{\alpha,k}$. The affine Weyl group $\widetilde{W}$ acts on the root system $\widehat{\Delta}$ by*

$$r_{(\alpha,k)}(\gamma, m) = (r_\alpha(\gamma), m - \alpha(h_\gamma)k) \quad \text{for } (\alpha, k), (\gamma, m) \in \widehat{\Delta}. \tag{2.22}$$

Proposition 2.13. *The Weyl group of LSU_n is the semidirect product $S_n \ltimes \mathbb{Z}^{n-1}$, where S_n acts by permutation action on coordinates of $\mathbb{Z}^{n-1}$.*

Actually the symmetric group S_n acts on $\mathbb{Z}^n$ by the permutation action, and $\mathbb{Z}^{n-1}$ is the fixed subgroup which corresponds to the eigenvalue action.

By Proposition 2.13, the Weyl group of LSU_3 is $S_3 \ltimes \mathbb{Z}^2$. Moreover, we explicitly give the group presentation of $S_3 \ltimes \mathbb{Z}^2$.

Proposition 2.14. *The Weyl group $\widetilde{W}$ of LSU_3 is isomorphic to the group defined by the presentation*

$$\{r_{a_i} : r_i^2 = 1,\ r_i r_j r_i = r_j r_i r_j,\ i \neq j,\ i,j = 0,1,2\}. \tag{2.23}$$

Proposition 2.15. *All elements of the Weyl group $\widetilde{W}$ of LSU_3 are classified as in the following matrices:*

$$\begin{pmatrix} \left(r_{a_i}r_{a_j}r_{a_k}\right)^n \\ \left(r_{a_i}r_{a_j}r_{a_k}\right)^n r_{a_i} \\ \left(r_{a_i}r_{a_j}r_{a_k}\right)^n r_{a_i}r_{a_j} \end{pmatrix}, \qquad \begin{pmatrix} \left(r_{a_i}r_{a_k}r_{a_j}\right)^n \\ \left(r_{a_i}r_{a_k}r_{a_j}\right)^n r_{a_i} \\ \left(r_{a_i}r_{a_k}r_{a_j}\right)^n r_{a_i}r_{a_k} \end{pmatrix},$$
$$\begin{pmatrix} \left(r_{a_i}r_{a_k}r_{a_j}\right)^{n_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{n_2} \\ \left(r_{a_i}r_{a_k}r_{a_j}\right)^{n_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{n_2} r_{a_j} \\ \left(r_{a_i}r_{a_k}r_{a_j}\right)^{n_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{n_2} r_{a_j}r_{a_k} \\ r_{a_j}\left(r_{a_i}r_{a_k}r_{a_j}\right)^{n_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{n_2} \\ r_{a_j}\left(r_{a_i}r_{a_k}r_{a_j}\right)^{n_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{n_2} r_{a_j} \\ r_{a_j}\left(r_{a_i}r_{a_k}r_{a_j}\right)^{n_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{n_2} r_{a_j}r_{a_k} \\ r_{a_k}r_{a_j}\left(r_{a_i}r_{a_k}r_{a_j}\right)^{n_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{n_2} \\ r_{a_k}r_{a_j}\left(r_{a_i}r_{a_k}r_{a_j}\right)^{n_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{n_2} r_{a_j} \\ r_{a_k}r_{a_j}\left(r_{a_i}r_{a_k}r_{a_j}\right)^{n_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{n_2} r_{a_j}r_{a_k} \end{pmatrix} \tag{2.24}$$

for every $\tau, \sigma \in S_3$, and $n, n_1, n_2 \in \mathbb{N}$.

Proof. If each class in the entries is acted by each reflection r_{a_i} from the left and right sides, by the relations in Proposition 2.14, we get new classes which are similar to one of the classes above. □

From [10], we have the following.

Theorem 2.16. *The affine Weyl group $\widetilde{W}$ of LG is a Coxeter group.*

We will give some properties of the affine Weyl group $\widetilde{W}$.

Definition 2.17. The *length* of an element $w \in \widetilde{W}$ is the least number of factors in the decomposition relative to the set of the reflections $\{r_{\mathbf{a}_i}\}$, and it is denoted by $\ell(w)$.

Definition 2.18. Let $w_1, w_2 \in \widetilde{W}$, $\gamma \in \Delta_{\text{re}}^+$. Then $w_1 \xrightarrow{\gamma} w_2$ indicates the fact that

$$r_\gamma w_1 = w_2, \qquad \ell(w_2) = \ell(w_1) + 1. \tag{2.25}$$

We put $w \leqslant w'$ if there is a chain

$$w = w_1 \longrightarrow w_2 \longrightarrow \cdots \longrightarrow w_k = w'. \tag{2.26}$$

The relation $\leqslant$ is called the *Bruhat order* on the affine Weyl group $\widetilde{W}$.

Proposition 2.19. *Let $w \in \widetilde{W}$ and let $w = r_{\mathbf{a}_1} r_{\mathbf{a}_2} \cdots r_{\mathbf{a}_l}$ be the reduced decomposition of w. If $1 \leq i_1 < \cdots < i_k \leq l$ and $w' = r_{\mathbf{a}_{i_1}} r_{\mathbf{a}_{i_2}} \cdots r_{\mathbf{a}_{i_k}}$, then $w' \leqslant w$. If $w' \leqslant w$, then w' can be represented as above for some indexing set $\{i_\xi\}$. If $w' \to w$, then there is a unique index i, $1 \leq i \leq l$ such that*

$$w' = r_{\mathbf{a}_1} \cdots r_{\mathbf{a}_{i-1}} r_{\mathbf{a}_{i+1}}. \tag{2.27}$$

The last proposition gives an alternative definition of the Bruhat ordering on $\widetilde{W}$.

Proposition 2.20. *In the Weyl group $\widetilde{W}$ of LSU_3, the number of elements with length s is $3s$.*

Proof. The proof will be done for the following cases:

$$s \equiv \begin{cases} 0 & \bmod 3, \\ 1 & \bmod 3, \\ 2 & \bmod 3. \end{cases} \tag{2.28}$$

Let $w \in \widetilde{W}_{LSU_3}$ be an element with length s.

For $s \equiv 0 \bmod 3$, there exists $k \in \mathbb{Z}^+$ such that $s = 3k$ and by Proposition 2.14, we have elements

$$\left(\begin{gathered} A^k_{ijk,0} = \left(r_{a_i} r_{a_j} r_{a_k}\right)^k, \\ B^k_{ikj,0} = \left(r_{a_i} r_{a_k} r_{a_j}\right)^k, \\ C^{k_1,k_2}_{ikj,0} = \left(r_{a_i} r_{a_k} r_{a_j}\right)^{k_1} \left(r_{a_i} r_{a_k} r_{a_i}\right) \left(r_{a_j} r_{a_k} r_{a_i}\right)^{k_2}, \\ D^{l_1,l_2}_{ikj,0} = r_{a_j} \left(r_{a_i} r_{a_k} r_{a_j}\right)^{l_1} \left(r_{a_i} r_{a_k} r_{a_i}\right) \left(r_{a_j} r_{a_k} r_{a_i}\right)^{l_2} r_{a_j} r_{a_k}, \\ E^{n_1,n_2}_{ikj,0} = r_{a_k} r_{a_j} \left(r_{a_i} r_{a_k} r_{a_j}\right)^{n_1} \left(r_{a_i} r_{a_k} r_{a_i}\right) \left(r_{a_j} r_{a_k} r_{a_i}\right)^{n_2} r_{a_j} \end{gathered}\right) \tag{2.29}$$

such that $[ijk] \in S_3$, and

$$\begin{gathered} k_1 + k_2 = k - 1, \\ l_1 + l_2 = k - 2, \\ n_1 + n_2 = k - 2, \\ 0 \leq k_1, \qquad k_2 \leq k - 1, \\ 0 \leq l_1, \qquad l_2 \leq k - 2, \\ 0 \leq n_1, \qquad n_2 \leq k - 2. \end{gathered} \tag{2.30}$$

There are 6 elements of the first- and second-kind classes, $3k$ elements of the third-kind class, $3k-3$ elements of the fourth-kind class, and $3k-3$ elements of the last class. So we have totally $9k = 3s$ elements with length $s = 3k$.

For $s \equiv 1 \mod 3$, there exists $k \in \mathbb{Z}^+$ such that $s = 3k+1$ and by Proposition 2.14, we have elements

$$\begin{pmatrix} A^{k}_{ijk,1} = \left(r_{a_i}r_{a_j}r_{a_k}\right)^{k} r_{a_i}, \\ B^{k}_{ikj,1} = \left(r_{a_i}r_{a_k}r_{a_j}\right)^{k} r_{a_i}, \\ C^{k_1,k_2}_{ikj,1} = \left(r_{a_i}r_{a_k}r_{a_j}\right)^{k_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{k_2} r_{a_j}, \\ D^{l_1,l_2}_{ikj,1} = r_{a_j}\left(r_{a_i}r_{a_k}r_{a_j}\right)^{l_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{l_2}, \\ E^{n_1,n_2}_{ikj,1} = r_{a_k}r_{a_j}\left(r_{a_i}r_{a_k}r_{a_j}\right)^{n_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{n_2} r_{a_j}r_{a_k} \end{pmatrix} \tag{2.31}$$

such that $[ijk] \in S_3$, and

$$\begin{aligned} k_1 + k_2 &= k-1, \\ l_1 + l_2 &= k-1, \\ n_1 + n_2 &= k-2, \\ 0 \le k_1, &\qquad k_2 \le k-1, \\ 0 \le l_1, &\qquad l_2 \le k-1, \\ 0 \le n_1, &\qquad n_2 \le k-2. \end{aligned} \tag{2.32}$$

There are 6 elements of the first- and second-kind classes, $3k$ elements of the third-kind class, $3k$ elements of the fourth-kind class, and $3k-3$ elements of the last class. So we have totally $9k+3 = 3s$ elements with length $s = 3k+1$.

For $s \equiv 2 \mod 3$, there exists $k \in \mathbb{Z}^+$ such that $s = 3k+2$ and by Proposition 2.14, we have elements

$$\begin{pmatrix} A^{k}_{ijk,2} = \left(r_{a_i}r_{a_j}r_{a_k}\right)^{k} r_{a_i}r_{a_j}, \\ B^{k}_{ikj,2} = \left(r_{a_i}r_{a_k}r_{a_j}\right)^{k} r_{a_i}r_{a_k}, \\ C^{k_1,k_2}_{ikj,2} = \left(r_{a_i}r_{a_k}r_{a_j}\right)^{k_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{k_2} r_{a_j}r_{a_k}, \\ D^{l_1,l_2}_{ikj,2} = r_{a_j}\left(r_{a_i}r_{a_k}r_{a_j}\right)^{l_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{l_2} r_{a_j}, \\ E^{n_1,n_2}_{ikj,2} = r_{a_k}r_{a_j}\left(r_{a_i}r_{a_k}r_{a_j}\right)^{n_1}\left(r_{a_i}r_{a_k}r_{a_i}\right)\left(r_{a_j}r_{a_k}r_{a_i}\right)^{n_2} \end{pmatrix} \tag{2.33}$$

such that $[ijk] \in S_3$, and

$$
\begin{aligned}
k_1 + k_2 &= k - 1, \\
l_1 + l_2 &= k - 1, \\
n_1 + n_2 &= k - 1, \\
0 \le k_1, &\qquad k_2 \le k - 1, \\
0 \le l_1, &\qquad l_2 \le k - 1, \\
0 \le n_1, &\qquad n_2 \le k - 1.
\end{aligned}
\tag{2.34}
$$

There are 6 elements of the first- and second-kind classes, $3k$ elements of the third-kind class, $3k$ elements of the fourth-kind class, and $3k$ elements of the last class. So we have totally $9k + 6 = 3s$ elements with length $s = 3k + 2$.

Then there are totally $3s$ elements with length s. □

Now we will define the subset $\widehat{W}$ of the affine Weyl group $\widetilde{W}$ which will be used in the text later. We know that the Weyl group $\widetilde{W}$ of the loop group LG is a split extension $T^\vee \to \widetilde{W} \to W$, where W is the Weyl group of the compact group Lie group G. Since the Weyl group W is a sub-Coxeter system of the affine Weyl group $\widetilde{W}$, we can define the set of cosets $\widetilde{W}/W$.

Lemma 2.21. *The subgroup of $\widetilde{W}$ fixing 0 is the Weyl group W.*

Corollary 2.22. *Let $w, w' \in \widetilde{W}$. Then, $w(0) = w'(0)$ if and only if $wW = w'W$ in $\widetilde{W}/W$.*

By the last corollary, the map $\widetilde{W}/W \to T^\vee$ given by $wW \to w(0)$ is well defined and has an inverse map given by $\chi_i \to r_{\alpha_i}W$, so the coset set $\widetilde{W}/W$ is identified to $T^\vee$ as a set. We have from [1] the following.

Theorem 2.23. *Each coset in $\widetilde{W}/W$ has a unique element of the minimal length.*

We will write $\overline{\ell(w)}$ for the minimal length element occuring in the coset wW, for $w \in \widetilde{W}$. We see that each coset wW, where $w \in \widetilde{W}$, has two distinguished representatives which are not in general the same. Let the subset $\widehat{W}$ of the affine Weyl group $\widetilde{W}$ be the set of the minimal representative elements $\overline{\ell(w)}$ in the coset wW for each $w \in \widetilde{W}$. The subset $\widehat{W}$ has the Bruhat order since it identifies the set of the minimal representative elements $\overline{\ell(w)}$. As an example, we calculate the subset $\widehat{W}$ of the Weyl group of LSU_3. Our aim is to find the minimal representative elements $\overline{\ell(w)}$ in the right coset wW for each element $w \in \widetilde{W}$, where

$$
\begin{aligned}
\widetilde{W} &= \{r_{a_i} : r_i^2 = 1,\ r_i r_j r_i = r_j r_i r_j,\ i \ne j,\ i, j = 0, 1, 2\}, \\
W &= \{r_{a_i} : r_i^2 = 1,\ r_i r_j r_i = r_j r_i r_j,\ i \ne j,\ i, j = 1, 2\}.
\end{aligned}
\tag{2.35}
$$

We have the minimal representative elements $\overline{\ell(w)}$ for each coset wW, $w \in \widetilde{W}$ as follows.

For $s \equiv 0 \mod 3$, there exists $k \in \mathbb{Z}^+$ such that $s = 3k$ and by Proposition 2.14, we have elements

$$A^{k}_{012,0},\ B^{k}_{021,0},\ C^{k_1,k_2}_{021,0},\ D^{l_1,l_2}_{210,0},\ E^{n_1,n_2}_{102,0} \tag{2.36}$$

such that

$$\begin{gathered} k_1 + k_2 = k - 1, \quad k_1 = \text{odd}, \\ l_1 + l_2 = k - 2, \quad l_1 = \text{even}, \\ n_1 + n_2 = k - 2, \quad n_1 = \text{odd}, \\ 0 \le k_1, \qquad k_2 \le k - 1, \\ 0 \le l_1, \qquad l_2 \le k - 2, \\ 0 \le n_1, \qquad n_2 \le k - 2. \end{gathered} \tag{2.37}$$

There are 2 elements of the first- and second-kind classes, $\lfloor k/2 \rfloor$ elements of the third-kind class, $\lfloor k/2 \rfloor$ elements of the fourth-kind class, and $\lfloor k/2 \rfloor - 1$ many elements of the last class if k is an even otherwise $\lfloor k/2 \rfloor$ elements of the last class. So we have totally $3\lfloor k/2 \rfloor + 1 = \lfloor s/2 \rfloor + 1$ if k is even otherwise $3\lfloor k/2 \rfloor + 2 = \lfloor s/2 \rfloor + 1$ elements with length $s = 3k$.

For $s \equiv 1 \mod 3$, there exists $k \in \mathbb{Z}^+$ such that $s = 3k + 1$ and by Proposition 2.14, we have elements $A^{k}_{012,1}$, $B^{k}_{021,1}$, $C^{k_1,k_2}_{021,1}$, $D^{l_1,l_2}_{210,1}$, $E^{n_1,n_2}_{102,1}$ such that

$$\begin{gathered} k_1 + k_2 = k - 1, \quad k_1 = \text{odd}, \\ l_1 + l_2 = k - 1, \quad l_1 = \text{even}, \\ n_1 + n_2 = k - 2, \quad n_1 = \text{odd}, \\ 0 \le k_1, \qquad k_2 \le k - 1, \\ 0 \le l_1, \qquad l_2 \le k - 1, \\ 0 \le n_1, \qquad n_2 \le k - 2. \end{gathered} \tag{2.38}$$

There are 2 elements of the first- and second-kind classes, $\lfloor k/2 \rfloor$ elements of the third-kind class. If k is even, there are $\lfloor k/2 \rfloor$ elements of the fourth-kind class, and $\lfloor k/2 \rfloor - 1$ elements of the last class. If k is odd, there are $\lfloor k/2 \rfloor + 1$ elements of the fourth-kind class and $\lfloor k/2 \rfloor$ elements of the last class. So we have totally $3\lfloor k/2 \rfloor + 1 = \lfloor s/2 \rfloor + 1$ if k is even otherwise $3\lfloor k/2 \rfloor + 3 = \lfloor s/2 \rfloor + 1$ elements with length $s = 3k + 1$.

For $s \equiv 2 \mod 3$, there exists $k \in \mathbb{Z}^+$ such that $s = 3k + 2$ and by Proposition 2.14, we have elements $A^{k}_{012,2}$, $B^{k}_{021,2}$, $C^{k_1,k_2}_{021,2}$, $D^{l_1,l_2}_{210,2}$, $E^{n_1,n_2}_{102,2}$ such that

$$\begin{gathered} k_1 + k_2 = k - 1, \quad k_1 = \text{odd}, \\ l_1 + l_2 = k - 1, \quad l_1 = \text{even}, \\ n_1 + n_2 = k - 1, \quad n_1 = \text{odd}, \\ 0 \le k_1, \qquad k_2 \le k - 1, \\ 0 \le l_1, \qquad l_2 \le k - 1, \\ 0 \le n_1, \qquad n_2 \le k - 1. \end{gathered} \tag{2.39}$$

There are 2 elements of the first- and second-kind classes, $\lfloor k/2 \rfloor$ elements of the third-kind class, $\lfloor k/2 \rfloor$ elements of the fifth-kind class and $\lfloor k/2 \rfloor$ elements of the last class if k is an even otherwise $\lfloor k/2 \rfloor + 1$ many elements of the fourth class. So we have totally $3\lfloor k/2 \rfloor + 2 = \lfloor s/2 \rfloor + 1$ if k is even otherwise $3\lfloor k/2 \rfloor + 3 = \lfloor s/2 \rfloor + 1$ elements with length $s = 3k + 2$. Then we have the subset $\widehat{W} = \{\overline{\ell(w)} : w \in \widetilde{W}\}$.

PROPOSITION 2.24. *In the Weyl group $\widehat{W} = \widetilde{W}/W$ of LSU_3, the number of elements with length s is $\lfloor s/2 \rfloor + 1$.*

Now we will describe the Lie algebra $L_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ and its universal central extension in terms of generators and relations. For a finite dimensional semisimple Lie algebra $\mathbf{g}_{\mathbb{C}}$, we can choose a nonzero element e_α in $\mathbf{g}_\alpha$ for each root α. From [11], we have the following.

THEOREM 2.25. *$\mathbf{g}_{\mathbb{C}}$ is a Kac-Moody Lie algebra generated by $e_i = e_{\alpha_i}$ and $f_i = e_{-\alpha_i}$ for $i = 1,\dots,l$ where the elements α_i are the simple roots and l is the rank of $\mathbf{g}_{\mathbb{C}}$ only if G is semisimple.*

Let us choose generators e_j and f_j of $L\mathbf{g}_{\mathbb{C}}$ corresponding to simple affine roots. Since $\mathbf{g}_{\mathbb{C}} \subset L\mathbf{g}_{\mathbb{C}}$, we can take

$$e_j = \begin{cases} z e_{-\alpha_0} & \text{for } j = 0, \\ e_i & \text{for } 1 \le j \le l, \end{cases}$$
$$f_j = \begin{cases} z^{-1} e_{\alpha_0} & \text{for } j = 0, \\ f_i & \text{for } 1 \le j \le l, \end{cases} \tag{2.40}$$

where α_0 is the highest root of the adjoint representation. From [22], we have the following.

THEOREM 2.26. *Let $\mathbf{g}_{\mathbb{C}}$ be a semisimple Lie algebra. Then, $L_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ is generated by the elements e_j and f_j corresponding to simple affine roots.*

The Cartan matrix $A_{(l+1)\times(l+1)}$ of $L\mathbf{g}_{\mathbb{C}}$ has the Cartan integers $a_{ij} = \mathbf{a}_j(h_{\mathbf{a}_i})$ as entries where $\mathbf{a}_0 = -\alpha_0$, and $\mathbf{a}_j = \alpha_j$ if $1 \le j \le l$. As an example, we have the following.

PROPOSITION 2.27. *Let $G = SU_3$. The Cartan matrix $A_{3\times 3}$ of $L\mathbf{g}_{\mathbb{C}}$ is the symmetric matrix*

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}. \tag{2.41}$$

Although the relations of the Kac-Moody algebra hold in $L_{\text{pol}}\mathbf{g}_{\mathbb{C}}$, they do not define it. By a theorem of Gabber and Kac [6], the relations define the universal central extension $\widehat{L}_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ of $L_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ by $\mathbb{C}$ which is described by the cocycle ω_K given by

$$\omega_K(\xi,\eta) = \frac{1}{2\pi}\int_0^{2\pi} \sigma(\xi(\theta),\eta'(\theta))\,d\theta. \tag{2.42}$$

As a vector space $\widehat{L}_{\text{pol}}\mathbf{g}_{\mathbb{C}}$ is $L_{\text{pol}}\mathbf{g}_{\mathbb{C}} \oplus \mathbb{C}$ and the bracket is given by

$$[(\xi,\lambda),(\eta,\mu)] = ([\xi,\eta],\omega_K(\xi,\eta)). \tag{2.43}$$

Theorem 2.28. *$\hat{L}\mathbf{g}_{\mathbb{C}}$ is an affine Kac-Moody algebra.*

3. Some homotopy equivalences for the loop group LG and its homogeneous spaces

From [8], we have the following.

Theorem 3.1. *The compact group G is a deformation retract of $G_{\mathbb{C}}$, and so the loop space LG is homotopic to the complexified loop space $LG_{\mathbb{C}}$.*

Now, we want to give a major result from [8].

Theorem 3.2. *The inclusion*

$$\iota : L_{\text{pol}}G_{\mathbb{C}} \longrightarrow LG_{\mathbb{C}} \tag{3.1}$$

is a homotopy equivalence.

Now we will give some useful notations. The parabolic subgroup P of $L_{\text{pol}}G_{\mathbb{C}}$ is the set of maps $\mathbb{C} \to G_{\mathbb{C}}$ which have nonnegative Laurent series expansions. Then $P = G_{\mathbb{C}}[z]$. The minimal parabolic subgroup B is the Iwahori subgroup

$$\{f \in P : f(0) \in \overline{B}\}, \tag{3.2}$$

where $\overline{B}$ is the finite-dimensional Borel subgroup of G. Note also that the minimal parabolic subgroup B corresponds to the positive roots and the parabolic subgroup P to the roots (α, n) with $n \geq 0$. From [8] we have the following.

Theorem 3.3. *The evaluation map at zero $e_0 : P \to G_{\mathbb{C}}$ is a homotopy equivalence with the homotopy inverse which is the inclusion of $G_{\mathbb{C}}$ as the constant loops.*

The following fact follows from the local rigidity of the trivial bundle on the projective line. From [9], we have the following.

Proposition 3.4. *The projection*

$$L_{\text{pol}}G_{\mathbb{C}} \longrightarrow L_{\text{pol}}G_{\mathbb{C}}/P \tag{3.3}$$

is a principal bundle with fiber P.

Now, as a consequence of Theorem 3.2, Proposition 3.4, and Theorem 3.3, we have the following.

Theorem 3.5. *$\Omega G_{\mathbb{C}}$ is homotopy equivalent to $L_{\text{pol}}G_{\mathbb{C}}/P$.*

Theorem 3.6 (see [19]). *The homogeneous space*

$$L_{\text{pol}}G_{\mathbb{C}}/P = \coprod_{w \in \widetilde{W}/W = \widehat{W}} BwP/P. \tag{3.4}$$

Corollary 3.7. *The homogeneous space*

$$L_{\text{pol}}G_{\mathbb{C}}/B = \coprod_{w \in \widetilde{W}} BwB/B. \tag{3.5}$$

4. Cohomology of flag manifolds of Kac-Moody groups

Now we discuss the cohomolgy of flag manifolds of Kac-Moody groups.

Let G be the group associated to the Kac-Moody Lie algebra $\mathbf{g}$. Then G may be of three different types: finite, affine, and wild. The finite type Kac-Moody groups are simply connected semisimple finite dimensional algebraic groups. The affine type Kac-Moody groups are the circle group extension of the group of polynomial maps from $\mathbb{S}^1$ to a group of finite type, or a twisted analogue. There is no concrete realization of the wild type groups. Now, we will introduce some subgroups of the Kac-Moody group G. For $e \in \mathbf{g}_\alpha$, we put $\exp(e) = q(i_\alpha(e))$ so that $U_\alpha = \exp \mathbf{g}_\alpha$ is an additive one parameter subgroup of G. We denote by U (resp., U^-) the subgroup of G generated by the U_α (resp., $U_{-\alpha}$) for $\alpha \in \Delta_+$. For $1 \le i \le l$, there exists a unique homomorphism $\varphi_i : SL_2(\mathbb{C}) \to G$, satisfying $\varphi\left(\begin{smallmatrix} 1 & z \\ 0 & 1 \end{smallmatrix}\right) = \exp(ze_i)$ and $\varphi\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right) = \exp(zf_i)$ for all $z \in \mathbb{C}$. We define

$$H_i = \left\{\varphi\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \in \mathbb{C}^*\right\}; \tag{4.1}$$

$G_i = \varphi(SL_2(\mathbb{C}))$. Let N_i be the normalizer of H_i in G_i, H the subgroup of G generated by all H_i, and N the subgroup of G generated by all N_i. There is an isomorphism $W \to N/H$. We put $B = HU$. B is called *standard Borel subgroup* of G. Also, we can define the negative Borel subgroup B^- as $B^- = HU^-$. G has Bruhat and Birkhoff decompositions. Details can be found in [21]. The conjugate linear involution ω_0 of $\mathbf{g}$ gives an involution $\widetilde{\omega_0}$ on G. Let K denote the set of fixed points of this involution. K is called the *standard real form* of G. Also, this involution preserves the subgroups G_i, H_i, and H; we denote by K_i, T_i, and T, respectively, the corresponding fixed point subgroups. Then, $K_i = \varphi(SU_2)$ and

$$T_i = \left\{\varphi\begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} : |u| = 1\right\} \tag{4.2}$$

is a maximal torus of K_i and $T = \prod T_i$ is a maximal torus in K.

Now, we will give some facts about the topology of K. Let D (resp., D°) be the unit disk (resp., its interior) in $\mathbb{C}$ and let $\mathbb{S}^1$ be the unit circle. Given $u \in D$, let

$$z(u) = \begin{pmatrix} u & \left(1 - |u|^2\right)^{1/2} \\ -\left(1 - |u|^2\right)^{1/2} & \bar{u} \end{pmatrix} \in SU_2, \tag{4.3}$$

and $z_i(u) = \varphi_i(z(u))$. We also set

$$Y_i = \left\{z_i(u) : u \in D^\circ\right\} \subset K_i. \tag{4.4}$$

Let $w = r_{i_1} \cdots r_{i_n}$ be a reduced expression of $w \in W$. We put $Y_w = Y_{i_1} \cdots Y_{i_n}$. We have a fibration $\pi : K \to K/T$. The topological space K/T is called the *flag variety* of the K and G. Now, we will give the topological structure in the infinite dimensional case. We define $C_w = \pi(Y_w)$. From [13], we have the following.

Proposition 4.1. *The decomposition*

$$K/T = \coprod_{w \in W} C_w \tag{4.5}$$

defines a CW structure on K/T.

The closure of C_w is given by

$$\overline{C}_w = \coprod_{w' \leqslant w} C_{w'}. \tag{4.6}$$

The closures $\overline{C}_w$ are called Schubert varieties and they are finite dimensional complex spaces. The infinite type flag variety K/T is the inductive limit of these spaces and by Iwasawa decomposition in [21], we have a homeomorphism $K/T \to G/B$. From [13], we have the following.

Proposition 4.2. *The flag variety K/T is an infinite dimensional complex projective variety.*

Proposition 4.3. *The elements $\overline{C}_w$ are a basic form of free $\mathbb{Z}$-module $H_*(K/T, \mathbb{Z})$.*

Now we will give the construction of the dual Schubert cocycles on the flag variety by using the relative Lie-algebra cohomology tools. This construction was done by Kostant [15] for finite type and extended by Kumar [17] for the Kac-Moody case.

$\mathscr{C}(\mathbf{g},\mathbf{h})$ denotes the standard cochain complex with differential d associated to the Lie algebra pair $(\mathbf{g},\mathbf{h})$ with trivial coefficients where $\mathbf{h}$ is the Cartan subalgebra of the Lie algebra $\mathbf{g}$. That is, $\mathscr{C}(\mathbf{g},\mathbf{h})$ is defined to be $\sum_{s\geq 0} \operatorname{Hom}_{\mathbf{h}}(\Lambda^s(\mathbf{g}/\mathbf{h}),\mathbb{C})$ such that $\mathbf{h}$ acts trivially on $\mathbb{C}$. We define

$$\widetilde{\mathscr{C}} = \sum_{s\geq 0} \widetilde{\mathscr{C}}^s, \tag{4.7}$$

where $\widetilde{\mathscr{C}}^s = \operatorname{Hom}_{\mathbb{C}}(\Lambda^s(\mathbf{g}/\mathbf{h}),\mathbb{C})$. We put the topology of pointwise convergence on $\widetilde{\mathscr{C}}^s$, that is, $f_n \to f$ in $\widetilde{\mathscr{C}}^s$ if and only if $f_n(x) \to f(x)$ in $\mathbb{C}$ with usual topology, for all $x \in \Lambda(\mathbf{g},\mathbf{h})$. From [3], we have the following.

Theorem 4.4. *$\widetilde{\mathscr{C}}^s$ is a complete, Hausdorff, topological vector space with respect to the pointwise topology.*

In [17], a continous map $\widetilde{\partial} : \widetilde{\mathscr{C}}^s \to \widetilde{\mathscr{C}}^{s-1}$ and a cochain map of $\widetilde{b}$ on $\widetilde{\mathscr{C}}$ are defined. We define ∂, b to be the restrictions of $\widetilde{\partial}$ and $\widetilde{b}$ to the subspace $\mathscr{C}(\mathbf{g},\mathbf{h})$. We define the following operators on $\mathscr{C}(\mathbf{g},\mathbf{h})$: $S = d\partial + \partial d$ and $L = b\partial + \partial b$. From [17], we have the following.

Proposition 4.5. $\ker S \oplus \operatorname{im} S = \mathscr{C}$.

Theorem 4.6. *d and ∂ on $\mathscr{C}(\mathbf{g},\mathbf{h})$ are disjoint.*

Proposition 4.7 (Hodge-type decomposition). *Let V be any vector space and let $d,\partial : V \to V$ be two disjoint operators such that $d^2 = \partial^2 = 0$. Further, assume that $\ker S \oplus \operatorname{im} S = V$, where $S = d\partial + \partial d$. Then, $\ker S \to \ker d/\operatorname{im} d$ and $\ker S \to \ker \partial/\operatorname{im}\partial$ are both isomorphisms.*

By the Hodge-type decomposition and Proposition 4.5, we have the following.

Theorem 4.8. *The canonical maps* $\psi_{d,S} : \ker S \to H(\mathscr{C}, d)$ *and* $\psi_{\partial,S} : \ker S \to H(\mathscr{C}, \partial)$ *are both isomorphisms.*

Now, we describe a basis for $\ker L$. We fix $w \in W$ of length s. We define $\Phi_w = w\Delta_- \cap \Delta_+$. Φ_w consists of real roots $\{\gamma_1, \dots, \gamma_s\}$. We pick $y_{\gamma_i} \in \mathbf{g}_{-\gamma_i}$ of unit norm with respect to the form $\{\cdot, \cdot\}$ and let $x_{\gamma_i} = -\omega_0(y_{\gamma_i})$. Let $M_{(w\rho-\rho)}$ be the irreducible $\mathbf{h}$-submodule with the highest weight $(w\rho - \rho)$. By [7, Proposition 2.5], the corresponding highest weight vector is $y_{\gamma_1} \wedge \cdots \wedge y_{\gamma_s}$. There exists a unique element $\overline{h^w} \in [M_{(w\rho-\rho)} \otimes \Lambda^s(\mathbf{n})]$ such that $\overline{h^w} = (2i)^s (y_{\gamma_1} \wedge \cdots \wedge y_{\gamma_s} \wedge x_{\gamma_1} \wedge \cdots \wedge x_{\gamma_s}) \mod P_w \otimes \Lambda^s(\mathbf{n})$, where P_w is the orthogonal complement of $y_{\gamma_1} \wedge \cdots \wedge y_{\gamma_s}$ in $M_{(w\rho-\rho)}$. Using the nondegenerate bilinear form $\langle \cdot \rangle$ on $\mathbf{g}$, we have the embedding

$$e : \bigoplus_{k\geq 0} \Lambda^s(\mathbf{n} \oplus \mathbf{n}^-) \longrightarrow \bigoplus_{k\geq 0} [\Lambda^s(\mathbf{n} \oplus \mathbf{n}^-)]^*, \quad \text{where } \mathbf{n} = \bigoplus_{\alpha\in\Delta^+} \mathbf{g}_\alpha,\ \mathbf{n}^- = \bigoplus_{\alpha\in\Delta^-} \mathbf{g}_\alpha. \tag{4.8}$$

Then $h_w = e(\overline{h_w}) \in \ker L$. These elements $\{h_w\}_{w\in W}$ are a $\mathbb{C}$-basis of $\ker L$. Then, we can define $s^w = \psi_{\partial,S}{}^{-1}([h^w]) \in H(\mathscr{C}, \partial)$. From [16, 17], we have the following.

Theorem 4.9. *Let* $\mathbf{g}$ *be the Kac-Moody Lie algebra, let* G *be the group associated to the Kac-Moody algebra* $\mathbf{g}$*, and let* B *be standard Borel subgroup of* G*. Then*

$$\int_{C_{w'}} s^w = \begin{cases} 0 & \text{if } w \neq w', \\ (4\pi)^{2\ell(w)} \displaystyle\prod_{\nu\in w^{-1}\Delta\cap\Delta_+} \sigma(\rho,\nu)^{-1} & \text{if } w = w'. \end{cases} \tag{4.9}$$

This gives the expression for the d, ∂ harmonic forms $s_0^w = s^w/d_w$ which are dual to the Schubert cells where $d_w = \int_{C_w} s^w$.

Theorem 4.10 (see [18]).

$$H\left(\int\right) : H^*(\mathbf{g}, \mathbf{h}) \longrightarrow H^*(G/B, \mathbb{C}) \tag{4.10}$$

is a graded algebra isomorphism.

Let ε^w denote the image of $s_0{}^w$ by the integral map in the last theorem. These cohomology classes are dual to the closure of the Schubert cells, hence we have the following.

Theorem 4.11. *The elements* ε^w, $w \in W$, *form a basis of the* $\mathbb{Z}$*-module* $H^*(G/B, \mathbb{Z})$.

Let $Q^\vee = \bigoplus_i \mathbb{Z}h_i$, where h_i is coroot, be the coroot lattice and let

$$P = \{\lambda \in \mathbf{h}'^* : \lambda(h_i) \in \mathbb{Z}\} \tag{4.11}$$

be the weight lattice dual to $Q^\vee$. Let $S(P) = \bigoplus_{j\geq 0} S^j(P)$ be the integral symmetric algebra over the lattice P, and $S(P)^+ = \bigoplus_{j>0} S^j(P)$ the augmentation ideal. Given a commutative ring $\mathbb{F}$ with unit, we denote $S(P)_{\mathbb{F}} = S(P)_{\mathbb{F}} \otimes_{\mathbb{Z}} \mathbb{F}$. We define the *characteristic homomorphism* $\psi : S(P) \to H^*(G/B, \mathbb{Z})$ as follows: given $\lambda \in P$, we have the corresponding character of B and the associated line bundle L_λ on G/B. We put $\psi(\lambda) \in H^2(G/B, \mathbb{Z})$ equal to the Chern class of L_λ and we extend this multiplicativity to the whole $S(P)$. We denote by $\psi_{\mathbb{F}}$ the extension of ψ by linearity to $S(P)_{\mathbb{F}}$. In order to describe the properties of $\psi_{\mathbb{F}}$, we define BGG-operator Δ_i for $1 \le i \le l$ on $S(P)$ by

$$\Delta_i(f) = \frac{f - r_i(f)}{\alpha_i} \tag{4.12}$$

and we extend this by linearity to $S(P)_{\mathbb{F}}$.

We will introduce certain operators on cohomology of the flag space G/B which are basic tools in the study of this theory. These operators are extension of action of the BGG-operators Δ_i from the image of ψ to the whole cohomology operators. We know that the Weyl group W acts by right multiplication on K/T and this action induces an action of W on homology and cohomology of flag space. On the other hand, we have a fibration $p_i : K/T \to K/K_iT$ with fiber K_i/T_i. Since the odd degree cohomologies of K_i/T_i and K/K_iT are trivial, then the Leray-Serre spectral sequence of the fibration degenerates after the second term. So, $H^*(K/T, \mathbb{Z})$ is generated by $\operatorname{im} p_i{}^*$, which is r_i invariant, and the element $\psi(\chi_i)$, where χ_i is fundamental weight. We define a $\mathbb{Z}$-linear operator A^i on $H^*(K/T, \mathbb{Z})$ lowering the degree by 2 such that r_i leaves the image of A^i invariant and

$$x - r_i(x) = A^i(x) \cup \psi(\alpha_i) \tag{4.13}$$

for $x \in H^*(K/T, \mathbb{Z})$.

Let ε^w be the dual basis of $H^*(K/T, \mathbb{Z})$. Then we have the following.

Proposition 4.12.

$$r_i(\varepsilon^w) = \begin{cases} \varepsilon^w & \text{if } \ell(r_i w) > \ell(w), \\ \varepsilon^w - \displaystyle\sum_{r_i w \xrightarrow{\gamma} w'} \langle \alpha_i, \gamma \rangle \varepsilon^{w'} & \text{otherwise.} \end{cases} \tag{4.14}$$

Proposition 4.13.

$$A^i(\varepsilon^w) = \begin{cases} \varepsilon^{r_i w} & \text{if } \ell(r_i w) < \ell(w), \\ 0 & \text{otherwise.} \end{cases} \tag{4.15}$$

Now, we will give the cup product formula in the cohomology of G/B where G is a Kac-Moody group.

Theorem 4.14.

$$\varepsilon^u \cdot \varepsilon^v = \sum_{u,v \leqslant w} p_{u,v}^w \varepsilon^w, \tag{4.16}$$

where $p_{u,v}^w$ is a homogeneous polynomial of degree 0 and $\ell(u) + \ell(v) = \ell(w)$.

Theorem 4.15. *Let $u,v \in W$. Denote $w^{-1} = r_{i_1} \cdots r_{i_n}$ as a reduced expression.*

$$p^w_{u,v} = \sum_{\substack{j_1 < \cdots < j_m \\ r_{j_1} \cdots r_{j_m} = v^{-1}}} A_{i_1} \circ \cdots \circ r_{i_{j_1}} \circ \cdots \circ r_{i_{j_m}} \circ \cdots \circ A_{i_n}(\varepsilon^u), \tag{4.17}$$

where $m = \ell(v)$.

Since $\widehat{L}\mathbf{g}_{\mathbb{C}}$ is an affine Kac-Moody algebra, we have the following isomorphism.

Theorem 4.16.

$$H^*(LG/T;\mathbb{C}) \cong H^*(L\mathbf{g}_{\mathbb{C}}, \mathbf{t}_{\mathbb{C}};\mathbb{C}) \cong H^*(\widehat{L}\mathbf{g}_{\mathbb{C}}, \hat{\mathbf{t}}_{\mathbb{C}};\mathbb{C}) \cong H^*(\widehat{L}_{\text{pol}} G_{\mathbb{C}}/\widehat{B};\mathbb{C}). \tag{4.18}$$

Then the $\mathbb{Z}$-cohomology ring of LG/T generated by the strata can be calculated using the cup product formula in Theorem 4.14. In the last section, we will work at an example.

5. Identities on combinatorial integers $\binom{m,n}{j}$

Now we introduce an interesting integer sequence which will play an important role in our calculations. Let

$$\binom{m,n}{j} = \sum_{k=0}^{j} (-1)^k \binom{m}{k}\binom{n}{j-k}, \tag{5.1}$$

where $n,m \geq 0$ and $0 \leq j \leq m+n$, and

$$\binom{m}{k} = \begin{cases} \dfrac{m!}{k!(m-k)!} & \text{if } m \geq k, \\ 0 & \text{if } m < k. \end{cases} \tag{5.2}$$

The generating function of the integer sequence of $\binom{m,n}{j}$ is the function $(1+x)^n(1-x)^m$. By definition of $\binom{m,n}{j}$ we have

$$\begin{aligned} \binom{m,n}{j} &= \sum_{k=0}^{j} (-1)^k \binom{m}{k}\binom{n}{j-k} = \sum_{k=0}^{j} (-1)^k \binom{n}{j-k}\binom{m}{k} \\ &= \sum_{k=0}^{j} (-1)^{k-j} \binom{n}{k}\binom{m}{j-k} = \begin{cases} \binom{n,m}{j} & \text{if } j \text{ even}, \\ -\binom{n,m}{j} & \text{if } j \text{ odd}, \end{cases} \end{aligned} \tag{5.3}$$

and hence $\binom{n,n}{j} = 0$ whenever j is odd.

THEOREM 5.1 (symmetry and antisymmetry). *Let n be a nonnegative integer. For $k = 0,1,2,\dots,n$,*

$$\binom{k,n-k}{k} = \begin{cases} \dbinom{n-k,k}{n-k} & \text{if } n \text{ even}, \\ -\dbinom{n-k,k}{n-k} & \text{if } n \text{ odd}. \end{cases} \tag{5.4}$$

Proof. By definition, for $k = 0,1,2,\dots,n$ we have

$$\begin{aligned} \binom{k,n-k}{k} &= \sum_{i=0}^{k}(-1)^i\binom{k}{i}\binom{n-k}{k-i} \\ &= \sum_{i=0}^{k}(-1)^i\binom{n-k}{k-i}\binom{k}{i} \\ &= \sum_{i=0}^{k}(-1)^i\binom{n-k}{n+i-2k}\binom{k}{k-i} \\ &= \sum_{i=n-2k}^{n-k}(-1)^{i-n+2k}\binom{n-k}{i}\binom{k}{n-k-i} \\ &= \begin{cases} \displaystyle\sum_{i=n-2k}^{n-k}(-1)^i\binom{n-k}{i}\binom{k}{n-k-i} & \text{if } n \text{ even}, \\ \displaystyle -\sum_{i=n-2k}^{n-k}(-1)^i\binom{n-k}{i}\binom{k}{n-k-i} & \text{if } n \text{ odd}. \end{cases} \end{aligned} \tag{5.5}$$

Since for $i < n-2k$, we have $n-k-i > k$ so it follows that $\binom{k}{n-k-i} = 0$ where $i = 0,1,\dots, n-2k-1$. Therefore we have

$$\binom{k,n-k}{k} = \begin{cases} \displaystyle\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}\binom{k}{n-k-i} & \text{if } n \text{ even}, \\ \displaystyle -\sum_{i=0}^{n-k}(-1)^i\binom{n-k}{i}\binom{k}{n-k-i} & \text{if } n \text{ odd}. \end{cases} \tag{5.6}$$

Hence we have the desired result. □

Also we have the following identities.

Theorem 5.2. *Let r, s, l, p be nonnegative integers. Then*

$$(\textit{right-shifting property})\quad \binom{r,s}{l} = \binom{r,s-1}{l} + \binom{r,s-1}{l-1}, \tag{5.7}$$

$$(\textit{left-shifting property})\quad \binom{r,s}{l} = \binom{r-1,s}{l} - \binom{r-1,s}{l-1}, \tag{5.8}$$

$$(\textit{right-shifting expansion})\quad \binom{r,s}{l} = \sum_{i=0}^{l} \binom{r,s-i-1}{l-i}, \tag{5.9}$$

$$(\textit{Vandermonde convolution})\quad \binom{r,s}{l} = \sum_{i=0}^{p} \binom{r,s-p}{l-i}\binom{p}{i}, \tag{5.10}$$

$$2\binom{r-1,s-1}{l} = \binom{r-1,s}{l} + \binom{r,s-1}{l}, \tag{5.11}$$

$$2\binom{r-1,s-1}{l-1} = \binom{r-1,s}{l} - \binom{r,s-1}{l}. \tag{5.12}$$

Proof. First we will prove that (5.7) holds. Then

$$\begin{aligned}\binom{r,s}{l} &= \sum_{i=0}^{l}(-1)^i\binom{r}{i}\binom{s}{l-i} = \sum_{i=0}^{l}(-1)^i\binom{r}{i}\left[\binom{s-1}{l-i} + \binom{s-1}{l-i-1}\right]\\ &= \sum_{i=0}^{l}(-1)^i\binom{r}{i}\binom{s-1}{l-i} + \sum_{i=0}^{l}(-1)^i\binom{r}{i}\binom{s-1}{l-1-i}.\end{aligned} \tag{5.13}$$

Since $\binom{s-1}{-1} = 0$, then

$$\binom{r,s}{l} = \sum_{i=0}^{l}(-1)^i\binom{r}{i}\binom{s-1}{l-i} + \sum_{i=0}^{l-1}(-1)^i\binom{r}{i}\binom{s-1}{l-1-i} = \binom{r,s-1}{l} + \binom{r,s-1}{l-1}. \tag{5.14}$$

Let l be even. Then we have

$$\binom{r,s}{l} = \binom{s,r}{l} = \binom{s,r-1}{l} + \binom{s,r-1}{l-1} = \binom{r-1,s}{l} - \binom{r-1,s}{l-1}. \tag{5.15}$$

Let l be odd. Then we have

$$\binom{r,s}{l} = -\binom{s,r}{l} = -\binom{s,r-1}{l} - \binom{s,r-1}{l-1} = \binom{r-1,s}{l} - \binom{r-1,s}{l-1}. \tag{5.16}$$

If we take the sum (difference) of both sides of (5.7) and (5.8), then we obtain (5.11) and (5.12). Equations (5.9) and (5.10) can be also obtained from (5.7). □

Theorem 5.3. *Let r, s, l be nonnegative integers. Then*

$$s\binom{r,s-1}{l-1} = (l-r)\binom{r,s}{l} + r\binom{r-1,s}{l}, \tag{5.17}$$

$$r\binom{r-1,s}{l-1} = -(l-s)\binom{r,s}{l} - s\binom{r,s-1}{l}, \tag{5.18}$$

$$(r+s-l)\binom{r,s}{l} = r\binom{r-1,s}{l} + s\binom{r,s-1}{l}. \tag{5.19}$$

Proof. Let us begin the proof of the first equation (5.17). Then

$$\begin{aligned}
(l-r)\binom{r,s}{l} + r\binom{r-1,s}{l} &= (l-r)\sum_{i=0}^{l}(-1)^i\binom{r}{i}\binom{s}{l-i} + r\sum_{i=0}^{l}(-1)^i\binom{r-1}{i}\binom{s}{l-i} \\
&= \sum_{i=0}^{l}(-1)^i\bigg[(l-r)\frac{r!}{i!(r-i)!}\frac{s!}{(l-i)!(s-l+i)!} \\
&\qquad + \frac{r!}{i!(r-i-1)!}\frac{s!}{(l-i)!(s-l+i)!}\bigg] \\
&= \sum_{i=0}^{l}(-1)^i(l-r+r-i)\frac{r!s!}{i!(r-i)!(l-i)!(s-l+i)!} \\
&= s\sum_{i=0}^{l}(-1)^i\frac{r!}{i!(r-i)!}\frac{(s-1)!}{(l-i-1)!(s-l+i)!} \\
&= s\sum_{i=0}^{l}(-1)^i\binom{r}{i}\binom{s-1}{l-1-i} \\
&= s\sum_{i=0}^{l-1}(-1)^i\binom{r}{i}\binom{s-1}{l-1-i}
\end{aligned} \tag{5.20}$$

since $\binom{s-1}{-1} = 0$. Therefore we have

$$(l-r)\binom{r,s}{l} + r\binom{r-1,s}{l} = s\binom{r,s-1}{l-1}. \tag{5.21}$$

Let l be odd. Then we have

$$r\binom{r-1,s}{l-1} = r\binom{s,r-1}{l-1} = (l-s)\binom{s,r}{l} + s\binom{s-1,r}{l} = -(l-s)\binom{r,s}{l} - s\binom{r,s-1}{l}. \tag{5.22}$$

Let l be even. Then we have

$$r\binom{r-1,s}{l-1} = -r\binom{s,r-1}{l-1} = -(l-s)\binom{s,r}{l} - s\binom{s-1,r}{l} = -(l-s)\binom{r,s}{l} - s\binom{r,s-1}{l}. \tag{5.23}$$

By (5.17) and (5.18),

$$\begin{aligned} r\binom{r-1,s}{l} + s\binom{r,s-1}{l} &= s\binom{r,s-1}{l-1} - (l-r)\binom{r,s}{l} + (s-l)\binom{r,s}{l} - r\binom{r-1,s}{l} \\ &= 2(r+s-l)\binom{r,s}{l} - s\binom{r,s-1}{l} - r\binom{r-1,s}{l} \end{aligned} \tag{5.24}$$

and hence we have

$$2\left\{r\binom{r-1,s}{l} + s\binom{r,s-1}{l}\right\} = 2(r+s-l)\binom{r,s}{l}. \tag{5.25}$$

Then we get our aim. □

Lemma 5.4. *Let n be a nonnegative integer. For $k = 0,1,2,\dots,n$,*

$$\sum_{j=0}^{n}\binom{k,n-k}{j} = \begin{cases} 2^n & \text{if } k = 0, \\ 0 & \text{if } k \neq 0. \end{cases} \tag{5.26}$$

Proof. For $k = 0$,

$$\sum_{j=0}^{n}\binom{k,n-k}{j} = \sum_{j=0}^{n}\binom{n}{j} = 2^n. \tag{5.27}$$

Let $k \neq 0$. Since

$$(1+x)^{n-k}(1-x)^k = \sum_{j=0}^{n}\binom{k,n-k}{j}x^j, \tag{5.28}$$

for $x = 1$, then we have

$$0 = \sum_{j=0}^{n}\binom{k,n-k}{j}. \tag{5.29}$$

□

Similarly we have the following result.

Lemma 5.5. *Let n be a nonnegative integer. For $k = 0, 1, 2, \ldots, n$,*

$$\sum_{j=0}^{n} (-1)^j \binom{k, n-k}{j} = \begin{cases} 2^n & \text{if } k = n, \\ 0 & \text{if } k \neq n. \end{cases} \tag{5.30}$$

In this section we give a result from Riordan [23]. Let $P_n(x)$ denote the Legendre polynomials of nth order. Then the function

$$P(x, y) = \sum_{n=0} P_n(x) y^n = (1 - 2xy + y^2)^{-1/2} \tag{5.31}$$

is the generating function for Legendre polynomials. Then we have

$$P(1 + 2x, y) = (1 - y)^{-1}\left[1 - 4xy(1 - y)^{-2}\right]^{-1/2}, \tag{5.32}$$

so that, if $Q_n(x) = P_n(1 + 2x)$, then $Q(x, y) = P(1 + 2x, y)$. Now we have two expansions

$$\begin{aligned} Q(x, y) &= (1 - y)^{-1}\left[1 - 4xy(1 - y)^{-2}\right]^{-1/2} \\ &= \sum_{k=0} \binom{2k}{k} x^k y^k (1 - y)^{-2k-1} \\ &= \sum_{n=0} y^n \sum_{k=0} \binom{n+k}{2k} \binom{2k}{k} x^k, \\ Q(x, y) &= \left(1 - (1 + 2x)y\right)^{-1}\left[1 - 4\left(x + x^2\right) y^2 \left(1 - y - 2xy\right)^{-2}\right]^{-1/2} \\ &= \sum_{n=0} y^n \sum_{k=0} \binom{n}{2k} \binom{2k}{k} (1 + 2x)^{n-2k} \left(x + x^2\right)^k, \end{aligned} \tag{5.33}$$

so that

$$Q_n(x) = \sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k} x^k = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} \binom{2k}{k} (1 + 2x)^{n-2k} \left(x + x^2\right)^k. \tag{5.34}$$

Then

$$Q_n(x) = \sum_{k=0}^{n} q_{n,k} x^k (1 + x)^{n-k}, \tag{5.35}$$

where

$$\begin{aligned} q_{n,k} &= \sum_{j=0}^{k} \binom{n}{2j} \binom{2j}{j} \binom{n-2j}{k-j} \\ &= \binom{n}{k} \sum_{j=0}^{k} \binom{k}{k-j} \binom{n-k}{j} = \binom{n}{k}^2 \quad \text{(by Vandermonde convolution)} \end{aligned} \tag{5.36}$$

so that

$$Q_n(x) = \sum_{k=0}^{n} \binom{n}{k}^2 x^k (1+x)^{n-k}. \tag{5.37}$$

Since

$$Q\left(-\frac{1}{2}, y\right) = (1+y^2)^{-1/2} = \sum_{n=0} (-1)^n \binom{2n}{n} 2^{-2n} y^{2n}, \tag{5.38}$$

then we have the following identities:

$$(-1)^n \binom{2n}{n} = \sum_{k=0}^{2n} (-1)^k \binom{2n+k}{2k} \binom{2k}{k} 2^{2n-k} = \sum_{k=0}^{2n} \binom{2n}{k}^2 (-1)^k. \tag{5.39}$$

Thus we can give our result.

Lemma 5.6 (twin pairs). *Let n be a nonnegative integer. Then,*

$$\binom{2n, 2n}{2n} = \sum_{j=0}^{2n} (-1)^j \binom{2n}{j}^2 = (-1)^n \binom{2n}{n}. \tag{5.40}$$

Theorem 5.7 (diagonal formula). *Let n be a nonnegative integer. Then,*

$$\sum_{k=0}^{n} \binom{k, n-k}{k} = \begin{cases} 2^{\lfloor n/2 \rfloor} & \text{if } n \text{ even}, \\ 0 & \text{if } n \text{ odd}. \end{cases} \tag{5.41}$$

Theorem 5.8 (orthogonality formula). *Let n be a nonnegative integer. For* $i, j = 0, 1, 2, \ldots, n$,

$$\frac{1}{2^n} \sum_{k=0}^{n} \binom{k, n-k}{i} \binom{j, n-j}{k} = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{5.42}$$

6. Schubert calculus in cohomology ring of the homogeneous spaces LSU_3/T and ΩSU_3

The integral cohomology of LSU_3/T is generated by the Schubert classes indexed:

$$\widetilde{W} = \left\{ A_{ijk}^{k}, B_{ikj}^{k}, C_{ikj}^{k_1,k_2}, D_{ikj}^{l_1,l_2}, E_{ikj}^{n_1,n_2} \right\}. \tag{6.1}$$

Let $\mathfrak{a}_{ijk}^{k}$, $\mathfrak{b}_{ikj}^{k}$, $\mathfrak{c}_{ikj}^{k_1,k_2}$, $\mathfrak{d}_{ikj}^{l_1,l_2}$, $\mathfrak{e}_{ikj,i}^{n_1,n_2}$ be Schubert classes indexed by elements A_{ijk}^{k}, B_{ikj}^{k}, $C_{ikj}^{k_1,k_2}$, $D_{ikj}^{l_1,l_2}$, $E_{ikj}^{n_1,n_2}$ of the Weyl group $\widetilde{W}$, respectively. Let $\mathfrak{X}_i = \varepsilon^{r_i} \in H^2(LSU_3/T, R)$ and let $\mathfrak{t} = \varepsilon^{r_0 r_1} - \varepsilon^{r_0 r_2} \in H^4(LSU_3/T, R)$.

By Theorems 4.14 and 4.15, we have the following identities.

Lemma 6.1. *Let $w \in \widehat{W}$ with $\ell(w) = s$. Then,*

$$\mathfrak{X} \cdot \mathfrak{a}_{012} = \begin{cases} \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{a}_{012} + \mathfrak{d}_{210}^{0} & \text{if } s \geq 3, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{a}_{012} + \mathfrak{b}_{021} & \text{if } s < 3, \end{cases}$$
$$\mathfrak{X} \cdot \mathfrak{b}_{021} = \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{b}_{021} + \xi_{\lfloor (s+1)/2 \rfloor}, \tag{6.2}$$

where

$$\xi_k = \begin{cases} \mathfrak{a}_{012} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor = 1, \\ \mathfrak{d}_{210}^{(2\lfloor (s+1)/2 \rfloor - 4)/3} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 2 \bmod 3, \\ \mathfrak{c}_{021}^{(2\lfloor (s+1)/2 \rfloor - 3)/3} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 0 \bmod 3, \\ \mathfrak{e}_{102}^{(2\lfloor (s+1)/2 \rfloor - 5)/3} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 1 \bmod 3, \end{cases}$$

$$\mathfrak{X} \cdot \mathfrak{c}_{021}^{m} = \begin{cases} 0 & \text{if } s < 3m+3, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{c}_{021}^{m} \\ \quad + \frac{3m+3}{2}\mathfrak{b}_{021} + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+1}{2}\right)\mathfrak{d}_{210}^{m-1} & \text{if } s = 3m+3, \\ \frac{3m+3}{2}\mathfrak{e}_{102}^{m} + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+1}{2}\right)\mathfrak{c}_{021}^{m} & \text{if } s > 3m+3,\ s \text{ is odd}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{c}_{021}^{m} \\ \quad + \frac{3m+3}{2}\mathfrak{e}_{102}^{m} + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+1}{2}\right)\mathfrak{d}_{210}^{m-1} & \text{if } s > 3m+3,\ s \text{ is even}, \end{cases} \quad m \text{ is odd},$$

$$\mathfrak{X} \cdot \mathfrak{d}_{210}^{m} = \begin{cases} 0 & \text{if } s < 3m+4, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{d}_{210}^{m} + \frac{3m+4}{2}\mathfrak{b}_{021} \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{e}_{102}^{m-1} & \text{if } s = 3m+4, \\ \frac{3m+4}{2}\mathfrak{c}_{021}^{m+1} \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{d}_{210}^{m} & \text{if } s > 3m+4,\ s \text{ is odd}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{d}_{210}^{m} + \frac{3m+4}{2}\mathfrak{c}_{021}^{m+1} \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{e}_{102}^{m-1} & \text{if } s > 3m+4,\ s \text{ is even}, \end{cases} \quad m \text{ is even},$$

$$\mathfrak{X}\cdot\mathfrak{e}_{102}^{m}=\begin{cases}0 & \text{if } s<3m+5,\\ \left(\left\lfloor\frac{s}{2}\right\rfloor+1\right)\mathfrak{e}_{102}^{m}+\frac{3m+5}{2}\mathfrak{b}_{021} \\ \quad +\left(\left\lfloor\frac{s}{2}\right\rfloor+1-\frac{3m+3}{2}\right)\mathfrak{c}_{021}^{m} & \text{if } s=3m+5,\\ \frac{3m+5}{2}\mathfrak{d}_{210}^{m+1} \\ \quad +\left(\left\lfloor\frac{s}{2}\right\rfloor+1-\frac{3m+3}{2}\right)\mathfrak{e}_{102}^{m} & \text{if } s>3m+5,\ s \text{ is odd},\\ \left(\left\lfloor\frac{s}{2}\right\rfloor+1\right)\mathfrak{e}_{102}^{m}+\frac{3m+5}{2}\mathfrak{d}_{021}^{m+1} \\ \quad +\left(\left\lfloor\frac{s}{2}\right\rfloor+1-\frac{3m+3}{2}\right)\mathfrak{c}_{021}^{m} & \text{if } s>3m+5,\ s \text{ is even},\end{cases}\quad m \text{ is odd}. \tag{6.3}$$

By Theorems 4.14 and 4.15, we have the following identities.

Lemma 6.2. *Let $w\in\widehat{W}$ with $\ell(w)=s$. Then,*

$$\mathfrak{t}\cdot\mathfrak{a}_{012}=\begin{cases}\left(\left\lfloor\frac{s}{2}\right\rfloor+1\right)\mathfrak{a}_{012}-\mathfrak{d}_{210}^{0} & \text{if } s>1,\\ \left(\left\lfloor\frac{s}{2}\right\rfloor+1\right)\mathfrak{a}_{012}-\mathfrak{b}_{021} & \text{if } s=1,\end{cases} \tag{6.4}$$

$$\mathfrak{t}\cdot\mathfrak{b}_{021}=-\left(\left\lfloor\frac{s}{2}\right\rfloor+1\right)\mathfrak{b}_{021}+\xi_{\lfloor s/2\rfloor},$$

where

$$\xi_k=\begin{cases}a_{012} & \text{if } \left\lfloor\frac{s}{2}\right\rfloor=0,\\ \mathfrak{d}_{210}^{(2\lfloor s/2\rfloor-2)/3} & \text{if } \left\lfloor\frac{s}{2}\right\rfloor\equiv 1 \bmod 3,\\ \mathfrak{c}_{021}^{(2\lfloor s/2\rfloor-1)/3} & \text{if } \left\lfloor\frac{s}{2}\right\rfloor\equiv 2 \bmod 3,\\ \mathfrak{e}_{102}^{(2\lfloor s/2\rfloor-3)/3} & \text{if } \left\lfloor\frac{s}{2}\right\rfloor\equiv 0 \bmod 3,\end{cases}$$

$$\mathfrak{t}\cdot\mathfrak{c}_{021}^{m}=\begin{cases}0 & \text{if } s<3m+3,\\ \frac{-3m-3}{2}\mathfrak{e}_{102}^{m}+\left(\left\lfloor\frac{s}{2}\right\rfloor+1-\frac{3m+1}{2}\right)\mathfrak{c}_{021}^{m} & \text{if } s\geq 3m+3,\end{cases}\quad m \text{ is odd}, \tag{6.5}$$

$$\mathfrak{t}\cdot\mathfrak{d}_{210}^{m}=\begin{cases}0 & \text{if } s<3m+4,\\ \frac{-3m-4}{2}\mathfrak{c}_{021}^{m+1}+\left(\left\lfloor\frac{s}{2}\right\rfloor+1-\frac{3m+2}{2}\right)\mathfrak{d}_{210}^{m} & \text{if } s\geq 3m+4,\end{cases}\quad m \text{ is even},$$

$$\mathfrak{t}\cdot\mathfrak{e}_{102}^{m}=\begin{cases}0 & \text{if } s<3m+5,\\ \frac{-3m-5}{2}\mathfrak{d}_{210}^{m+1}+\left(\left\lfloor\frac{s}{2}\right\rfloor+1-\frac{3m+3}{2}\right)\mathfrak{e}_{102}^{m} & \text{if } s\geq 3m+5,\end{cases}\quad m \text{ is odd}.$$

LEMMA 6.3. *Let $w \in \widehat{W}$ with $\ell(w) = s$. Then,*

$$\mathfrak{X}_1 \cdot \mathfrak{a}_{012} = \begin{cases} \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{a}_{012} + \mathfrak{c}'^{0}_{102} & \text{if } s \text{ is even}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 \right) \mathfrak{a}_{012} + \mathfrak{c}'^{0}_{102} & \text{if } s \text{ is odd}, \end{cases} \tag{6.6}$$

$$\mathfrak{X}_1 \cdot \mathfrak{b}_{021} = \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{b}_{021} + \mathfrak{b}'_{102} + \xi_{\lfloor (s+1)/2 \rfloor},$$

where

$$\xi_{\lfloor (s+1)/2 \rfloor} = \begin{cases} \mathfrak{a}_{012} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor = 1, \\ \mathfrak{d}_{210}^{(2\lfloor (s+1)/2 \rfloor - 4)/3} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 2 \bmod 3, \\ \mathfrak{c}_{021}^{(2\lfloor (s+1)/2 \rfloor - 3)/3} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 0 \bmod 3, \\ \mathfrak{e}_{102}^{(2\lfloor (s+1)/2 \rfloor - 5)/3} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 1 \bmod 3. \end{cases} \tag{6.7}$$

Let m be odd,

$$\mathfrak{X}_1 \cdot \mathfrak{c}_{021}^{m} = \begin{cases} 0 & \text{if } s < 3m+3, \\ \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{c}_{021}^{m} + \frac{3m+1}{2} \mathfrak{b}_{021}\left(\mathfrak{e}_{102}^{m}\right) & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m-1}{2} \right) \mathfrak{d}_{210}^{m-1} + \mathfrak{d}'^{m}_{021} & \text{if } s \geq 3m+3,\ s \text{ is even}, \\ \frac{3m+1}{2} \mathfrak{e}_{102}^{m} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m-1}{2} \right) \mathfrak{c}_{210}^{m} + \mathfrak{d}'^{m}_{021} & \text{if } s > 3m+3,\ s \text{ is odd}, \end{cases}$$

$$\mathfrak{X}_1 \cdot \mathfrak{d}_{210}^{m} = \begin{cases} 0 & \text{if } s < 3m+4, \\ \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{d}_{210}^{m} + \frac{3m+3}{2} \mathfrak{b}_{021}\left(\mathfrak{c}_{021}^{m+1}\right) & \\ \quad + \mathfrak{b}''_{210}\left(\mathfrak{d}''^{m+1}_{102}\right) + \mathfrak{c}_{021}^{m} + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m+1}{2} \right) \mathfrak{e}_{102}^{m-1} + \mathfrak{e}'^{m}_{210} & \text{if } s \geq 3m+4,\ s \text{ is odd}, \\ \frac{3m+3}{2} \mathfrak{c}_{021}^{m+1} + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m+3}{2} \right) \mathfrak{d}_{210}^{m} & \\ \quad + \mathfrak{c}_{021}^{m} + \mathfrak{d}''^{m+1}_{102} + \mathfrak{e}'^{m}_{210} & \text{if } s > 3m+4,\ s \text{ is even}, \end{cases}$$

$$\mathfrak{X}_1 \cdot \mathfrak{e}_{102}^{m} = \begin{cases} 0 & \text{if } s < 3m+5, \\ \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{e}_{102}^{m} + \frac{3m+3}{2} \mathfrak{b}_{021}\left(\mathfrak{d}_{210}^{m+1}\right) & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m+1}{2} \right) \mathfrak{c}_{021}^{m} + \mathfrak{c}'^{m+1}_{102} & \text{if } s \geq 3m+5,\ s \text{ is even}, \\ \frac{3m+3}{2} \mathfrak{d}_{021}^{m+1} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m+1}{2} \right) \mathfrak{e}_{102}^{m} + \mathfrak{c}'^{m+1}_{102} & \text{if } s > 3m+5,\ s \text{ is odd}. \end{cases} \tag{6.8}$$

Let m be even,

$$\mathfrak{X}_1 \cdot \mathfrak{c}_{021}^{m} = \begin{cases} 0 & \text{if } s < 3m+3, \\ \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{c}_{021}^{m} + \frac{3m+2}{2} \mathfrak{b}_{021}\left(\mathfrak{e}_{102}^{m}\right) + \mathfrak{b}''_{210}\left(\mathfrak{c}''^{m+1}_{210}\right) & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m}{2} \right) \mathfrak{d}_{210}^{m-1} + \mathfrak{d}'^{m}_{021} + \mathfrak{e}_{102}^{m-1} & \text{if } s \geq 3m+3,\ s \text{ is odd}, \\ \frac{3m+2}{2} \mathfrak{e}_{102}^{m} + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m+2}{2} \right) \mathfrak{c}_{210}^{m} & \\ \quad + \mathfrak{d}'^{m}_{021} + \mathfrak{c}''^{m+1}_{210} + \mathfrak{e}_{102}^{m-1} & \text{if } s > 3m+3,\ s \text{ is even}, \end{cases}$$

$$\mathfrak{X}_1 \cdot \mathfrak{d}_{210}^{m} = \begin{cases} 0 & \text{if } s < 3m+4, \\ \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{d}_{210}^{m} + \frac{3m+2}{2} \mathfrak{b}_{021}\left(\mathfrak{c}_{021}^{m+1}\right) & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m}{2} \right) \mathfrak{e}_{102}^{m-1} + \mathfrak{e}'^{m}_{210} & \text{if } s \geq 3m+4,\ s \text{ is even}, \\ \frac{3m+2}{2} \mathfrak{c}_{021}^{m+1} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m}{2} \right) \mathfrak{d}_{210}^{m} + \mathfrak{e}'^{m}_{210} & \text{if } s > 3m+4,\ s \text{ is odd}, \end{cases}$$

$$\mathfrak{X}_1 \cdot \mathfrak{e}_{102}^{m} = \begin{cases} 0 & \text{if } s < 3m+5, \\ \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{e}_{102}^{m} + \frac{3m+4}{2} \mathfrak{b}_{021}\left(\mathfrak{d}_{210}^{m+1}\right) & \\ \quad + \mathfrak{d}_{210}^{m} + \mathfrak{b}''_{210}\left(\mathfrak{e}''^{m+1}_{021}\right) + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m+2}{2} \right) \mathfrak{c}_{021}^{m} + \mathfrak{c}'^{m+1}_{102} & \text{if } s \geq 3m+5,\ s \text{ is odd}, \\ \frac{3m+4}{2} \mathfrak{d}_{021}^{m+1} + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m+4}{2} \right) \mathfrak{e}_{102}^{m} & \\ \quad + \mathfrak{c}'^{m+1}_{102} + \mathfrak{d}_{210}^{m} + \mathfrak{e}''^{m+1}_{021} & \text{if } s > 3m+5,\ s \text{ is even}. \end{cases} \tag{6.9}$$

Lemma 6.4. *Let $w \in \widehat{W}$ with $\ell(w) = s$. Then,*

$$\mathfrak{X}_1 \cdot \mathfrak{a}'_{120} = \begin{cases} \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{a}'_{120} + \mathfrak{d}'^{0}_{021} & \text{if } s \geq 3, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{a}'_{120} + \mathfrak{b}'_{102} & \text{if } s < 3, \end{cases}$$
$$\mathfrak{X}_1 \cdot \mathfrak{b}'_{102} = \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{b}'_{102} + \xi_{\lfloor (s+1)/2 \rfloor}, \tag{6.10}$$

where

$$\xi_{\lfloor (s+1)/2 \rfloor} = \begin{cases} \mathfrak{a}'_{120} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor = 1, \\ \mathfrak{d}'^{(2\lfloor (s+1)/2 \rfloor - 4)/3}_{021} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 2 \bmod 3, \\ \mathfrak{c}'^{(2\lfloor (s+1)/2 \rfloor - 3)/3}_{102} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 0 \bmod 3, \\ \mathfrak{e}'^{(2\lfloor (s+1)/2 \rfloor - 5)/3}_{210} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 1 \bmod 3. \end{cases} \tag{6.11}$$

Let m be odd,

$$\mathfrak{X}_1 \cdot \mathfrak{c}'^{m}_{102} = \begin{cases} 0 & \text{if } s < 3m+3, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{c}'^{m}_{102} + \frac{3m+3}{2}\mathfrak{b}'_{102} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+1}{2}\right)\mathfrak{d}'^{m-1}_{021} & \text{if } s = 3m+3, \\ \frac{3m+3}{2}\mathfrak{e}'^{m}_{210} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+1}{2}\right)\mathfrak{c}'^{m}_{102} & \text{if } s > 3m+3,\ s \text{ is odd}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{c}'^{m}_{102} + \frac{3m+3}{2}\mathfrak{e}'^{m}_{210} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+1}{2}\right)\mathfrak{d}'^{m-1}_{021} & \text{if } s > 3m+3,\ s \text{ is even}, \end{cases}$$

$$\mathfrak{X}_1 \cdot \mathfrak{d}'^{m}_{021} = \begin{cases} 0 & \textit{if } s < 3m+4, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{d}'^{m}_{021} + \frac{3m+5}{2}\mathfrak{b}'_{102} \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+3}{2}\right)\mathfrak{e}'^{m-1}_{210} + \mathfrak{c}'^{m}_{102} & \textit{if } s = 3m+4, \\ \frac{3m+5}{2}\mathfrak{c}'^{m+1}_{102} \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+5}{2}\right)\mathfrak{d}'^{m}_{021} + \mathfrak{c}'^{m}_{102} & \textit{if } s > 3m+4,\ s \textit{ is even}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{d}'^{m}_{021} + \frac{3m+5}{2}\mathfrak{c}'^{m+1}_{102} \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+3}{2}\right)\mathfrak{e}'^{m-1}_{210} + \mathfrak{c}'^{m}_{102} & \textit{if } s > 3m+4,\ s \textit{ is odd}, \end{cases}$$

$$\mathfrak{X}_1 \cdot \mathfrak{e}'^{m}_{210} = \begin{cases} 0 & \textit{if } s < 3m+5, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{e}'^{m}_{210} + \frac{3m+5}{2}\mathfrak{b}'_{102} \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+3}{2}\right)\mathfrak{c}'^{m}_{102} & \textit{if } s = 3m+5, \\ \frac{3m+5}{2}\mathfrak{d}'^{m+1}_{021} \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+3}{2}\right)\mathfrak{e}'^{m}_{210} & \textit{if } s > 3m+5,\ s \textit{ is odd}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{e}'^{m}_{210} + \frac{3m+5}{2}\mathfrak{d}'^{m+1}_{102} \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+3}{2}\right)\mathfrak{c}'^{m}_{102} & \textit{if } s > 3m+5,\ s \textit{ is even}. \end{cases} \tag{6.12}$$

Let m be even,

$$\mathfrak{X}_1 \cdot \mathfrak{c}'^{m}_{102} = \begin{cases} 0 & \textit{if } s < 3m+3, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{c}'^{m}_{102} + \frac{3m+4}{2}\mathfrak{b}'_{102} \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{d}'^{m-1}_{021} + \mathfrak{e}'^{m-1}_{210} & \textit{if } s = 3m+3, \\ \frac{3m+4}{2}\mathfrak{e}'^{m}_{210} \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+4}{2}\right)\mathfrak{c}'^{m}_{102} + \mathfrak{e}'^{m-1}_{210} & \textit{if } s > 3m+3,\ s \textit{ is even}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{c}'^{m}_{102} + \frac{3m+4}{2}\mathfrak{e}'^{m}_{210} \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{d}'^{m-1}_{021} + \mathfrak{e}'^{m-1}_{210} & \textit{if } s > 3m+3,\ s \textit{ is odd}, \end{cases}$$

$$\mathfrak{X}_1 \cdot \mathfrak{d}'^{m}_{021} = \begin{cases} 0 & \text{if } s < 3m+4, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{d}'^{m}_{021} + \frac{3m+4}{2}\mathfrak{b}'_{102} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{e}'^{m-1}_{210} & \text{if } s = 3m+4, \\ \frac{3m+4}{2}\mathfrak{c}'^{m+1}_{102} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{d}'^{m}_{021} & \text{if } s > 3m+4,\ s \text{ is odd}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{d}'^{m}_{021} + \frac{3m+4}{2}\mathfrak{c}'^{m+1}_{102} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{e}'^{m-1}_{210} & \text{if } s > 3m+4,\ s \text{ is even}, \end{cases}$$

$$\mathfrak{X}_1 \cdot \mathfrak{e}'^{m}_{210} = \begin{cases} 0 & \text{if } s < 3m+5, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{e}'^{m}_{210} + \frac{3m+6}{2}\mathfrak{b}'_{102} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+4}{2}\right)\mathfrak{c}'^{m}_{102} + \mathfrak{d}'^{m}_{021} & \text{if } s = 3m+5, \\ \frac{3m+6}{2}\mathfrak{d}'^{m+1}_{021} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+6}{2}\right)\mathfrak{e}'^{m}_{210} + \mathfrak{d}'^{m}_{021} & \text{if } s > 3m+5,\ s \text{ is even}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{e}'^{m}_{210} + \frac{3m+6}{2}\mathfrak{d}'^{m+1}_{102} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+4}{2}\right)\mathfrak{c}'^{m}_{102} + \mathfrak{d}'^{m}_{021} & \text{if } s > 3m+5,\ s \text{ is odd}. \end{cases} \tag{6.13}$$

By Theorems 4.14 and 4.15, we have the following identities.

Lemma 6.5. *Let $w \in \widehat{W}$ with $\ell(w) = s$. Then,*

$$\mathfrak{X}_2 \cdot \mathfrak{a}'_{120} = \begin{cases} \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{a}'_{120} + \mathfrak{c}''^{0}_{210} & \text{if } s \text{ is even}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{a}'_{120} + \mathfrak{c}''^{0}_{210} & \text{if } s \text{ is odd}, \end{cases} \tag{6.14}$$

$$\mathfrak{X}_2 \cdot \mathfrak{b}'_{102} = \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{b}'_{102} + \mathfrak{b}''_{210} + \xi_{\lfloor (s+1)/2 \rfloor},$$

where

$$\xi_{\lfloor (s+1)/2 \rfloor} = \begin{cases} \mathfrak{a}'_{120} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor = 1, \\ \mathfrak{d}'^{(2\lfloor (s+1)/2 \rfloor - 4)/3}_{021} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 2 \bmod 3, \\ \mathfrak{c}'^{(2\lfloor (s+1)/2 \rfloor - 3)/3}_{102} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 0 \bmod 3, \\ \mathfrak{e}'^{(2\lfloor (s+1)/2 \rfloor - 5)/3}_{210} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 1 \bmod 3. \end{cases} \tag{6.15}$$

Let m be odd,

$$\mathfrak{X}_2 \cdot \mathfrak{c}'^{m}_{102} = \begin{cases} 0 & \text{if } s < 3m+3, \\ \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{c}'^{m}_{102} + \frac{3m+1}{2} \mathfrak{b}'_{102}\left(\mathfrak{e}'^{m}_{210}\right) & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m-1}{2} \right) \mathfrak{d}'^{m-1}_{021} + \mathfrak{d}''^{m}_{102} & \text{if } s \geq 3m+3,\ s \text{ is even}, \\ \frac{3m+1}{2} \mathfrak{e}'^{m}_{210} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m-1}{2} \right) \mathfrak{c}'^{m}_{102} + \mathfrak{d}''^{m}_{102} & \text{if } s > 3m+3,\ s \text{ is odd}, \end{cases}$$

$$\mathfrak{X}_2 \cdot \mathfrak{d}'^{m}_{021} = \begin{cases} 0 & \text{if } s < 3m+4, \\ \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{d}'^{m}_{021} + \frac{3m+3}{2} \mathfrak{b}'_{102}\left(\mathfrak{c}'^{m+1}_{102}\right) & \\ \quad + \mathfrak{b}_{021}\left(\mathfrak{d}^{m+1}_{210}\right) + \mathfrak{c}'^{m}_{102} + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m+1}{2} \right) \mathfrak{e}'^{m-1}_{210} + \mathfrak{e}''^{m}_{0210} & \text{if } s \geq 3m+4,\ s \text{ is odd}, \\ \frac{3m+3}{2} \mathfrak{c}'^{m+1}_{102} + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m+3}{2} \right) \mathfrak{d}'^{m}_{021} & \\ \quad + \mathfrak{c}'^{m}_{102} + \mathfrak{d}^{m+1}_{210} + \mathfrak{e}''^{m}_{021} & \text{if } s > 3m+4,\ s \text{ is even}, \end{cases}$$

$$\mathfrak{X}_2 \cdot \mathfrak{e}'^{m}_{210} = \begin{cases} 0 & \text{if } s < 3m+5, \\ \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{e}'^{m}_{210} + \frac{3m+3}{2} \mathfrak{b}'_{102}\left(\mathfrak{d}'^{m+1}_{021}\right) & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m+1}{2} \right) \mathfrak{c}'^{m}_{102} + \mathfrak{c}''^{m+1}_{210} & \text{if } s \geq 3m+5,\ s \text{ is even}, \\ \frac{3m+3}{2} \mathfrak{d}'^{m+1}_{021} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor - \frac{3m+1}{2} \right) \mathfrak{e}'^{m}_{210} + \mathfrak{c}''^{m+1}_{210} & \text{if } s > 3m+5,\ s \text{ is odd}. \end{cases} \tag{6.16}$$

Let m be even,

$$\mathfrak{X}_2\cdot\mathfrak{c}'^{m}_{102}=\begin{cases}0 & \text{if } s<3m+3,\\ \left\lfloor\frac{s}{2}\right\rfloor\mathfrak{c}'^{m}_{102}+\frac{3m+2}{2}\mathfrak{b}'_{102}\left(\mathfrak{e}'^{m}_{210}\right)+\mathfrak{b}_{021}\left(\mathfrak{c}^{m+1}_{021}\right) & \\ \quad+\left(\left\lfloor\frac{s}{2}\right\rfloor-\frac{3m}{2}\right)\mathfrak{d}'^{m-1}_{021}+\mathfrak{d}''^{m}_{102}+\mathfrak{e}'^{m-1}_{210} & \text{if } s\geq 3m+3,\ s \text{ is odd},\\ \frac{3m+2}{2}\mathfrak{e}'^{m}_{210}+\left(\left\lfloor\frac{s}{2}\right\rfloor-\frac{3m+2}{2}\right)\mathfrak{c}'^{m}_{102} & \\ \quad+\mathfrak{d}''^{m}_{102}+\mathfrak{c}^{m+1}_{021}+\mathfrak{e}'^{m-1}_{210} & \text{if } s>3m+3,\ s \text{ is even},\end{cases}$$

$$\mathfrak{X}_2\cdot\mathfrak{d}'^{m}_{021}=\begin{cases}0 & \text{if } s<3m+4,\\ \left\lfloor\frac{s}{2}\right\rfloor\mathfrak{d}'^{m}_{021}+\frac{3m+2}{2}\mathfrak{b}'_{102}\left(\mathfrak{c}'^{m+1}_{102}\right) & \\ \quad+\left(\left\lfloor\frac{s}{2}\right\rfloor-\frac{3m}{2}\right)\mathfrak{e}'^{m-1}_{210}+\mathfrak{e}''^{m}_{021} & \text{if } s\geq 3m+4,\ s \text{ is even},\\ \frac{3m+2}{2}\mathfrak{c}'^{m+1}_{102} & \\ \quad+\left(\left\lfloor\frac{s}{2}\right\rfloor-\frac{3m}{2}\right)\mathfrak{d}'^{m}_{021}+\mathfrak{e}''^{m}_{021} & \text{if } s>3m+4,\ s \text{ is odd},\end{cases}$$

$$\mathfrak{X}_2\cdot\mathfrak{e}'^{m}_{210}=\begin{cases}0 & \text{if } s<3m+5,\\ \left\lfloor\frac{s}{2}\right\rfloor\mathfrak{e}'^{m}_{210}+\frac{3m+4}{2}\mathfrak{b}'_{102}\left(\mathfrak{d}'^{m+1}_{021}\right) & \\ \quad+\mathfrak{d}'^{m}_{021}+\mathfrak{b}_{021}\left(\mathfrak{e}^{m+1}_{102}\right)+\left(\left\lfloor\frac{s}{2}\right\rfloor-\frac{3m+2}{2}\right)\mathfrak{c}'^{m}_{102}+\mathfrak{c}''^{m+1}_{210} & \text{if } s\geq 3m+5,\ s \text{ is odd},\\ \frac{3m+4}{2}\mathfrak{d}'^{m+1}_{021}+\left(\left\lfloor\frac{s}{2}\right\rfloor-\frac{3m+4}{2}\right)\mathfrak{e}'^{m}_{210} & \\ \quad+\mathfrak{c}''^{m+1}_{210}+\mathfrak{d}'^{m}_{021}+\mathfrak{e}^{m+1}_{102} & \text{if } s>3m+5,\ s \text{ is even}.\end{cases}\tag{6.17}$$

Lemma 6.6. *Let $w\in\widehat{W}$ with $\ell(w)=s$. Then,*

$$\mathfrak{X}_2\cdot\mathfrak{a}''_{201}=\begin{cases}\left(\left\lfloor\frac{s}{2}\right\rfloor+1\right)\mathfrak{a}''_{201}+\mathfrak{d}''^{0}_{102} & \text{if } s\geq 3,\\ \left(\left\lfloor\frac{s}{2}\right\rfloor+1\right)\mathfrak{a}''_{201}+\mathfrak{b}''_{210} & \text{if } s<3,\end{cases}\tag{6.18}$$

$$\mathfrak{X}_2\cdot\mathfrak{b}''_{210}=\left(\left\lfloor\frac{s}{2}\right\rfloor+1\right)\mathfrak{b}''_{210}+\xi_{\lfloor (s+1)/2\rfloor},$$

where

$$\xi_{\lfloor (s+1)/2 \rfloor} = \begin{cases} \mathfrak{a}''_{201} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor = 1, \\ \mathfrak{d}''^{(2\lfloor (s+1)2 \rfloor -4)/3}_{102} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 2 \bmod 3, \\ \mathfrak{c}''^{(2\lfloor (s+1)/2 \rfloor -3)/3}_{210} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 0 \bmod 3, \\ \mathfrak{e}''^{(2\lfloor (s+1)/2 \rfloor -5)/3}_{021} & \text{if } \left\lfloor \frac{s+1}{2} \right\rfloor \equiv 1 \bmod 3. \end{cases} \tag{6.19}$$

Let m be odd,

$$\mathfrak{X}_2 \cdot \mathfrak{c}''^{m}_{210} = \begin{cases} 0 & \text{if } s < 3m+3, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{c}''^{m}_{210} + \frac{3m+3}{2}\mathfrak{b}''_{210} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+1}{2}\right)\mathfrak{d}''^{m-1}_{102} & \text{if } s = 3m+3, \\ \frac{3m+3}{2}\mathfrak{e}''^{m}_{021} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+1}{2}\right)\mathfrak{c}''^{m}_{210} & \text{if } s > 3m+3, \ s \text{ is odd}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{c}''^{m}_{210} + \frac{3m+3}{2}\mathfrak{e}''^{m}_{021} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+1}{2}\right)\mathfrak{d}''^{m-1}_{102} & \text{if } s > 3m+3, \ s \text{ is even}, \end{cases}$$

$$\mathfrak{X}_2 \cdot \mathfrak{d}''^{m}_{102} = \begin{cases} 0 & \text{if } s < 3m+4, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{d}''^{m}_{102} + \frac{3m+5}{2}\mathfrak{b}''_{210} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+3}{2}\right)\mathfrak{e}''^{m-1}_{021} + \mathfrak{c}''^{m}_{210} & \text{if } s = 3m+4, \\ \frac{3m+5}{2}\mathfrak{c}''^{m+1}_{210} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+5}{2}\right)\mathfrak{d}''^{m}_{102} + \mathfrak{c}''^{m}_{210} & \text{if } s > 3m+4, \ s \text{ is even}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{d}''^{m}_{102} + \frac{3m+5}{2}\mathfrak{c}''^{m+1}_{210} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+3}{2}\right)\mathfrak{e}''^{m-1}_{021} + \mathfrak{c}''^{m}_{210} & \text{if } s > 3m+4, \ s \text{ is odd}, \end{cases}$$

$$\mathfrak{X}_2 \cdot \mathfrak{e}''^{m}_{021} = \begin{cases} 0 & \textit{if } s < 3m+5, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{e}''^{m}_{021} + \frac{3m+5}{2}\mathfrak{b}''_{210} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+3}{2}\right)\mathfrak{c}''^{m}_{210} & \textit{if } s = 3m+5, \\ \frac{3m+5}{2}\mathfrak{d}''^{m+1}_{102} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+3}{2}\right)\mathfrak{e}''^{m}_{021} & \textit{if } s > 3m+5,\ s \textit{ is odd}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{e}''^{m}_{021} + \frac{3m+5}{2}\mathfrak{d}''^{m+1}_{102} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+3}{2}\right)\mathfrak{c}''^{m}_{210} & \textit{if } s > 3m+5,\ s \textit{ is even}. \end{cases} \tag{6.20}$$

Let m be even,

$$\mathfrak{X}_2 \cdot \mathfrak{c}''^{m}_{210} = \begin{cases} 0 & \textit{if } s < 3m+3, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{c}''^{m}_{210} + \frac{3m+4}{2}\mathfrak{b}''_{210} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{d}''^{m-1}_{102} + \mathfrak{e}''^{m-1}_{021} & \textit{if } s = 3m+3, \\ \frac{3m+4}{2}\mathfrak{e}''^{m}_{021} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+4}{2}\right)\mathfrak{c}''^{m}_{210} + \mathfrak{e}''^{m-1}_{021} & \textit{if } s > 3m+3,\ s \textit{ is even}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{c}''^{m}_{210} + \frac{3m+4}{2}\mathfrak{e}''^{m}_{021} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{d}''^{m-1}_{102} + \mathfrak{e}''^{m-1}_{021} & \textit{if } s > 3m+3,\ s \textit{ is odd}, \end{cases}$$

$$\mathfrak{X}_2 \cdot \mathfrak{d}''^{m}_{102} = \begin{cases} 0 & \textit{if } s < 3m+4, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{d}''^{m}_{102} + \frac{3m+4}{2}\mathfrak{b}''_{210} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{e}''^{m-1}_{021} & \textit{if } s = 3m+4, \\ \frac{3m+4}{2}\mathfrak{c}''^{m+1}_{210} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{d}''^{m}_{102} & \textit{if } s > 3m+4,\ s \textit{ is odd}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{d}''^{m}_{102} + \frac{3m+4}{2}\mathfrak{c}''^{m+1}_{210} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{e}''^{m-1}_{021} & \textit{if } s > 3m+4,\ s \textit{ is even}, \end{cases}$$

$$\mathfrak{X}_2 \cdot \mathfrak{e}''^{m}_{021} = \begin{cases} 0 & \text{if } s < 3m+5, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{e}''^{m}_{021} + \frac{3m+6}{2}\mathfrak{b}''_{210} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+4}{2}\right)\mathfrak{c}''^{m}_{210} + \mathfrak{d}''^{m}_{102} & \text{if } s = 3m+5, \\ \frac{3m+6}{2}\mathfrak{d}''^{m+1}_{102} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+6}{2}\right)\mathfrak{e}''^{m}_{021} + \mathfrak{d}''^{m}_{102} & \text{if } s > 3m+5,\ s \text{ is even}, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{e}''^{m}_{021} + \frac{3m+6}{2}\mathfrak{d}''^{m+1}_{102} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+4}{2}\right)\mathfrak{c}''^{m}_{210} + \mathfrak{d}''^{m}_{102} & \text{if } s > 3m+5,\ s \text{ is odd}. \end{cases} \tag{6.21}$$

LEMMA 6.7. *Let $w \in \widehat{W}$ with $\ell(w) = s$. Then,*

$$\mathfrak{X}_2 \cdot \mathfrak{a}_{012} = \begin{cases} \mathfrak{a}''_{201} + \mathfrak{b}_{021} & \text{if } s = 1, \\ \mathfrak{a}_{012} + \mathfrak{a}''_{201} + \mathfrak{b}_{021} & \text{if } s = 2, \\ \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{a}_{012} + \mathfrak{a}''_{201} + \mathfrak{d}^{0}_{210} & \text{if } s \geq 3, \end{cases} \tag{6.22}$$

$$\mathfrak{X}_2 \cdot \mathfrak{b}_{021} = \left\lfloor \frac{s+1}{2} \right\rfloor \mathfrak{b}_{021} + \xi_{\lfloor s/2 \rfloor},$$

where

$$\xi_{\lfloor s/2 \rfloor} = \begin{cases} \mathfrak{a}''_{201} & \text{if } \left\lfloor \frac{s}{2} \right\rfloor = 0, \\ \mathfrak{c}^{(2\lfloor s/2 \rfloor - 2)/3}_{021} & \text{if } \left\lfloor \frac{s}{2} \right\rfloor \equiv 1 \bmod 3, \\ \mathfrak{e}^{(2\lfloor s/2 \rfloor - 4)/3}_{102} & \text{if } \left\lfloor \frac{s}{2} \right\rfloor \equiv 2 \bmod 3, \\ \mathfrak{d}^{(2\lfloor s/2 \rfloor - 3)/3}_{210} & \text{if } \left\lfloor \frac{s}{2} \right\rfloor \equiv 0 \bmod 3. \end{cases} \tag{6.23}$$

Let m be odd,

$$\mathfrak{X}_2 \cdot \mathfrak{c}^{m}_{021} = \begin{cases} 0 & \text{if } s < 3m+3, \\ \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{c}^{m}_{021} + \frac{3m+3}{2}\mathfrak{b}_{021}\left(\mathfrak{e}^{m}_{102}\right) & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+3}{2}\right)\mathfrak{d}^{m-1}_{210} + \mathfrak{e}^{m-1}_{102} & \text{if } s \geq 3m+3,\ s \text{ is even}, \\ \frac{3m+3}{2}\mathfrak{e}^{m}_{102} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+3}{2}\right)\mathfrak{c}^{m}_{210} + \mathfrak{e}^{m-1}_{102} & \text{if } s > 3m+3,\ s \text{ is odd}, \end{cases}$$

$$\mathfrak{X}_2 \cdot \mathfrak{d}_{210}^m = \begin{cases} 0 & \text{if } s < 3m+4, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{d}_{210}^m + \frac{3m+3}{2}\mathfrak{b}_{021}\left(\mathfrak{c}_{021}^{m+1}\right) & \\ \quad + \mathfrak{b}_{210}''\left(\mathfrak{d}''^{m+1}_{102}\right) + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+1}{2}\right)\mathfrak{e}_{102}^{m-1} & \text{if } s \geq 3m+4,\ s \text{ is odd}, \\ \frac{3m+3}{2}\mathfrak{c}_{021}^{m+1} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+3}{2}\right)\mathfrak{d}_{210}^m + \mathfrak{d}''^{m+1}_{102} & \text{if } s > 3m+4,\ s \text{ is even}, \end{cases}$$

$$\mathfrak{X}_2 \cdot \mathfrak{e}_{102}^m = \begin{cases} 0 & \text{if } s < 3m+5, \\ \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{e}_{102}^m + \frac{3m+5}{2}\mathfrak{b}_{021}\left(\mathfrak{d}_{210}^{m+1}\right) & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+5}{2}\right)\mathfrak{c}_{021}^m + \mathfrak{d}_{210}^m & \text{if } s \geq 3m+5,\ s \text{ is even}, \\ \frac{3m+5}{2}\mathfrak{d}_{021}^{m+1} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+5}{2}\right)\mathfrak{e}_{102}^m + \mathfrak{d}_{210}^m & \text{if } s > 3m+5,\ s \text{ is odd}. \end{cases} \tag{6.24}$$

Let m be even,

$$\mathfrak{X}_2 \cdot \mathfrak{c}_{021}^m = \begin{cases} 0 & \text{if } s < 3m+3, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{c}_{021}^m + \frac{3m+2}{2}\mathfrak{b}_{021}\left(\mathfrak{e}_{102}^m\right) & \\ \quad + \mathfrak{b}''_{210}\left(\mathfrak{c}''^{m+1}_{210}\right) + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m}{2}\right)\mathfrak{d}_{210}^{m-1} & \text{if } s \geq 3m+3,\ s \text{ is odd}, \\ \frac{3m+2}{2}\mathfrak{e}_{102}^m & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{c}_{210}^m + \mathfrak{c}''^{m+1}_{210} & \text{if } s > 3m+3,\ s \text{ is even}, \end{cases}$$

$$\mathfrak{X}_2 \cdot \mathfrak{d}_{210}^m = \begin{cases} 0 & \text{if } s < 3m+4, \\ \left\lfloor \frac{s}{2} \right\rfloor \mathfrak{d}_{210}^m + \frac{3m+4}{2}\mathfrak{b}_{021}\left(\mathfrak{c}_{021}^{m+1}\right) & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+4}{2}\right)\mathfrak{e}_{102}^{m-1} + \mathfrak{c}_{021}^m & \text{if } s \geq 3m+4,\ s \text{ is even}, \\ \frac{3m+4}{2}\mathfrak{c}_{021}^{m+1} & \\ \quad + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+4}{2}\right)\mathfrak{d}_{210}^m + \mathfrak{c}_{021}^m & \text{if } s > 3m+4,\ s \text{ is odd}, \end{cases}$$

$$\mathfrak{X}_2 \cdot \mathfrak{e}_{102}^{m} = \begin{cases} 0 & \textit{if } s < 3m+5, \\ \left(\left\lfloor \frac{s}{2} \right\rfloor + 1\right)\mathfrak{e}_{102}^{m} + \frac{3m+4}{2}\mathfrak{b}_{021}(\mathfrak{d}_{210}^{m+1}) & \\ \quad +\mathfrak{b}''_{210}(\mathfrak{e}''^{m+1}_{021}) + \left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+2}{2}\right)\mathfrak{c}_{021}^{m} & \textit{if } s \geq 3m+5,\ s \textit{ is odd}, \\ \frac{3m+4}{2}\mathfrak{d}_{021}^{m+1} & \\ \quad +\left(\left\lfloor \frac{s}{2} \right\rfloor + 1 - \frac{3m+4}{2}\right)\mathfrak{e}_{102}^{m} + \mathfrak{e}''^{m+1}_{021} & \textit{if } s > 3m+5,\ s \textit{ is even}. \end{cases} \tag{6.25}$$

Let $\mathfrak{X} = \varepsilon^{r_0} \in H^2(\Omega SU_3, R)$ and let $\mathfrak{t} = \varepsilon^{r_0 r_1} - \varepsilon^{r_0 r_2} \in H^4(\Omega SU_3, R)$. The following calculations will be done in $\mathbb{Z}[1/2]$. Let $\mathfrak{X}^{[n]} = \mathfrak{X}^n/n!$ and $\mathfrak{t}^{[n]} = \mathfrak{t}^n/n!$.

Lemma 6.8. *For* $n, m \in \mathbb{N}$,

$$\mathfrak{X}^{[n]}\mathfrak{t}^{[m]} = \frac{1}{2^{\lfloor n/2 \rfloor}} \sum_{k=0}^{m+\lfloor n/2 \rfloor} \binom{k, m + \lfloor \frac{n}{2} \rfloor - k}{m} \xi_k, \tag{6.26}$$

where

$$\xi_k = \begin{cases} \mathfrak{a}_{012,i} & \textit{if } k = 0, \\ \mathfrak{d}_{210}^{(2k-2)/3} & \textit{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}_{021}^{(2k-1)/3} & \textit{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-3)/3} & \textit{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}_{021} & \textit{if } k = m + \left\lfloor \frac{n}{2} \right\rfloor. \end{cases} \tag{6.27}$$

Proof. The proof will be done by induction on $n \in \mathbb{N}$. For $n = 0$, we will prove that the equality

$$\mathfrak{t}^{[m]} = \sum_{k=0}^{m} \binom{k, m-k}{m} \xi_k, \tag{6.28}$$

where

$$\xi_k = \begin{cases} \mathfrak{a}_{012} & \text{if } k = 0, \\ \mathfrak{d}_{210}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}_{021}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-3)/3} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}_{021} & \text{if } k = m, \end{cases} \tag{6.29}$$

is true. For $m = 0$, the equality holds. Suppose that for $m = q$ the equality

$$\mathfrak{t}^{[q]} = \sum_{k=0}^{q} \binom{k, q-k}{q} \xi_k, \tag{6.30}$$

where

$$\xi_k = \begin{cases} \mathfrak{a}_{012} & \text{if } k = 0, \\ \mathfrak{d}_{210}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}_{021}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-3)/3} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}_{021} & \text{if } k = q, \end{cases} \tag{6.31}$$

holds. Then we have

$$\begin{aligned} t \cdot \mathfrak{t}^{[q]} &= \sum_{k=0}^{q} \binom{k, q-k}{q} t \cdot \xi_k = \sum_{k=0}^{q} (-1)^k t \cdot \xi_k \\ &= \sum_{k=0}^{q+1} (-1)^k k \xi_k + \sum_{k=0}^{q+1} (-1)^k (q+1-k) \xi_k = \sum_{k=0}^{q+1} (-1)^k (q+1) \xi_k \\ &= \sum_{k=0}^{q+1} \binom{k, q+1-k}{q+1} (q+1) \xi_k, \end{aligned} \tag{6.32}$$

where

$$\xi_k = \begin{cases} \mathfrak{a}_{012} & \text{if } k = 0, \\ \mathfrak{d}_{210}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}_{021}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-3)/3} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}_{021} & \text{if } k = q+1. \end{cases} \tag{6.33}$$

Hence the equality

$$\mathfrak{t}^{[q+1]} = \sum_{k=0}^{q+1} \binom{k, q+1-k}{q+1} \xi_k, \tag{6.34}$$

where

$$\xi_k = \begin{cases} \mathfrak{a}_{012} & \text{if } k = 0, \\ \mathfrak{d}_{210}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}_{021}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-3)/3} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}_{021} & \text{if } k = q+1, \end{cases} \tag{6.35}$$

holds. Suppose that for $n = l$, the equality

$$\mathfrak{X}^{[l]}\mathfrak{t}^{[m]} = \frac{1}{2^{\lfloor l/2 \rfloor}} \sum_{k=0}^{m+\lfloor l/2 \rfloor} \binom{k, m + \left\lfloor \frac{l}{2} \right\rfloor - k}{m} \xi_k, \tag{6.36}$$

where

$$\xi_k = \begin{cases} \mathfrak{a}_{012,i} & \text{if } k = 0, \\ \mathfrak{d}_{210}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}_{021}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-3)/3} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}_{021} & \text{if } k = m + \left\lfloor \frac{l}{2} \right\rfloor, \end{cases} \tag{6.37}$$

holds. Let l be even. Then we have

$$\begin{aligned} \mathfrak{X} \cdot \mathfrak{X}^{[l]}\mathfrak{t}^{[m]} &= \frac{1}{2^{\lfloor l/2 \rfloor}} \sum_{k=0}^{m+\lfloor l/2 \rfloor} \binom{k, m + \left\lfloor \frac{l}{2} \right\rfloor - k}{m} \mathfrak{X} \cdot \xi_k \\ &= \frac{1}{2^{\lfloor l/2 \rfloor}} \sum_{k=0}^{m+\lfloor l/2 \rfloor} \left[\binom{k, m + \left\lfloor \frac{l}{2} \right\rfloor - k}{m} \left(m + \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) + \binom{k-1, m + \left\lfloor \frac{l}{2} \right\rfloor - k + 1}{m} k \right. \\ &\quad \left. + \binom{k+1, m + \left\lfloor \frac{l}{2} \right\rfloor - k - 1}{m} \left(m + \left\lfloor \frac{l}{2} \right\rfloor - k \right) \right] \xi_k \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2^{\lfloor l/2\rfloor}} \sum_{k=0}^{m+\lfloor l/2\rfloor} \sum_{i=0}^{m} (-1)^i \left[\binom{k}{i} \binom{m+\left\lfloor \frac{l}{2} \right\rfloor - k}{m-i} \left(m+\left\lfloor \frac{l}{2}\right\rfloor + 1\right) \right. \\
&\qquad + \binom{k-1}{i} \binom{m+\left\lfloor \frac{l}{2} \right\rfloor - k + 1}{m-i} k \\
&\qquad \left. + \binom{k+1}{i} \binom{m+\left\lfloor \frac{l}{2} \right\rfloor - k - 1}{m-i} \left(m+\left\lfloor \frac{l}{2}\right\rfloor - k\right) \right] \xi_k \\
&= \frac{1}{2^{\lfloor l/2\rfloor}} \sum_{k=0}^{m+\lfloor l/2\rfloor} \sum_{i=0}^{m} (-1)^i \left[\binom{k}{i} \binom{m+\left\lfloor \frac{l}{2} \right\rfloor - k}{m-i} (l+1) \right] \xi_k \\
&= \frac{1}{2^{\lfloor l/2\rfloor}} \sum_{k=0}^{m+\lfloor l/2\rfloor} (l+1) \binom{k, m+\left\lfloor \frac{l}{2} \right\rfloor - k}{m} \xi_k,
\end{aligned}
\tag{6.38}
$$

where

$$
\xi_k = \begin{cases}
\mathfrak{a}_{012,i} & \text{if } k = 0, \\
\mathfrak{d}_{210}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \\
\mathfrak{c}_{021}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\
\mathfrak{e}_{102}^{(2k-3)/3} & \text{if } k \equiv 0 \bmod 3, \\
\mathfrak{b}_{021} & \text{if } k = m + \left\lfloor \frac{l}{2} \right\rfloor.
\end{cases}
\tag{6.39}
$$

Since l is even, we have

$$
\mathfrak{X}^{[l+1]} \mathfrak{t}^{[m]} = \frac{1}{2^{\lfloor (l+1)/2\rfloor}} \sum_{k=0}^{m+\lfloor (l+1)/2\rfloor} \binom{k, m+\left\lfloor \frac{l+1}{2} \right\rfloor - k}{m} \xi_k,
\tag{6.40}
$$

where

$$
\xi_k = \begin{cases}
\mathfrak{a}_{012,i} & \text{if } k = 0, \\
\mathfrak{d}_{210}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \\
\mathfrak{c}_{021}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\
\mathfrak{e}_{102}^{(2k-3)/3} & \text{if } k \equiv 0 \bmod 3, \\
\mathfrak{b}_{021} & \text{if } k = m + \left\lfloor \frac{l+1}{2} \right\rfloor.
\end{cases}
\tag{6.41}
$$

Let l be odd. Then we have

$$
\begin{aligned}
\mathfrak{X}\cdot\mathfrak{X}^{[l]}\mathfrak{t}^{[m]} &= \frac{1}{2^{\lfloor l/2\rfloor}}\sum_{k=0}^{m+\lfloor l/2\rfloor}\begin{pmatrix} k, m+\left\lfloor \frac{l}{2}\right\rfloor - k \\ m\end{pmatrix}\mathfrak{X}\cdot\xi_k \\
&= \frac{1}{2^{\lfloor l/2\rfloor}}\sum_{k=0}^{m+\lfloor (l+1)/2\rfloor}\left[\begin{pmatrix} k-1, m+\left\lfloor \frac{l}{2}\right\rfloor - k+1 \\ m\end{pmatrix}k\right. \\
&\qquad\left. + \begin{pmatrix} k, m+\left\lfloor \frac{l}{2}\right\rfloor - k \\ m\end{pmatrix}\left(m+\left\lfloor\frac{l+1}{2}\right\rfloor-k\right)\right]\xi_k \\
&= \frac{1}{2^{\lfloor l/2\rfloor}}\sum_{k=0}^{m+\lfloor (l+1)/2\rfloor}\sum_{i=0}^{m}(-1)^i\left[\binom{k-1}{i}\begin{pmatrix} m+\left\lfloor \frac{l}{2}\right\rfloor - k+1 \\ m-i\end{pmatrix}k\right. \\
&\qquad\left. + \binom{k}{i}\begin{pmatrix} m+\left\lfloor \frac{l}{2}\right\rfloor - k \\ m-i\end{pmatrix}\left(m+\left\lfloor\frac{l+1}{2}\right\rfloor-k\right)\right]\xi_k \\
&= \frac{1}{2^{\lfloor (l+1)/2\rfloor}}\sum_{k=0}^{m+\lfloor (l+1)/2\rfloor}\sum_{i=0}^{m}(-1)^i\left[\binom{k}{i}\begin{pmatrix} m+\left\lfloor \frac{l+1}{2}\right\rfloor - k \\ m-i\end{pmatrix}(l+1)\right]\xi_k \\
&= \frac{1}{2^{\lfloor (l+1)/2\rfloor}}\sum_{k=0}^{m+\lfloor (l+1)/2\rfloor}(l+1)\begin{pmatrix} k, m+\left\lfloor \frac{l+1}{2}\right\rfloor - k \\ m\end{pmatrix}\xi_k,
\end{aligned}
\tag{6.42}
$$

where

$$
\xi_k = \begin{cases}
\mathfrak{a}_{012,i} & \text{if } k=0, \\
\mathfrak{d}_{210}^{(2k-2)/3} & \text{if } k\equiv 1 \bmod 3, \\
\mathfrak{c}_{021}^{(2k-1)/3} & \text{if } k\equiv 2 \bmod 3, \\
\mathfrak{e}_{102}^{(2k-3)/3} & \text{if } k\equiv 0 \bmod 3, \\
\mathfrak{b}_{021} & \text{if } k = m+\left\lfloor\frac{l+1}{2}\right\rfloor.
\end{cases}
\tag{6.43}
$$

By the last equation, we have

$$
\mathfrak{X}^{[l+1]}\mathfrak{t}^{[m]} = \frac{1}{2^{\lfloor (l+1)/2\rfloor}}\sum_{k=0}^{m+\lfloor (l+1)/2\rfloor}\begin{pmatrix} k, m+\left\lfloor \frac{l+1}{2}\right\rfloor - k \\ m\end{pmatrix}\xi_k,
\tag{6.44}
$$

where

$$\xi_k = \begin{cases} \mathfrak{a}_{012,i} & \text{if } k = 0, \\ \mathfrak{d}_{210}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}_{021}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-3)/3} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}_{021} & \text{if } k = m + \left\lfloor \dfrac{l+1}{2} \right\rfloor. \end{cases} \tag{6.45}$$

By the induction on n, we complete the proof of Lemma 6.8. □

LEMMA 6.9. *For all nonnegative integers n and m,*

$$\mathfrak{X}_0{}^{[n]}\mathfrak{X}_2\mathfrak{t}^{[m]} = \frac{1}{2^{\lfloor (n+1)/2 \rfloor}} \left\{ \sum_{k=0}^{m+\lfloor (n+1)/2 \rfloor} u_{(1,k)}\xi_{(1,k)} + \sum_{k=0}^{m+\lfloor n/2 \rfloor} u_{(2,k)}\xi_{(2,k)} \right\}, \tag{6.46}$$

where

$$\xi_{(1,k)} = \begin{cases} \mathfrak{d}_{210}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}_{021}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-3)/3} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}_{021} & \text{if } k = m + \left\lfloor \dfrac{n+1}{2} \right\rfloor, \end{cases}$$

$$u_{(1,k)} = \begin{cases} (n+1)\dbinom{k, j}{m} & \text{if } k = m + \left\lfloor \dfrac{n+1}{2} \right\rfloor,\ n \text{ odd}, \\ n\dbinom{k, j}{m} & \text{if } k = m + \left\lfloor \dfrac{n+1}{2} \right\rfloor,\ n \text{ even}, \\ (n+1)\dbinom{k, j}{m} - 2\dbinom{k, j-1}{m} & \text{if } k < m + \left\lfloor \dfrac{n+1}{2} \right\rfloor, \\ & j = m + \left\lfloor \dfrac{n+1}{2} \right\rfloor - k, \end{cases} \tag{6.47}$$

$$\xi_{(2,k)} = \begin{cases} \mathfrak{d}_{210}^{(2k-3)/3}(\mathfrak{a}_{201}'') & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{c}_{021}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-4)/3} & \text{if } k \equiv 2 \bmod 3, \end{cases}$$

$$u_{(2,k)} = \begin{cases} 2\dbinom{k, j}{m} & \text{if } n \text{ odd}, \\ \dbinom{k, j}{m} & \text{if } n \text{ even},\ j = m + \left\lfloor \dfrac{n}{2} \right\rfloor - k. \end{cases}$$

Lemma 6.10. *For all nonnegative integers n and m,*

$$\mathfrak{X}_0{}^{[n]}\mathfrak{X}_1\mathfrak{t}^{[m]} = \frac{1}{2^{\lfloor (n+1)/2 \rfloor}}\left\{\sum_{k=0}^{m+\lfloor (n+1)/2 \rfloor} u_k\xi_k + \sum_{k=0}^{m+\lfloor n/2 \rfloor} u'_k\xi'_k\right\}, \tag{6.48}$$

where

$$\xi_k = \begin{cases} \mathfrak{d}_{210}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}_{021}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-3)/3} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}_{021} & \text{if } k = m + \left\lfloor \dfrac{n+1}{2} \right\rfloor, \end{cases}$$

$$u_k = \begin{cases} (n+1)\dbinom{k,j}{m} & \text{if } k = 0,\ n \text{ odd}, \\ n\dbinom{k,j}{m} & \text{if } k = 0,\ n \text{ even}, \\ (n+1)\dbinom{k,j}{m} - 2\dbinom{k-1,j}{m} & \text{if } k \geq 1,\ j = m + \left\lfloor \dfrac{n+1}{2} \right\rfloor - k, \end{cases} \tag{6.49}$$

$$\xi'_k = \begin{cases} \mathfrak{e}'^{(2k-2)/3}_{102} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{d}'^{(2k-1)/3}_{021} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{c}'^{2k/3}_{210} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}'_{102} & \text{if } k = m + \left\lfloor \dfrac{n}{2} \right\rfloor, \end{cases}$$

$$u'_k = \begin{cases} 2\dbinom{k,j}{m} & \text{if } n \text{ odd}, \\ \dbinom{k,j}{m} & \text{if } n \text{ even},\ j = m + \left\lfloor \dfrac{n}{2} \right\rfloor - k. \end{cases}$$

Lemma 6.11. *For all nonnegative integers n and m,*

$$\mathfrak{X}_0{}^{[n]}\mathfrak{X}_1{}^{[2]}\mathfrak{t}^{[m]} = \frac{1}{2^{\lfloor (n+2)/2 \rfloor}}\left\{\sum_{k=0}^{m+\lfloor (n+2)/2 \rfloor} u_k\xi_k + \sum_{k=0}^{m+\lfloor (n+1)/2 \rfloor} u'_{(1,k)}\xi'_{(1,k)} + \sum_{k=0}^{m+\lfloor n/2 \rfloor} u'_{(2,k)}\xi'_{(2,k)}\right\}, \tag{6.50}$$

where

$$\xi_k = \begin{cases} \mathfrak{d}_{210}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}_{021}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-3)/3} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}_{021} & \text{if } k = m + \left\lfloor \frac{n+2}{2} \right\rfloor, \end{cases}$$

$$u_k = \begin{cases} n(n+1)\binom{k,j}{m} & \text{if } k = 0,\ n \text{ odd}, \\ (n+1)(n+2)\binom{k,j}{m} - 2\binom{k,j-1}{m} & \text{if } k = 0,\ n \text{ even}, \\ (n+1)(n+2)\binom{k,j}{m} - 2\binom{k-1,j}{m} & \text{if } k \geq 1,\ n \text{ odd}, \\ (n+1)(n+2)\binom{k,j}{m} - 2(2n+1)\binom{k-1,j}{m} & \text{if } k \geq 1,\ n \text{ even}, \\ & j = m + \left\lfloor \frac{n+2}{2} \right\rfloor - k, \end{cases}$$

$$\xi'_{(1,k)} = \begin{cases} \mathfrak{e}'^{(2k-2)/3}_{102} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{d}'^{(2k-1)/3}_{021} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{c}'^{(2k)/3}_{210} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}'_{102} & \text{if } k = m + \left\lfloor \frac{n+1}{2} \right\rfloor, \end{cases} \tag{6.51}$$

$$u'_{(1,k)} = \begin{cases} 2(n+1)\binom{k,j}{m} - 2\binom{k,j-1}{m} & \text{if } k \neq m + \left\lfloor \frac{n+1}{2} \right\rfloor,\ n \text{ odd}, \\ 2\left\{2(n+1)\binom{k,j}{m} - 2\binom{k,j-1}{m}\right\} & \text{if } k \neq m + \left\lfloor \frac{n+1}{2} \right\rfloor,\ n \text{ even}, \\ 2(n+1)\binom{k,j}{m} & \text{if } k = m + \left\lfloor \frac{n+1}{2} \right\rfloor,\ n \text{ odd}, \\ 2(2n+1)\binom{k,j}{m} & \text{if } k = m + \left\lfloor \frac{n+1}{2} \right\rfloor,\ n \text{ even}, \\ & j = m + \left\lfloor \frac{n+1}{2} \right\rfloor - k, \end{cases}$$

$$\xi'_{(2,k)} = \begin{cases} \mathfrak{d}'^{(2k-2)/3}_{021} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}'^{(2k-1)/3}_{210} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}'^{(2k-3)/3}_{102} & \text{if } k \equiv 0 \bmod 3, \end{cases} \qquad u'_{(2,k)} = 2\binom{k,j}{m}, \quad j = m + \left\lfloor \frac{n}{2} \right\rfloor - k.$$

Lemma 6.12. *For all nonnegative integers n and m,*

$$\mathfrak{X}_0{}^{[n]}\mathfrak{X}_1\mathfrak{X}_2\mathfrak{t}^{[m]} = \frac{1}{2^{\lfloor (n+2)/2\rfloor}}\Bigg\{\sum_{k=0}^{m+\lfloor (n+2)/2\rfloor} u_{(1,k)}\xi_{(1,k)} + \sum_{k=0}^{m+\lfloor (n-1)/2\rfloor} u_{(2,k)}\xi_{(2,k)} + \sum_{k=0}^{m+\lfloor (n+1)/2\rfloor} u'_{(1,k)}\xi'_{(1,k)} + \sum_{k=0}^{m+\lfloor n/2\rfloor} u'_{(2,k)}\xi'_{(2,k)} + \sum_{k=0}^{m+\lfloor n/2\rfloor} u''_k\xi''_k\Bigg\}, \tag{6.52}$$

where

$$\xi_{(1,k)} = \begin{cases} \mathfrak{d}_{210}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}_{021}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-3)/3} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}_{021} & \text{if } k = m + \left\lfloor \dfrac{n+2}{2} \right\rfloor, \end{cases}$$

$$u_{(1,k)} = \begin{cases} (n+1)^2\dbinom{k,j}{m} - 2n\dbinom{k,j-1}{m} & \text{if } k = 0 \text{ or } m + \left\lfloor \dfrac{n+2}{2} \right\rfloor,\ n \text{ odd}, \\ (n+1)(n+2)\dbinom{k,j}{m} - 2(n+1)\dbinom{k,j-1}{m} & \text{if } k = 0 \text{ or } m + \left\lfloor \dfrac{n+2}{2} \right\rfloor,\ n \text{ even}, \\ (n+1)(n+2)\dbinom{k,j}{m} - 4n\dbinom{k-1,j-1}{m} & \text{otherwise} \end{cases} \quad j = m + \left\lfloor \frac{n+2}{2} \right\rfloor - k,$$

$$\xi_{(2,k)} = \begin{cases} \mathfrak{d}_{210}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{c}_{021}^{2k/3} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \end{cases}$$

$$u_{(2,k)} = \begin{cases} (n+1)\dbinom{k+1,j}{m} - 2\dbinom{k,j}{m} & \text{if } n \text{ odd}, \\ 2\left\{(n+1)\dbinom{k+1,j}{m} - 2\dbinom{k,j}{m}\right\} & \text{if } n \text{ even}, \end{cases} \quad j = m + \left\lfloor \frac{n-1}{2} \right\rfloor - k,$$

$$\xi'_{(1,k)} = \begin{cases} \mathfrak{e}'^{(2k-2)/3}_{210} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{d}'^{(2k-1)/3}_{021} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{c}'^{2k/3}_{102} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}'_{102} & \text{if } k = m + \left\lfloor \dfrac{n+1}{2} \right\rfloor, \end{cases}$$

$$u'_{(1,k)} = \begin{cases} (n+1)\binom{k,j}{m} - 2\binom{k,j-1}{m} & \text{if } n \text{ odd}, \\ & \text{if } n \text{ even}, \\ 2\left\{n+1\binom{k,j}{m} - 2\binom{k,j-1}{m}\right\} & j = m + \left\lfloor \frac{n+1}{2} \right\rfloor - k, \end{cases}$$

$$\xi'_{(2,k)} = \begin{cases} \mathfrak{d}'^{(2k-2)/3}_{021} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}'^{(2k-1)/3}_{102} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}'^{(2k-3)/3}_{210} & \text{if } k \equiv 0 \bmod 3, \end{cases} \qquad u'_{(2,k)} = 2\binom{k,j}{m}, \quad j = m + \left\lfloor \frac{n}{2} \right\rfloor - k,$$

$$\xi''_k = \begin{cases} \mathfrak{d}''^{(2k-2)/3}_{102} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}''^{(2k-1)/3}_{210} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}''^{(2k-3)/3}_{021} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}''_{210} & k = m + \left\lfloor \frac{n}{2} \right\rfloor, \end{cases}$$

$$u''_k = \begin{cases} (n+1)\binom{k,j}{m} & \text{if } k = 0,\ n \text{ odd},\ j = m + \left\lfloor \frac{n+1}{2} \right\rfloor, \\ 4n\binom{k,j}{m} & \text{if } k = 0,\ n \text{ even},\ j = m + \left\lfloor \frac{n+1}{2} \right\rfloor, \\ 2\binom{k,j}{m} & k = m + \left\lfloor \frac{n}{2} \right\rfloor, \\ & \text{otherwise} \\ 2\binom{k-1,j+1}{m} & j = m + \left\lfloor \frac{n}{2} \right\rfloor - k. \end{cases} \tag{6.53}$$

Lemma 6.13. *For all nonnegative integers n and m,*

$$\begin{aligned} &\mathfrak{X}_0{}^{[n]}\mathfrak{X}_1{}^{[2]}\mathfrak{X}_2\mathfrak{t}^{[m]} \\ &\quad = \frac{1}{2^{\lfloor (n+3)/2 \rfloor}}\left\{\sum_{k=0}^{m+\lfloor (n+3)/2\rfloor} u_{(1,k)}\xi_{(1,k)} + \sum_{k=0}^{m+\lfloor n/2\rfloor} u_{(2,k)}\xi_{(2,k)} + \sum_{k=0}^{m+\lfloor (n+2)/2\rfloor} u'_{(1,k)}\xi'_{(1,k)}\right. \\ &\qquad\qquad \left. + \sum_{k=0}^{m+\lfloor (n+1)/2\rfloor} , u'_{(2,k)}\xi'_{(2,k)} + \sum_{k=0}^{m+\lfloor (n+1)/2\rfloor} u''_{(1,k)}\xi''_{(1,k)} + \sum_{k=0}^{m+\lfloor (n+2)/2\rfloor} , u''_{(2,k)}\xi''_{(2,k)}\right\}, \end{aligned} \tag{6.54}$$

where

$$\xi_{(1,k)} = \begin{cases} \mathfrak{d}_{210}^{(2k-2)/3} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}_{021}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-3)/3} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}_{021} & \text{if } k = m + \left\lfloor \frac{n+3}{2} \right\rfloor, \end{cases}$$

$$u_{(1,k)} = \begin{cases} (n+1)(n+2)(n+3)\binom{k,j}{m} \\ \quad -2(n+1)(n+3)\binom{k,j-1}{m} & \text{if } k = 0,\ n \text{ odd}, \\ (n+1)^2(n+2)\binom{k,j}{m} - 2(n^2+n+1)\binom{k,j-1}{m} & \text{if } k = 0,\ n \text{ even}, \\ (n+1)(n+2)(n+3)\binom{k,j}{m} \\ \quad -2(n+1)(2n+3)\binom{k-1,j}{m} & \text{if } k = m + \left\lfloor \frac{n+3}{2} \right\rfloor,\ n \text{ odd}, \\ (n+1)(n+2)^2\binom{k,j}{m} \\ \quad -2(2n^2+3n+2)\binom{k-1,j}{m} & \text{if } k = m + \left\lfloor \frac{n+3}{2} \right\rfloor,\ n \text{ even}, \\ (n+1)(n+2)(n+3)\binom{k,j}{m} & \text{otherwise} \\ \quad -2(n^2+n+1)\binom{k,j-1}{m} & j = m + \left\lfloor \frac{n+3}{2} \right\rfloor - k, \end{cases}$$

$$\xi_{(2,k)} = \begin{cases} \mathfrak{d}_{210}^{(2k-1)/3} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{c}_{021}^{2k/3} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{e}_{102}^{(2k-1)/3} & \text{if } k \equiv 1 \bmod 3, \end{cases}$$

$$u_{(2,k)} = \begin{cases} (n+1)(n+2)\binom{k+1,j}{m} - 2(2n+1)\binom{k,j}{m} & \text{if } n \text{ even}, \\ 2\left\{(n+1)(n+2)\binom{k+1,j}{m} - 2(2n+1)\binom{k,j}{m}\right\} & \text{if } n \text{ odd}, \\ & j = m + \left\lfloor \frac{n}{2} \right\rfloor - k, \end{cases}$$

$$\xi'_{(1,k)} = \begin{cases} \mathfrak{e}'^{(2k-2)/3}_{210} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{d}'^{(2k-1)/3}_{021} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{c}'^{2k/3}_{102} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{b}'_{102} & \text{if } k = m + \left\lfloor \frac{n+2}{2} \right\rfloor, \end{cases}$$

$$u'_{(1,k)} = \begin{cases} 2\left\{(n^2+3n+2)\binom{k,j}{m} - \binom{k-1,j}{m}\right\} & \text{if } n \text{ even},\ k = m + \left\lfloor \dfrac{n+2}{2} \right\rfloor, \\ (2n^2+3n+1)\binom{k,j}{m} \\ \quad -(3n+3)\binom{k+1,j-1}{m} + 2\binom{k,j-1}{m} & \text{if } n \text{ even},\ k < m + \left\lfloor \dfrac{n+2}{2} \right\rfloor, \\ 2\left\{(2n^2+3n+1)\binom{k,j}{m} \right. & \text{if } n \text{ odd}, \\ \quad \left. -(3n+3)\binom{k+1,j-1}{m} + 2\binom{k,j-1}{m}\right\} & j = m + \left\lfloor \dfrac{n+2}{2} \right\rfloor - k, \end{cases}$$

$$\xi'_{(2,k)} = \begin{cases} \mathfrak{d}'^{(2k-2)/3}_{021} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{c}'^{(2k-1)/3}_{102} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{e}'^{(2k-3)/3}_{210} & \text{if } k \equiv 0 \bmod 3, \end{cases}$$

$$u'_{(2,k)} = \begin{cases} 2(3n+2)\binom{k,j}{m} & \text{if } k = 0,\ n \text{ even}, \\ & \text{otherwise}, \\ 2(3n+3)\binom{k,j}{m} - 2\binom{k-1,j}{m} & j = m + \left\lfloor \dfrac{n+1}{2} \right\rfloor - k, \end{cases}$$

$$\xi''_{(1,k)} = \begin{cases} \mathfrak{d}''^{2k/3}_{102} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{c}''^{(2k+1)/3}_{210} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{e}''^{(2k-1)/3}_{021} & \text{if } k \equiv 2 \bmod 3, \\ \mathfrak{b}''_{210} & \text{if } k = m + \left\lfloor \dfrac{n+1}{2} \right\rfloor, \end{cases}$$

$$u''_{(1,k)} = \begin{cases} 2(2n+1)\binom{k,j}{m} & \text{if } k = m + \left\lfloor \dfrac{n+1}{2} \right\rfloor,\ n \text{ even}, \\ & \text{otherwise} \\ 2\left\{2(n+1)\binom{k,j}{m} - 2\binom{k,j-1}{m}\right\} & j = m + \left\lfloor \dfrac{n+1}{2} \right\rfloor - k, \end{cases}$$

$$\xi''_{(2,k)} = \begin{cases} \mathfrak{a}''_{201} & \text{if } k = 0, \\ \mathfrak{d}''^{(2k-3)/3}_{102} & \text{if } k \equiv 0 \bmod 3, \\ \mathfrak{c}''^{(2k-2)/3}_{210} & \text{if } k \equiv 1 \bmod 3, \\ \mathfrak{e}''^{(2k-4)/3}_{021} & \text{if } k \equiv 2 \bmod 3, \end{cases}$$

$$u''_{(2,k)} = \begin{cases} 2n(n+1)\binom{k,j}{m} & \textit{if } k=0,\ n \textit{ odd},\\ (n^2+3n+1)\binom{k,j}{m} - \binom{k+1,j-1}{m} & \textit{if } k=0,\ n \textit{ even},\\ 4\binom{k-1,j}{m} & \textit{if } k\neq 0,\ n \textit{ odd},\\ 2\binom{k-1,j}{m} & \textit{if } k\neq 0,\ n \textit{ even},\\ & j = m + \left\lfloor \frac{n+2}{2} \right\rfloor - k. \end{cases} \tag{6.55}$$

Also the inverse relations can be given as follows.

Lemma 6.14. *Let $w \in \widehat{W}$ with $\ell(w) = s$. Then,*

$$\begin{aligned} \mathfrak{a}_{012} &= \frac{1}{2^{\lfloor s/2\rfloor}} \sum_{k=0}^{\lfloor s/2\rfloor} \binom{k, \left\lfloor \frac{s}{2} \right\rfloor - k}{0} 2^{\lfloor s/2\rfloor - k} \mathfrak{X}_0{}^{[s-2k]} \mathfrak{t}^{[k]},\\ \mathfrak{b}_{021} &= \frac{1}{2^{\lfloor s/2\rfloor}} \sum_{k=0}^{\lfloor s/2\rfloor} \binom{k, \left\lfloor \frac{s}{2} \right\rfloor - k}{\left\lfloor \frac{s}{2} \right\rfloor} 2^{\lfloor s/2\rfloor - k} \mathfrak{X}_0{}^{[s-2k]} \mathfrak{t}^{[k]},\\ \mathfrak{c}^m_{021} &= \frac{1}{2^{\lfloor s/2\rfloor}} \sum_{k=0}^{\lfloor s/2\rfloor} \binom{k, \left\lfloor \frac{s}{2} \right\rfloor - k}{\frac{3m+1}{2}} 2^{\lfloor s/2\rfloor - k} \mathfrak{X}_0{}^{[s-2k]} \mathfrak{t}^{[k]}, \quad m \textit{ is odd},\ s \geq 3m+3,\\ \mathfrak{d}^m_{210} &= \frac{1}{2^{\lfloor s/2\rfloor}} \sum_{k=0}^{\lfloor s/2\rfloor} \binom{k, \left\lfloor \frac{s}{2} \right\rfloor - k}{\frac{3m+2}{2}} 2^{\lfloor s/2\rfloor - k} \mathfrak{X}_0{}^{[s-2k]} \mathfrak{t}^{[k]}, \quad m \textit{ is even},\ s \geq 3m+4,\\ \mathfrak{e}^m_{102} &= \frac{1}{2^{\lfloor s/2\rfloor}} \sum_{k=0}^{\lfloor s/2\rfloor} \binom{k, \left\lfloor \frac{s}{2} \right\rfloor - k}{\frac{3m+3}{2}} 2^{\lfloor s/2\rfloor - k} \mathfrak{X}_0{}^{[s-2k]} \mathfrak{t}^{[k]}, \quad m \textit{ is odd},\ s \geq 3m+5. \end{aligned} \tag{6.56}$$

Lemma 6.15. *Let $w \in \widehat{W}$ with $\ell(w) = s$. Then,*

$$\begin{aligned} \mathfrak{c}^m_{021} &= \frac{1}{2^{\lfloor s/2\rfloor}} \sum_{k=0}^{\lfloor s/2\rfloor} \left(\binom{k, \left\lfloor \frac{s}{2} \right\rfloor - k}{\frac{3m+2}{2}} - (s-2k) \binom{k, \left\lfloor \frac{s}{2} \right\rfloor - k - 1}{\frac{3m+2}{2}} \right) 2^{\lfloor s/2\rfloor - k} \mathfrak{X}_0{}^{[s-2k]} \mathfrak{t}^{[k]}\\ &\quad + \frac{1}{2^{\lfloor s/2\rfloor - 1}} \sum_{k=0}^{\lfloor s/2\rfloor - 1} \binom{k, \left\lfloor \frac{s}{2} \right\rfloor - 1 - k}{\frac{3m+2}{2}} 2^{\lfloor s/2\rfloor - 1 - k} \mathfrak{X}_0{}^{[s-2k-1]} \mathfrak{X}_2 \mathfrak{t}^{[k]}, \quad m \textit{ is even},\ s \geq 3m+3, \end{aligned}$$

$$\mathfrak{d}_{210}^{m} = \frac{1}{2^{\lfloor s/2\rfloor}} \sum_{k=0}^{\lfloor s/2\rfloor} \left(\begin{pmatrix} k, \left\lfloor \frac{s}{2} \right\rfloor - k \\ \frac{3m+3}{2} \end{pmatrix} - (s-2k) \begin{pmatrix} k, \left\lfloor \frac{s}{2} \right\rfloor - k - 1 \\ \frac{3m+3}{2} \end{pmatrix} \right) 2^{\lfloor s/2\rfloor - k} \mathfrak{X}_0{}^{[s-2k]} \mathfrak{t}^{[k]}$$
$$+ \frac{1}{2^{\lfloor s/2\rfloor - 1}} \sum_{k=0}^{\lfloor s/2\rfloor - 1} \begin{pmatrix} k, \left\lfloor \frac{s}{2} \right\rfloor - 1 - k \\ \frac{3m+3}{2} \end{pmatrix} 2^{\lfloor s/2\rfloor - 1 - k} \mathfrak{X}_0{}^{[s-2k-1]} \mathfrak{X}_2 \mathfrak{t}^{[k]}, \quad m \text{ is odd, } s \geq 3m+4,$$
$$\mathfrak{e}_{102}^{m} = \frac{1}{2^{\lfloor s/2\rfloor}} \sum_{k=0}^{\lfloor s/2\rfloor} \left(\begin{pmatrix} k, \left\lfloor \frac{s}{2} \right\rfloor - k \\ \frac{3m+4}{2} \end{pmatrix} - (s-2k) \begin{pmatrix} k, \left\lfloor \frac{s}{2} \right\rfloor - k - 1 \\ \frac{3m+4}{2} \end{pmatrix} \right) 2^{\lfloor s/2\rfloor - k} \mathfrak{X}_0{}^{[s-2k]} \mathfrak{t}^{[k]}$$
$$+ \frac{1}{2^{\lfloor s/2\rfloor - 1}} \sum_{k=0}^{\lfloor s/2\rfloor - 1} \begin{pmatrix} k, \left\lfloor \frac{s}{2} \right\rfloor - 1 - k \\ \frac{3m+4}{2} \end{pmatrix} 2^{\lfloor s/2\rfloor - 1 - k} \mathfrak{X}_0{}^{[s-2k-1]} \mathfrak{X}_2 \mathfrak{t}^{[k]}, \quad m \text{ is even, } s \geq 3m+5. \tag{6.57}$$

Let $R = \mathbb{Z}[1/2]$ and let $\Gamma_R(x_0,x_1,x_2,y)$ be the divided power algebra over R, where $\deg x_0 = \deg x_1 = \deg x_2 = 2$ and $\deg y = 4$.

Theorem 6.16. *$H^*(LSU_3/T,R)$ is graded isomorphic to $\Gamma_R(x_0,x_1,x_2,y)/I_R$, where*

$$I_R = \left\langle 2\left(x_0{}^{[2]} + x_1{}^{[2]} + x_2{}^{[2]}\right) - x_0x_1 - x_1x_2 - x_2x_0, \left(x_0 - x_1\right)^{[3]} \right\rangle. \tag{6.58}$$

Proof. Since the odd cohomology is trivial, by the universal coefficient theorem, the calculations can be done for $R = \mathbb{Z}[1/2]$.

Let

$$f_1 = 2\left(x_0{}^{[2]} + x_1{}^{[2]} + x_2{}^{[2]}\right) - x_0x_1 - x_1x_2 - x_2x_0 = 0,$$
$$f_2 = \left(x_0 - x_1\right)^{[3]} = 0 \tag{6.59}$$

be two relations. Let us consider the ideal $I_R = \langle f_1, f_2\rangle$. If we select the graded monomial order with $x_2 > x_1 > x_0$, then the leading monomial of f_1 is $x_2{}^{[2]}$ and the leading monomial of f_2 is $x_1{}^{[3]}$. Since the leading monomials f_1 and f_2 are relatively prime, we can say that $G = \{f_1, f_2\}$ is a Groebner basis for I_R [4]. Hence $\Gamma_R(x_0,x_1,x_2,y)/I_R$ and $\Gamma_R(x_0,x_1,x_2,y)/\langle x_1{}^{[3]}, x_2{}^{[2]}\rangle$ have the same basis as vector spaces [5]. The basis of $\Gamma_R(x_0,x_1,x_2,y)/\langle x_1{}^{[3]}, x_2{}^{[2]}\rangle$ is the monomials not involving the third and higher powers of x_1, and the second and higher powers of x_2. These are exactly the monomials which are in one of the following forms $x_0{}^{[n]}y^{[m]}$, $x_0{}^{[n]}x_1y^{[m]}$, $x_0{}^{[n]}x_2y^{[m]}$, $x_0{}^{[n]}x_1x_2y^{[m]}$, $x_0{}^{[n]}x_1{}^{[2]}y^{[m]}$, and $x_0{}^{[n]}x_1{}^{[2]}x_2y^{[m]}$ where $n, m \geq 0$.

We can show that there are exactly $3s$ monomial of the degree $2s$ with $s \geq 0$ satisfying the above forms.

Let s be even. Then for each degree $2s$, there are $(\lfloor s/2 \rfloor + 1)$ monomials of the type of the form ${x_0}^{[n]}y^{[m]}$ and $\lfloor s/2 \rfloor$ monomials of each type of the forms ${x_0}^{[n]}x_1y^{[m]}$, ${x_0}^{[n]}x_2y^{[m]}$, ${x_0}^{[n]}x_1x_2y^{[m]}$, ${x_0}^{[n]}{x_1}^{[2]}y^{[m]}$, and ${x_0}^{[n]}{x_1}^{[2]}x_2y^{[m]}$ and $(\lfloor s/2 \rfloor - 1)$ monomials of the type of the form ${x_0}^{[n]}{x_1}^{[2]}x_2y^{[m]}$, respectively. So we have totally $3s$ monomials.

Let s be odd. Then for each degree $2s$, there are $(\lfloor s/2 \rfloor + 1)$ monomials of each type of the forms ${x_0}^{[n]}y^{[m]}$, ${x_0}^{[n]}x_1y^{[m]}$, ${x_0}^{[n]}x_2y^{[m]}$, and $\lfloor s/2 \rfloor$ monomials of each type of the forms

$$ {x_0}^{[n]}x_1x_2y^{[m]},\ {x_0}^{[n]}{x_1}^{[2]}y^{[m]},\ {x_0}^{[n]}{x_1}^{[2]}x_2y^{[m]}, \tag{6.60} $$

respectively. So we have totally $3s$ monomials again.

Now let us consider the integral cohomology of LSU_3/T. By the lemmas above, we have two relations in $H^*(LSU_3/T,R)$ as follows:

$$ \begin{gathered} F_1 = 2\left({\mathfrak{X}_0}^{[2]} + {\mathfrak{X}_1}^{[2]} + {\mathfrak{X}_2}^{[2]}\right) - \mathfrak{X}_0\mathfrak{X}_1 - \mathfrak{X}_1\mathfrak{X}_2 - \mathfrak{X}_2\mathfrak{X}_0 = 0, \\ F_2 = \left(\mathfrak{X}_0 - \mathfrak{X}_1\right)^{[3]} = 0. \end{gathered} \tag{6.61} $$

Then we can define an algebra morphism $\phi : H^*(LSU_3/T,R) \to \Gamma_R(x_0,x_1,x_2,y)/I_R$ by

$$ \begin{aligned} \mathfrak{X}_0 &\longrightarrow x_0, \\ \mathfrak{X}_1 &\longrightarrow x_1, \\ \mathfrak{X}_2 &\longrightarrow x_2, \\ \mathfrak{t} &\longrightarrow y \end{aligned} \tag{6.62} $$

which is an isomorphism by the lemmas above. □

Now we will discuss cohomology of ΩG with respect to LG/T and G/T, where G is a compact semisimple Lie group. Since ΩG is homotopic to Ω_{pol}, the discussion can be restricted to the Kac-Moody groups and homogeneous spaces. The Lie algebras of $L_{\text{pol}}G_{\mathbb{C}}/B^+$, $L_{\text{pol}}G_{\mathbb{C}}/G_{\mathbb{C}}$ and $G_{\mathbb{C}}/B$ are $\mathbf{g}[t,t^{-1}]/\mathbf{b}^+$, $\mathbf{g}[t,t^{-1}]/\mathbf{g}$, and $\mathbf{g}/\mathbf{b}$, respectively. There is a surjective homomorphism

$$ \text{ev}_{t=1} : \mathbf{g}[t,t^{-1}]/\mathbf{b}^+ \longrightarrow \mathbf{g}/\mathbf{b}, \tag{6.63} $$

with $\ker \text{ev}_{t=1} = \mathbf{g}[t,t^{-1}]/\mathbf{g}$. Since the odd cohomology groups of $\mathbf{g}[t,t^{-1}]/\mathbf{b}^+$ and $\mathbf{g}/\mathbf{b}$ are trivial, the second term E_2^{**} of the Leray-Serre spectral sequence collapses and hence we have the following.

Theorem 6.17. *Let R be a commutative ring with unit. Then there exist an injective homomorphism $j : H^*(G/T,R) \to H^*(LG/T,R)$ and a surjective homomorphism $i : H^*(LG/T, R) \to H^*(\Omega G,R)$. In particular, $J = \operatorname{im} j^+$ is an ideal of $H^*(LG/T,R)$ and*

$$ H^*(\Omega G,R) \cong H^*(LG/T,R)//J. \tag{6.64} $$

Corollary 6.18. *Let $R = \mathbb{Z}[1/2]$. Then,*

$$ H^*(\Omega SU_3,R) \cong \Gamma_R(x_0,x_1,x_2,y)/(I_R,x_1,x_2) \cong \Gamma_R(x_0,x_1,x_2,y)//\langle x_1,x_2\rangle \cong \Gamma_R(x_0,y). \tag{6.65} $$

Now we will give a different approach to determine the cohomology ring of based loop group ΩG using the Schubert calculus. For a compact simply connected semisimple Lie group G, we have the following theorem from [22].

Theorem 6.19. *The natural map*

$$G \longrightarrow LG \longrightarrow LG/G \cong \Omega G \tag{6.66}$$

is a split extension of Lie groups.

Theorem 6.20. *Let G be a compact simply connected semisimple Lie group and let T be a maximal torus of G. Then $\pi : LG/T \to LG/G$ is a fiber bundle with the fiber G/T.*

Proof. Since $LG \to LG/G$ is a principal G-bundle and G/T is a left G-space by the action $g_1 \cdot g_2 T = g_1 g_2 T$ for $g_1, g_2 \in G$, we have a fibration

$$G/T \longrightarrow LG \times_G G/T \longrightarrow \Omega G. \tag{6.67}$$

Therefore, we have to show that $LG \times_G G/T$ is diffeomorphic to LG/T. Since $LG \times_G G/T$ is equal to

$$\{[\gamma, gT] : [\gamma, gT] = [\gamma h, h^{-1} gT] \ \forall g, h \in G, \gamma \in LG\}, \tag{6.68}$$

we define a smooth map $\tau : LG \times_G G/T \to LG/T$ given by $[\gamma, gT] \to \gamma gT$. It is well defined because for $h \in G$,

$$\tau([\gamma h, h^{-1} gT]) = \gamma h h^{-1} gT = \gamma gT = \tau([\gamma, gT]). \tag{6.69}$$

For every γT, we can find an element $[\gamma, T] \in LG \times_G G/T$ such that $\tau([\gamma, T]) = \gamma T$. So, τ is a surjective map. Now, we will show that τ is an injective map. Let $[\gamma_1, g_1 T], [\gamma_2, g_2 T] \in LG \times_G G/T$ such that

$$\tau([\gamma_1, g_1 T]) = \tau([\gamma_2, g_2 T]). \tag{6.70}$$

Equation (6.70) gives

$$\gamma_1 g_1 T = \gamma_2 g_2 T. \tag{6.71}$$

So, $(\gamma_1 g_1)^{-1}(\gamma_2 g_2), (\gamma_2 g_2)^{-1}(\gamma_1 g_1) \in T$. Then,

$$\begin{aligned} [\gamma_1, g_1 T] &= [\gamma_1 g_1, g_1^{-1} g_1 T] = [\gamma_1 g_1, T] = [(\gamma_1 g_1)(\gamma_1 g_1)^{-1}(\gamma_2 g_2), (\gamma_2 g_2)^{-1}(\gamma_1 g_1) T] \\ &= [\gamma_2 g_2, T] = [\gamma_2 g_2 g_2^{-1}, g_2 T] = [\gamma_2, g_2 T]. \end{aligned} \tag{6.72}$$

Thus, we have proved that τ is an injective map and its inverse is given by $\gamma T \to [\gamma, T]$ which is smooth map. Then $\pi : LG/T \to LG/G = \Omega G$, given by $\gamma T \to \gamma G$, is a fiber bundle map. □

Since LG/T is a fiber bundle over ΩG with the fiber G/T, by the Leray-Serre spectral sequence of the fibration and Kostant and Kumar [16, Corollary (5.13)], $\theta : H^*(\Omega G, \mathbb{Z}) \to H^*(LG/T, \mathbb{Z})$ is injective and $\theta(H^*(\Omega G, \mathbb{Z}))$ is generated by the Schubert classes $\{\varepsilon^w\}_{w\in\widehat{W}}$ in the cohomology of $LG/T$and hence we can determine the cohomology ring of ΩG.

Let $R = \mathbb{Z}[1/2]$ and let $\Gamma_R(\gamma,\beta)$ be the divided power algebra with $\deg\gamma = 2$ and $\deg\beta = 4$.

Theorem 6.21. *$H^*(\Omega SU(3), R)$ is isomorphic to $\Gamma_R(\gamma,\beta)$ with the R-module basis*

$$\gamma^{[s-2k]}\beta^{[k]}, \quad 0 \le k \le \left\lfloor \frac{s}{2} \right\rfloor \tag{6.73}$$

in each degree $2s$ for $s \ge 1$.

Proof. Since the odd cohomology is trivial, by the universal coefficient theorem, the calculations can be done for $R = \mathbb{Z}[1/2]$. The integral cohomology of ΩSU_3 is generated by the Schubert classes indexed:

$$\begin{aligned}\widehat{W} &= \{\overline{\ell(w)} : w \in \widetilde{W}\}\\ &= \left\{A^k_{012,i}, B^k_{021,i}, C^{k_1,k_2}_{021,i}, D^{l_1,l_2}_{210,i}, E^{n_1,n_2}_{102,i} : k \ge 0,\ k_1 \text{ and } n_1 \text{ odd},\ l_1 \text{ even},\ i = 0,1,2\right\}.\end{aligned} \tag{6.74}$$

Let $\mathfrak{a}^k_{012,i}$, $\mathfrak{b}^k_{021,i}$, $\mathfrak{c}^{k_1,k_2}_{021,i}$, $\mathfrak{d}^{l_1,l_2}_{210,i}$, $\mathfrak{e}^{n_1,n_2}_{102,i}$ be Schubert classes indexed by elements $A^k_{012,i}$, $B^k_{021,i}$, $C^{k_1,k_2}_{021,i}$, $D^{l_1,l_2}_{210,i}$, $E^{n_1,n_2}_{102,i}$ of the Weyl group $\widehat{W}$, respectively. □

Lemma 6.22. *Let $w \in \widehat{W}$ with $\ell(w) = s$. Then,*

$$\begin{aligned}
\mathfrak{a}_{012} &= \frac{1}{2^{\lfloor s/2\rfloor}} \sum_{k=0}^{\lfloor s/2\rfloor} \begin{pmatrix} k, \left\lfloor \frac{s}{2} \right\rfloor - k \\ 0 \end{pmatrix} 2^{\lfloor s/2\rfloor - k} \mathfrak{X}^{[s-2k]} \mathfrak{t}^{[k]},\\
\mathfrak{b}_{021} &= \frac{1}{2^{\lfloor s/2\rfloor}} \sum_{k=0}^{\lfloor s/2\rfloor} \begin{pmatrix} k, \left\lfloor \frac{s}{2} \right\rfloor - k \\ \left\lfloor \frac{s}{2} \right\rfloor \end{pmatrix} 2^{\lfloor s/2\rfloor - k} \mathfrak{X}^{[s-2k]} \mathfrak{t}^{[k]},\\
\mathfrak{c}^m_{021} &= \frac{1}{2^{\lfloor s/2\rfloor}} \sum_{k=0}^{\lfloor s/2\rfloor} \begin{pmatrix} k, \left\lfloor \frac{s}{2} \right\rfloor - k \\ \frac{3m+1}{2} \end{pmatrix} 2^{\lfloor s/2\rfloor - k} \mathfrak{X}^{[s-2k]} \mathfrak{t}^{[k]}, \quad m \text{ is odd, } s \ge 3m+3,\\
\mathfrak{d}^m_{210} &= \frac{1}{2^{\lfloor s/2\rfloor}} \sum_{k=0}^{\lfloor s/2\rfloor} \begin{pmatrix} k, \left\lfloor \frac{s}{2} \right\rfloor - k \\ \frac{3m+2}{2} \end{pmatrix} 2^{\lfloor s/2\rfloor - k} \mathfrak{X}^{[s-2k]} \mathfrak{t}^{[k]}, \quad m \text{ is even, } s \ge 3m+4,\\
\mathfrak{e}^m_{102} &= \frac{1}{2^{\lfloor s/2\rfloor}} \sum_{k=0}^{\lfloor s/2\rfloor} \begin{pmatrix} k, \left\lfloor \frac{s}{2} \right\rfloor - k \\ \frac{3m+3}{2} \end{pmatrix} 2^{\lfloor s/2\rfloor - k} \mathfrak{X}^{[s-2k]} \mathfrak{t}^{[k]}, \quad m \text{ is odd, } s \ge 3m+5.
\end{aligned} \tag{6.75}$$

Now we define a graded algebra isomorphism $\varphi : \Gamma_R(\gamma,\beta) \to H^*(\Omega SU(3),R)$ by

$$\varphi^{2s}\left(\sum_{k=0}^{\lfloor s/2\rfloor} u_k\gamma^{[s-2k]}\beta^{[k]}\right) = \frac{1}{2^{\lfloor s/2\rfloor -k}}\sum_{k=0}^{\lfloor s/2\rfloor}\sum_{j=0}^{\lfloor s/2\rfloor} u_k\binom{j,\left\lfloor \frac{s}{2}\right\rfloor - j}{k}\xi_j, \tag{6.76}$$

where

$$\xi_j = \begin{cases} \mathfrak{a}_{012,i} & \text{if } j = 0, \\ \mathfrak{d}_{210}^{(2j-2)/3} & \text{if } j \equiv 1 \bmod 3, \\ \mathfrak{c}_{021}^{(2j-1)3} & \text{if } j \equiv 2 \bmod 3, \\ \mathfrak{e}_{102}^{(2j-3)/3} & \text{if } j \equiv 0 \bmod 3, \\ \mathfrak{b}_{021} & \text{if } j = \left\lfloor \frac{s}{2}\right\rfloor. \end{cases} \tag{6.77}$$

Then $H^*(\Omega SU(3),R)$ is isomorphic to $\Gamma_R(\gamma,\beta)$ with the R-module basis $\gamma^{[s-2k]}\beta^{[k]}$, $0 \le k \le \lfloor s/2\rfloor$ in each degree $2s$ for $s \ge 1$.

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