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Research Article

Coefficient Bounds for Some Families of Starlike and Convex Functions of Reciprocal Order

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The aim of the present paper is to investigate coefficient estimates, Fekete-Szegő inequality, and upper bound of third Hankel determinant for some families of starlike and convex functions of reciprocal order.

1. Introduction

Let $\mathscr{A}$ denote the class of functions $f(z)$ which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U}). \tag{1}$$

Also let $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ denote the usual classes of starlike and convex functions of order α, $0 \le \alpha < 1$, respectively. In 1975, Silverman [1] proved that $f(z) \in \mathcal{S}^*(\alpha)$ if it satisfies the condition

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1 - \alpha, \quad (z \in \mathbb{U}). \tag{2}$$

Geometrical meaning of inequality (2) is that $zf'(z)/f(z)$ maps $\mathbb{U}$ onto the interior of the circle with center at 1 and radius $1 - \alpha$.

By $\mathcal{S}_*(\alpha)$ and $\mathcal{K}_*(\alpha)$, we mean the classes of starlike and convex functions of reciprocal order α, $0 \le \alpha < 1$ which are defined, respectively, by

$$\mathcal{S}_*(\alpha) = \left\{ f(z) \in \mathscr{A} : \operatorname{Re} \frac{f(z)}{zf'(z)} > \alpha,\ (z \in \mathbb{U}) \right\},$$

$$\mathcal{K}_*(\alpha) = \left\{ f(z) \in \mathscr{A} : \operatorname{Re} \frac{f'(z)}{zf''(z) + f'(z)} > \alpha,\ (z \in \mathbb{U}) \right\}. \tag{3}$$

Recently in 2008, Nunokawa and his coauthors [2] improved inequality (2) for the class $\mathcal{S}_*(\alpha)$ and they proved that, for $f(z) \in \mathcal{S}_*(\alpha)$, $0 < \alpha < 1/2$, if and only if the following inequality holds:

$$\left| \frac{zf'(z)}{f(z)} - \frac{1}{2\alpha} \right| < \frac{1}{2\alpha}, \quad (z \in \mathbb{U}). \tag{4}$$

In view of these results we now define the following subclass of analytic functions of reciprocal order and investigate its various properties.

Definition 1. A function $f(z) \in \mathscr{A}$ is said to be in the class $\mathscr{L}(\lambda, \gamma)$, with $\gamma \in \mathbb{C} \setminus \{0\}$ and $\lambda \in [0,1]$, if it satisfies the inequality

$$\operatorname{Re}\left(1 + \frac{1}{\gamma}\left(\frac{F_\lambda(z)}{zF'_\lambda(z)} - 1\right)\right) > 0, \tag{5}$$

where

$$F_\lambda(z) = (1-\lambda) f(z) + \lambda z f'(z). \tag{6}$$

Example 2. Let us define the functions $F_\lambda(z)$ by

$$F_\lambda(z) = \frac{z}{(1 + (2\gamma - 1) z)^{2\gamma/(2\gamma-1)}}. \tag{7}$$

This implies that

$$\frac{zF'_\lambda(z)}{F_\lambda(z)} = \frac{1-z}{1 + (2\gamma - 1) z}. \tag{8}$$

Hence

$$1 + \frac{1}{\gamma}\left(\frac{F_\lambda(z)}{zF'_\lambda(z)} - 1\right) = \frac{1+z}{1-z} \tag{9}$$

and this further implies that

$$\operatorname{Re}\left(1 + \frac{1}{\gamma}\left(\frac{F_\lambda(z)}{zF'_\lambda(z)} - 1\right)\right) = \operatorname{Re}\frac{1+z}{1-z} > 0, \quad (z \in \mathbb{U}). \tag{10}$$

The qth Hankel determinant $H_q(n)$, $q \geq 1$, $n \geq 1$, for a function $f(z) \in \mathscr{A}$ is studied by Noonan and Thomas [3] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{11}$$

In literature many authors have studied the determinant $H_q(n)$. For example, Arif et al. [4, 5] studied the qth Hankel determinant for some subclasses of analytic functions. Hankel determinant of exponential polynomials is obtained by Ehrenborg in [6]. The Hankel transform of an integer sequence and some of its properties were discussed by Layman [7]. It is well known that the Fekete-Szegő functional $|a_3 - a_2^2|$ is $H_2(1)$. Fekete-Szegő then further generalized the estimate $|a_3 - \lambda a_2^2|$ with λ real and $f(z) \in \mathscr{S}$. Moreover, we also know that the functional $|a_2 a_4 - a_3^2|$ is equivalent to $H_2(2)$. The sharp upper bounds of the second Hankel determinant for the familiar classes of starlike and convex functions were studied by Janteng et al. [8]; that is, for $f(z) \in \mathscr{S}^*$ and $f(z) \in \mathscr{C}$, they obtained $|a_2 a_4 - a_3^2| \leq 1$ and $8|a_2 a_4 - a_3^2| \leq 1$, respectively. In 2007, Babalola [9] considered the third Hankel determinant $H_3(1)$ and obtained the upper bound of the well-known classes of bounded-turning, starlike and convex functions. In 2013 Raza and Malik [10] studied the Hankel third determinant related with lemniscate of Bernoulli. In the present investigation, we study the upper bound of $H_3(1)$ for a subclass of analytic functions of reciprocal order by using Toeplitz determinants.

In this paper we study some useful results including coefficient estimates, Fekete-Szegő inequality, and upper bound of third Hankel determinant for the functions belonging to the class $\mathscr{L}(\lambda, \gamma)$.

Throughout in this paper we assume that $\gamma \in \mathbb{C} \setminus \{0\}$ and $\lambda \in [0,1]$ unless otherwise stated.

For our results we will need the following Lemmas.

Lemma 3 (see [11]). *If $q(z)$ is a function with $\operatorname{Re} q(z) > 0$ and is of the form*

$$q(z) = 1 + c_1 z + c_2 z^2 + \cdots \tag{12}$$

then

$$|c_n| \leq 2, \quad \text{for } n \geq 1. \tag{13}$$

Lemma 4 (see [12]). *If $q(z)$ is of the form (12) with positive real part, then the following sharp estimate holds:*

$$\left|c_2 - \nu c_1^2\right| \leq 2 \max\{1, |2\nu - 1|\}, \quad \text{for all } \nu \in \mathbb{C}. \tag{14}$$

Lemma 5 (see [13]). *If $q(z)$ is of the form (12) with positive real part, then*

$$\begin{aligned} 2c_2 &= c_1^2 + x\left(4 - c_1^2\right), \\ 4c_3 &= c_1^3 + 2\left(4 - c_1^2\right) c_1 x - c_1\left(4 - c_1^2\right) x^2 \\ &\quad + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right) z, \end{aligned} \tag{15}$$

for some x, z with $|x| \leq 1$ and $|z| \leq 1$.

2. Some Properties of the Class $\mathscr{L}(\lambda,\gamma)$

Theorem 6. *Let $f(z) \in \mathscr{L}(\lambda, \gamma)$. Then*

$$|a_2| \leq \frac{2|\gamma|}{1+\lambda}, \tag{16}$$

and for all $n = 3, 4, 5, \ldots$

$$|a_n| \leq \frac{2|\gamma|}{(n-1)(1 + \lambda(n-1))} \prod_{k=2}^{n-1}\left(1 + \frac{2|\gamma| k}{k-1}\right). \tag{17}$$

Proof. Let us define the function $q(z)$ by

$$q(z) = 1 + \frac{1}{\gamma}\left(\frac{F_\lambda(z)}{zF'_\lambda(z)} - 1\right), \tag{18}$$

where $F_\lambda(z)$ is given by (6) with

$$F_\lambda(z) = z + \sum_{k=2}^{\infty} [1 + \lambda(k-1)] a_k z^k, \tag{19}$$

and $q(z)$ is analytic in $\mathbb{U}$ with $q(0) = 1$, $\operatorname{Re} q(z) > 0$.

Now using (1) and (12), we have

$$z + \sum_{k=2}^{\infty} A_k z^k = \left[1 + \gamma\left(\sum_{k=1}^{\infty} c_k z^k\right)\right]\left(z + \sum_{k=2}^{\infty} k A_k z^k\right), \tag{20}$$

where

$$A_k = [1 + \lambda(k-1)]\, a_k. \quad (21)$$

Comparing coefficient of like power of z^n, we obtain

$$(1-n) A_n = \gamma \{c_{n-1} + 2A_2 c_{n-2} + \cdots + (n-1) A_{n-1} c_1\}. \quad (22)$$

Using triangle inequality and Lemma 3, we get

$$|(1-n) A_n| \le 2|\gamma| \{1 + 2|A_2| + \cdots + (n-1)|A_{n-1}|\}. \quad (23)$$

For $n = 2$ and $n = 3$ in (23), we easily obtain that

$$|a_2| \le \frac{2|\gamma|}{1+\lambda}, \qquad |a_3| \le \frac{|\gamma|(1+4|\gamma|)}{1+2\lambda}. \quad (24)$$

Making $n = 4$ in (23), we see that

$$2|A_3| \le 2|\gamma|(1 + 2|A_2|) \le 2|\gamma| \left\{1 + \frac{4|\gamma|(1+2\lambda)}{1+\lambda}\right\}; \quad (25)$$

equivalently, we have

$$|a_4| \le \frac{2|\gamma|(1+3|\gamma|)(1+4|\gamma|)}{3(1+3\lambda)}. \quad (26)$$

Using the principal of mathematical induction, we obtain

$$|A_n| \le \frac{2|\gamma|}{(n-1)} \prod_{k=2}^{n-1} \left(1 + \frac{2|\gamma| k}{k-1}\right). \quad (27)$$

Now from the use of relation (21), we obtain the required result. □

If we take $\lambda = 0$ and $\gamma = 1 - \alpha$, we get the following result.

Corollary 7 (see [14]). *Let $f(z) \in \mathcal{S}_*(\alpha)$. Then, for $n = 3, 4, 5, \ldots$, one has*

$$|a_n| \le \frac{2(1-\alpha)}{(n-1)} \prod_{k=2}^{n-1} \left(1 + \frac{2(1-\alpha) k}{k-1}\right), \quad (28)$$

with $|a_2| \le 2(1-\alpha)$.

Making $\lambda = 1$ and $\gamma = 1 - \alpha$, we get the following result.

Corollary 8 (see [14]). *Let $f(z) \in \mathcal{K}_*(\alpha)$. Then, for $n = 3, 4, 5, \ldots$, one has*

$$|a_n| \le \frac{2(1-\alpha)}{n(n-1)} \prod_{k=2}^{n-1} \left(1 + \frac{2(1-\alpha) k}{k-1}\right) \quad (29)$$

with $|a_2| \le (1-\alpha)$.

Theorem 9. *Let $f(z) \in \mathcal{L}(\lambda, \gamma)$ and be of the form (1). Then*

$$|a_3 - \mu a_2^2| \le \frac{|\gamma|}{(1+2\lambda)} \max\{1, |2\nu - 1|\}, \quad (30)$$

where

$$\nu = 2\gamma(1+2\lambda)\left(\frac{1}{1+2\lambda} - \frac{\mu}{(1+\lambda)^2}\right). \quad (31)$$

Proof. Let $f(z) \in \mathcal{L}(\lambda, \gamma)$. Then from (22) we have

$$a_2 = \frac{-\gamma c_1}{(1+\lambda)}, \qquad a_3 = \frac{-\gamma}{2(1+2\lambda)}\left(c_2 - 2\gamma c_1^2\right). \quad (32)$$

We now consider

$$|a_3 - \mu a_2^2| = \frac{|\gamma|}{2(1+2\lambda)} \left| c_2 - 2\gamma(1+2\lambda)\left(\frac{1}{1+2\lambda} - \frac{\mu}{(1+\lambda)^2}\right) c_1^2 \right|. \quad (33)$$

Using Lemma 4, we obtain

$$|a_3 - \mu a_2^2| \le \frac{|\gamma|}{(1+2\lambda)} \max\{1, |2\nu - 1|\}, \quad (34)$$

where ν is given by (31). □

Putting $\mu = 1$, we obtain the following result.

Corollary 10. *Let $f(z) \in \mathcal{L}(\lambda, \gamma)$. Then*

$$|a_3 - a_2^2| \le \frac{|\gamma|}{(1+2\lambda)}. \quad (35)$$

Theorem 11. *Let $f(z) \in \mathcal{L}(\lambda, \gamma)$ and be of the form (1). Then*

$$|a_2 a_4 - a_3^2| \le \left[\frac{7 + 28\lambda + 25\lambda^2 + 4(1 + 4\lambda + 10\lambda^2)|\gamma| + 48\lambda^2 |\gamma|^2}{3(1+2\lambda)^2(1 + 4\lambda + 3\lambda^2)}\right] \times |\gamma|^2. \quad (36)$$

Proof. Let $f(z) \in \mathcal{L}(\lambda, \gamma)$. Then, from (22), we have

$$a_2 = \frac{-\gamma c_1}{(1+\lambda)},$$
$$a_3 = \frac{-\gamma}{2(1+2\lambda)}\left(c_2 - 2\gamma c_1^2\right), \quad (37)$$
$$a_4 = \frac{-\gamma}{3(1+3\lambda)}\left(c_3 - \frac{7\gamma}{2} c_1 c_2 + 3\gamma^2 c_1^3\right).$$

Consider

$$\begin{aligned}&\left|a_2a_4-a_3^2\right|\\&=\left|\frac{\gamma^2}{12(1+\lambda)(1+2\lambda)^2(1+3\lambda)}\right.\\&\quad\times\left(4(1+2\lambda)^2c_1c_3-2\gamma\left(1+4\lambda+10\lambda^2\right)c_1^2c_2\right.\\&\quad\left.\left.+12\gamma^2\lambda^2c_1^4-3\left(1+4\lambda+3\lambda^2\right)c_2^2\right)\right|.\end{aligned}\tag{38}$$

Now using values of c_2 and c_3 from Lemma 5, we obtain

$$\begin{aligned}&\left|a_2a_4-a_3^2\right|\\&=\frac{|\gamma|^2}{12(1+\lambda)(1+2\lambda)^2(1+3\lambda)}\\&\quad\times\left|\left\{(1+2\lambda)^2-\gamma\left(1+4\lambda+10\lambda^2\right)\right.\right.\\&\quad\left.+12\gamma^2\lambda^2-\frac{3}{4}\left(1+4\lambda+3\lambda^2\right)\right\}c_1^4\\&\quad+\left\{2(1+2\lambda)^2-\gamma\left(1+4\lambda+10\lambda^2\right)-\frac{3}{2}\left(1+4\lambda+3\lambda^2\right)\right\}\\&\quad\times\left(4-c_1^2\right)c_1^2x\\&\quad-\left\{(1+2\lambda)^2c_1^2+\frac{3}{4}\left(1+4\lambda+3\lambda^2\right)\left(4-c_1^2\right)\right\}\\&\quad\left.\times\left(4-c_1^2\right)x^2+2c_1(1+2\lambda)^2\left(4-c_1^2\right)\left(1-|x|^2\right)z\right|.\end{aligned}\tag{39}$$

Applying triangle inequality and replacing c_1 by c, $|x|$ by ρ, and $|z|$ by 1, we get

$$\begin{aligned}&\left|a_2a_4-a_3^2\right|\\&\le\frac{|\gamma|^2}{12(1+\lambda)(1+2\lambda)^2(1+3\lambda)}\\&\quad\times\left[\left\{(1+2\lambda)^2+|\gamma|\left(1+4\lambda+10\lambda^2\right)\right.\right.\\&\quad\left.+12|\gamma|^2\lambda^2+\frac{3}{4}\left(1+4\lambda+3\lambda^2\right)\right\}c^4\\&\quad+\left\{2(1+2\lambda)^2+|\gamma|\left(1+4\lambda+10\lambda^2\right)+\frac{3}{2}\left(1+4\lambda+3\lambda^2\right)\right\}\\&\quad\times\left(4-c^2\right)c^2\rho\\&\quad+\left\{(1+2\lambda)^2c^2+\frac{3}{4}\left(1+4\lambda+3\lambda^2\right)\left(4-c^2\right)\right\}\\&\quad\left.\times\left(4-c^2\right)\rho^2+2c(1+2\lambda)^2\left(4-c^2\right)\left(1-\rho^2\right)\right]=F(c,\rho).\end{aligned}\tag{40}$$

Differentiating with respect to ρ, we get

$$\begin{aligned}&\frac{\partial F(c,\rho)}{\partial\rho}\\&=\frac{|\gamma|^2}{12(1+\lambda)(1+2\lambda)^2(1+3\lambda)}\\&\quad\times\left[\left\{2(1+2\lambda)^2+|\gamma|\left(1+4\lambda+10\lambda^2\right)\right.\right.\\&\quad\left.+\frac{3}{2}\left(1+4\lambda+3\lambda^2\right)\right\}\left(4-c^2\right)c^2\\&\quad+\left\{2(1+2\lambda)^2c^2+\frac{3}{2}\left(1+4\lambda+3\lambda^2\right)\left(4-c^2\right)\right\}\\&\quad\left.\times\left(4-c^2\right)\rho-4c(1+2\lambda)^2\left(4-c^2\right)\rho\right].\end{aligned}\tag{41}$$

Now since $\partial F(c,\rho)/\partial\rho>0$ for $c\in[0,2]$ and $\rho\in[0,1]$, maximum of $F(c,\rho)$ will exist at $\rho=1$ and let $F(c,1)=G(c)$. Then

$$\begin{aligned}G(c)&=\frac{|\gamma|^2}{12(1+\lambda)(1+2\lambda)^2(1+3\lambda)}\\&\quad\times\left[\left\{(1+2\lambda)^2+|\gamma|\left(1+4\lambda+10\lambda^2\right)\right.\right.\\&\quad\left.+12|\gamma|^2\lambda^2+\frac{3}{4}\left(1+4\lambda+3\lambda^2\right)\right\}c^4\\&\quad+\left\{2(1+2\lambda)^2+|\gamma|\left(1+4\lambda+10\lambda^2\right)\right.\\&\quad\left.+\frac{3}{2}\left(1+4\lambda+3\lambda^2\right)\right\}\left(4-c^2\right)c^2\\&\quad+\left\{(1+2\lambda)^2c^2+\frac{3}{4}\left(1+4\lambda+3\lambda^2\right)\left(4-c^2\right)\right\}\\&\quad\left.\times\left(4-c^2\right)\right].\end{aligned}\tag{42}$$

Now by differentiating with respect to c, we obtain

$$\begin{aligned}G'(c)&=\frac{|\gamma|^2}{12(1+\lambda)(1+2\lambda)^2(1+3\lambda)}\\&\quad\times\left[4\left\{(1+2\lambda)^2+|\gamma|\left(1+4\lambda+10\lambda^2\right)\right.\right.\\&\quad\left.+12|\gamma|^2\lambda^2+\frac{3}{4}\left(1+4\lambda+3\lambda^2\right)\right\}c^3\\&\quad+\left\{2(1+2\lambda)^2+|\gamma|\left(1+4\lambda+10\lambda^2\right)\right.\\&\quad\left.+\frac{3}{2}\left(1+4\lambda+3\lambda^2\right)\right\}\left(8c-4c^3\right)\end{aligned}$$

$$+\left\{(1+2\lambda)^2\left(8c-4c^3\right) - 3\left(1+4\lambda+3\lambda^2\right)\left(4c-c^3\right)\right\}\Big]. \tag{43}$$

Since $\partial G(c)/\partial c > 0$ for $c \in [0,2]$, $G(c)$ has a maximum value at $c = 2$ and hence

$$\left|a_2a_4 - a_3^2\right| \le \left[\frac{7+28\lambda+25\lambda^2+4\left(1+4\lambda+10\lambda^2\right)|\gamma|+48\lambda^2|\gamma|^2}{3(1+2\lambda)^2(1+4\lambda+3\lambda^2)}\right] \times |\gamma|^2. \tag{44}$$

□

Theorem 12. *Let $f(z) \in \mathscr{L}(\lambda,\gamma)$ and be of the form (1). Then*

$$\left|a_2a_3 - a_4\right| \le \frac{2|\gamma|\left[(4|\gamma|+1)\left((6|\gamma|+2)\lambda^2+(3\lambda+1)\right)\right]}{3(1+\lambda)(1+2\lambda)(1+3\lambda)}. \tag{45}$$

Proof. From (37), we can write

$$\left|a_2a_3 - a_4\right| = \left|\frac{12\gamma^2\lambda^2c_1^3 - 2\gamma\left(2+6\lambda+7\lambda^2\right)c_1c_2 + 2\left(1+3\lambda+2\lambda^2\right)c_3}{6(1+\lambda)(1+2\lambda)(1+3\lambda)}\right| \times |\gamma|. \tag{46}$$

Using Lemma 5 for the values of c_2 and c_3, we have

$$\begin{aligned}\left|a_2a_3 - a_4\right| &= \frac{|\gamma|}{6(1+\lambda)(1+2\lambda)(1+3\lambda)} \\ &\times \Big|\frac{1}{2}(4\gamma-1)\left(2(3\gamma-1)\lambda^2-(3\lambda+1)\right)c_1^3 \\ &+ \left(\left(1+3\lambda+2\lambda^2\right)-\gamma\left(2+6\lambda+7\lambda^2\right)\right) \\ &\times \left(4-c_1^2\right)c_1x \\ &- \frac{1}{2}\left(1+3\lambda+2\lambda^2\right)\left(4-c_1^2\right)c_1x^2 \\ &+ \left(1+3\lambda+2\lambda^2\right)\left(4-c_1^2\right)\left(1-|x|^2\right)z\Big|.\end{aligned} \tag{47}$$

Applying triangle inequality and then putting $|z| = 1$, $|x| = \rho$, and $c_1 = c$, we have

$$\begin{aligned}\left|a_2a_3 - a_4\right| &\le \frac{|\gamma|}{6(1+\lambda)(1+2\lambda)(1+3\lambda)} \\ &\times \Big[(4|\gamma|+1)\left((6|\gamma|+2)\lambda^2+(3\lambda+1)\right)\frac{c^3}{2} \\ &+ \left(\left(1+3\lambda+2\lambda^2\right)+|\gamma|\left(2+6\lambda+7\lambda^2\right)\right) \\ &\times \left(4-c^2\right)c\rho + \frac{1}{2}\left(1+3\lambda+2\lambda^2\right)\left(4-c^2\right)c\rho^2 \\ &+ \left(1+3\lambda+2\lambda^2\right)\left(4-c^2\right)\left(1-\rho^2\right)\Big] = F(c,\rho).\end{aligned} \tag{48}$$

Now by using the same procedure as we did in the proof of Theorem 11, we obtain the required result. □

Theorem 13. *If $f(z) \in \mathscr{L}(\lambda,\gamma)$ and is of the form (1), then*

$$\begin{aligned}\left|H_3(1)\right| &\le \left[\frac{7+28\lambda+25\lambda^2+4\left(1+4\lambda+10\lambda^2\right)|\gamma|+48\lambda^2|\gamma|^2}{3(1+2\lambda)^3(1+4\lambda+3\lambda^2)}\right] \\ &\times |\gamma|^3(1+4|\gamma|) \\ &+ \left[\frac{4\left[(4|\gamma|+1)^2\left((3|\gamma|+1)2\lambda^2+(3\lambda+1)\right)\right]}{9(1+\lambda)(1+2\lambda)(1+3\lambda)^2}\right] \\ &\times |\gamma|^2(1+3|\gamma|) \\ &+ \frac{(1+4|\gamma|)(1+3|\gamma|)(3+8|\gamma|)|\gamma|^2}{6(1+2\lambda)(1+4\lambda)}.\end{aligned} \tag{49}$$

Proof. Since

$$\left|H_3(1)\right| \le \left|a_3\right|\left|a_2a_4 - a_3^2\right| + \left|a_4\right|\left|a_2a_3 - a_4\right| + \left|a_5\right|\left|a_1a_3 - a_2^2\right|, \tag{50}$$

using Theorem 6, Corollary 10, and Theorems 11 and 12, we have

$$\begin{aligned}\left|H_3(1)\right| &\le \frac{|\gamma|(1+4|\gamma|)}{(1+2\lambda)} \\ &\times \left[\frac{7+28\lambda+25\lambda^2+4\left(1+4\lambda+10\lambda^2\right)|\gamma|+48\lambda^2|\gamma|^2}{3(1+2\lambda)^2(1+4\lambda+3\lambda^2)}\right]\end{aligned}$$

$$\times |\gamma|^2 + \frac{2|\gamma|(1+4|\gamma|)(1+3|\gamma|)}{3(1+3\lambda)}$$
$$\times \left[\frac{2|\gamma|\left[(4|\gamma|+1)\left((3|\gamma|+1)2\lambda^2+(3\lambda+1)\right)\right]}{3(1+\lambda)(1+2\lambda)(1+3\lambda)}\right]$$
$$+ \frac{|\gamma|(1+4|\gamma|)(1+3|\gamma|)(3+8|\gamma|)}{6(1+4\lambda)}\frac{|\gamma|}{(1+2\lambda)}$$
$$= \left[\frac{7+28\lambda+25\lambda^2+4\left(1+4\lambda+10\lambda^2\right)|\gamma|+48\lambda^2|\gamma|^2}{3(1+2\lambda)^3\left(1+4\lambda+3\lambda^2\right)}\right]$$
$$\times |\gamma|^3(1+4|\gamma|)$$
$$+ \left[\frac{4\left[(4|\gamma|+1)^2\left((3|\gamma|+1)2\lambda^2+(3\lambda+1)\right)\right]}{9(1+\lambda)(1+2\lambda)(1+3\lambda)^2}\right]$$
$$\times |\gamma|^2(1+3|\gamma|) + \frac{(1+4|\gamma|)(1+3|\gamma|)(3+8|\gamma|)|\gamma|^2}{6(1+2\lambda)(1+4\lambda)}. \quad (51)$$

This completes the proof of this result. □

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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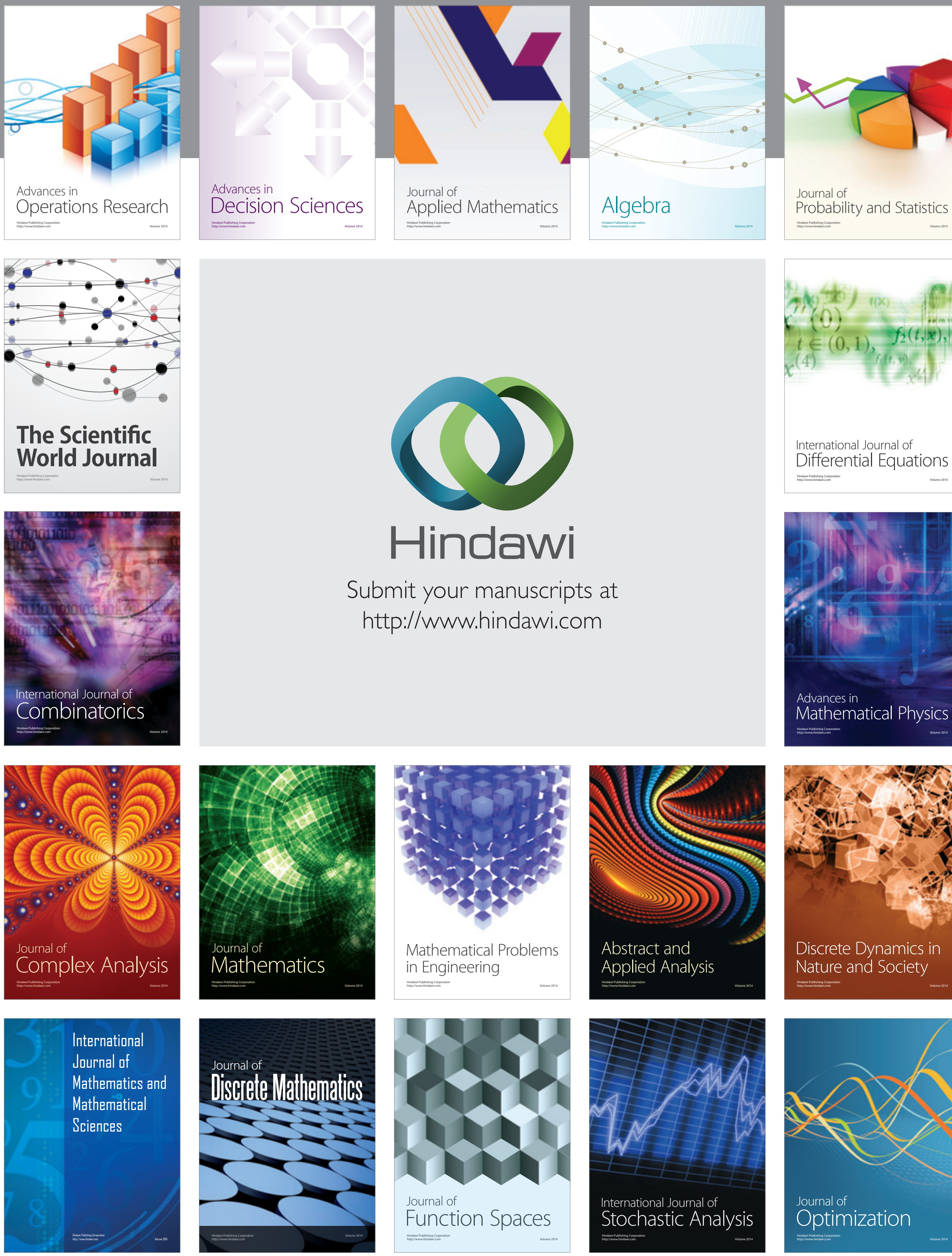
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