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Research Article

Local Stabilization of Time-Delay Nonlinear Discrete-Time Systems Using Takagi-Sugeno Models and Convex Optimization

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A convex condition in terms of linear matrix inequalities (LMIs) is developed for the synthesis of stabilizing fuzzy state feedback controllers for nonlinear discrete-time systems with time-varying delays. A Takagi-Sugeno (T-S) fuzzy model is used to represent exactly the nonlinear system in a restricted domain of the state space, called region of validity. The proposed stabilization condition is based on a Lyapunov-Krasovskii (L-K) function and it takes into account the region of validity to determine a set of initial conditions for which the actual closed-loop system trajectories are asymptotically stable and do not evolve outside the region of validity. This set of allowable initial conditions is determined from the level set associated to a fuzzy L-K function as a Cartesian product of two subsets: one characterizing the set of states at the initial instant and another for the delayed state sequence necessary to characterize the initial conditions. Finally, we propose a convex programming problem to design a fuzzy controller that maximizes the set of initial conditions taking into account the shape of the region of validity of the T-S fuzzy model. Numerical simulations are given to illustrate this proposal.

1. Introduction

Fuzzy logic technique has been widely and successfully used in nonlinear system modeling and control. In a large number of model-based fuzzy control studies and applications the Takagi-Sugeno (T-S) fuzzy model approach has been shown to be quite popular and a convenient tool to handle complex nonlinear systems. Such an approach consists of a set of local linear models that are smoothly connected by nonlinear fuzzy membership functions [1]. Some successful applications in controller synthesis can be found in [2–7].

In many real applications, the controlled system is required to work inside a subregion of the state space due, for instance, to safe operational conditions, physical constraints, or some desired level of energy consumptions. Thus, an important characteristic of the T-S fuzzy model is that it can represent exactly or approximately the original nonlinear system in a state space region of interest [8]. In particular, using the technique described in [8] the obtained T-S fuzzy model represents exactly, that is, without modeling error, a nonlinear system in a restricted domain of the state space, here called region of validity. This region can be chosen according to the mentioned operational, physical, or energy consumption constraints. Once this region of validity is assigned and the associated T-S fuzzy model is determined, it is relevant to consider the local behavior in terms of performance and stability of the actual closed-loop system formed by the feedback connection of the nonlinear system with a controller designed using this T-S fuzzy model. Just to cite a few, see [9–14] and the references therein. Few results are known where the region validity of the T-S fuzzy model is taken into account, as example in [15]. However, no delay is considered in the last reference. Therefore, the control laws synthesized without taking into account such a region of validity may yield some trajectories of the actual controlled system to go outside of the region of validity.

As a consequence, malfunctioning, performance deterioration, or even instability of the actual controlled closed-loop system may be verified.

Because the considered T-S fuzzy model represents exactly the controlled nonlinear system only inside the above mentioned region of validity, in general, the actual closed-loop system is not globally asymptotically stable and we have to deal with the corresponding region of attraction. This region is the set of all initial conditions such that the corresponding closed-loop trajectories converge to the origin. The exact characterization of the region of attraction is not an easy task. Hence, it is relevant to characterize subsets of this region with well-defined analytical representation and use them for analysis and synthesis of the nonlinear closed-loop system. These subsets are called regions of stability, according to [16].

Dynamic systems with delay are often found in industrial processes especially when there is transfer of mass, energy, and/or information. The delay usually causes performance deterioration and even loss of stability [17, 18]. In the recent years, the academic community has given great attention to the control problem of systems with delayed states, as can be seen, for example, in [10, 19–23].

Recently, T-S fuzzy control of time-delay systems has been studied in context of systems with delayed states, as can be seen in [13, 24, 25]. In these papers, the stability analysis and the control synthesis problems for time-varying delay discrete-time systems represented by T-S fuzzy model is treated and delay-dependent conditions are obtained by using Lyapunov-Krasovskii (L-K) functions. Besides, in [26, 27] the influence of exogenous perturbations is included. In all these works, the region of validity is not taken into account.

In the context briefly described above, the main contribution of the present work is to propose convex conditions in terms of linear matrix inequalities (LMIs) for the synthesis of fuzzy stabilizing feedback controllers that maximize the set of initial conditions for which the corresponding trajectories evolve inside the region of validity of the T-S fuzzy model. The proposed results are based on an L-K function to guarantee the convergence of the trajectories for any sequence of initial conditions that belong to some level sets defined from the L-K function and contained in the region of validity of the T-S model. To determine the regions of stability, we split the sequence of the initial states into two parts: the first one is composed only by the initial state vector at $k = 0$. The other part encompasses the remaining delayed state vectors required for the uniqueness of the solutions. Using this decomposition, we can characterize the region of stability through the Cartesian product of two sets. In particular, if the delayed sequence of initial state vectors is identically null, the set of initial conditions at $k = 0$ is obtained as the biggest possible level set associated to the L-K contained in the region of validity. Otherwise, if delayed initial states are not null, the mentioned level set shrinks in function of a given measure of these delayed states.

Based on the developed conditions an LMI based optimization problem is proposed to synthesize a fuzzy state feedback controller that maximizes the associate region of stability with respect to the region of validity of the T-S fuzzy model, where the dynamics of the nonlinear closed-loop system are allowed to evolve.

In Section 2 some definitions and the problem formulation are provided. In Section 3 we present the main result: a convex condition for the synthesis of T-S state feedback control gains and an optimization convex problem is also presented to compute the control law maximizing the region of stability. In Section 4, numerical examples are given to illustrate the relevance of the present proposal. Comparisons with recent conditions presented in the literature are also presented, showing the importance of considering the region of validity. Some conclusions are presented in Section 5.

Notations. The rth row of the matrix L is denoted by $L_{(r)}$. The symbol $\star$ represents the symmetric blocks in relation to the diagonal. The matrices $\mathbf{I}$ and $\mathbf{0}$ denote, respectively, identity and null matrices of appropriate dimensions. For $d \in \mathbb{N}^*$ and $k \in \mathbb{N}$, $\phi_{d,k} = \{x_{k-d}, x_{k-(d-1)}, \ldots, x_{k-1}\}$ denotes a sequence of d vectors $x_j \in \mathbb{R}^n$, $j \in [-d, -1]$, where $[a, b]$ is the interval of the integer numbers starting in "a" and ending in "b." Consider that $\varphi_{d,k}$ defines a sequence of $d + 1$ vectors $x_j \in \mathbb{R}^n$, $j \in [-d, 0]$, such that $\varphi_{d,k} = \{\phi_{d,k}, x_k\}$. The space of the vector sequence $\varphi_{d,k} = \{\phi_{d,k}, x_k\}$, which maps $[-d, 0]$ in $\mathbb{R}^n$, is $\mathscr{D}_d = \mathscr{D}([-d, 0], \mathbb{R}^n)$, with the norm $\|\phi_{d,k}\|_d = \sup_{-d \le j \le -1} \|x(k + j)\|$, where $\| \cdot \|$ is the Euclidian norm. The function $Y = \mathtt{round}(X)$ rounds the elements of X to the nearest integers.

2. Problem Statement

Consider a discrete-time nonlinear system with a time-varying delay in the state such that in a specified subset of the state space including the origin it can be represented by

$$x_{k+1} = f(x_k) + f_d(x_{k-d_k}) + g(x_k) u_k, \tag{1}$$

where $x_k \in \mathscr{X} \subset \mathbb{R}^n$ is the state vector with the initial condition given by a sequence $\varphi_{\overline{d},0}$, with $\varphi_{\overline{d},0} = \{\phi_{\overline{d},0}, x_0\}$, and $\varphi_{\overline{d},0} \in \mathscr{D}_{\overline{d}}$, $\phi_{\overline{d},0}(j) = x_j$, $j \in [-\overline{d}, -1]$, and $u_k \in \mathscr{U} \subset \mathbb{R}^m$ is the control input vector. Functions $f(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, $f_d(\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, and $g(\cdot) : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous and bounded for all $x_k \in \mathscr{X}$. Besides, the origin is assumed to be the equilibrium of the system; that is, $f(\mathbf{0}) = f_d(\mathbf{0}) = g(\mathbf{0}) = \mathbf{0}$. The time-varying delay is denoted by d_k and it is subject to

$$|d_{k+1} - d_k| \le \delta, \tag{2}$$

where $\delta \in \mathbb{N}$ denotes the maximum modulus variation admissible by d_k between two samples and $d_k \le \overline{d}$. The value of $\overline{d}$ is arbitrated through of knowledge of the plant.

The nonlinear system (1) is represented exactly as a Takagi-Sugeno (T-S) fuzzy model with $N = 2^p$ rules inside a region $\mathscr{V}_0 \subseteq \mathscr{X}$ to be specified later in this section. Each rule of such a model is given by

$$\begin{aligned} &\text{Rule } i: \\ &\text{IF } z_1(k) \text{ is } M_{i1} \text{ and } \cdots \text{ and } z_p(k) \text{ is } M_{ip}, \\ &\text{THEN } x_{k+1} = A_i x_k + A_{di} x_{k-d_k} + B_i u_k, \end{aligned} \tag{3}$$

where $z_j(k)$, $j = 1, \ldots, p$, are the scalar premise variables supposed to be dependent only on the states, M_{ij} are the fuzzy sets, and p is the number of premise variables. The matrices of the systems, $A_i \in \mathbb{R}^{n\times n}$, $A_{di} \in \mathbb{R}^{n\times n}$, and $B_i \in \mathbb{R}^{n\times m}$, are known.

In this paper $\mathscr{V}_0$ is called region of validity and it is defined by a polyhedral set as

$$\mathscr{V}_0 = \{x_k \in \mathbb{R}^n; |L_{(r)}x_k| \leq \eta_{(r)}\} \subseteq \mathscr{X}, \tag{4}$$

where $\eta_{(r)} > 0$ and $L_{(r)} \in \mathbb{R}^{1\times n}$, for $r = 1, \ldots, \kappa$, with κ representing the number of constraints that characterize the allowed region for the closed-loop system in the state space. Due to the use of the modulus function in (4), the model (3) has symmetric limitations in the states relative to the origin.

Each linear delayed system shown in the fuzzy rule (3) represents a subsystem. By using a standard fuzzy inference method, for example, a center-average defuzzifier, product fuzzy inference, and singleton fuzzifier, the dynamic fuzzy model (3) can be expressed by the following model [28, 29]:

$$x_{k+1} = A(\alpha_k)x_k + A_d(\alpha_k)x_{k-d_k} + B(\alpha_k)u_k, \tag{5}$$

where $\alpha_{k(i)} = w_i(z(k))/\sum_{j=1}^{N} w_j(z(k))$ with $w_i = \prod_{j=1}^{p} M_{ij}(z_j(k))$ and $z(k) = [z_1(k) \;\; z_2(k) \;\; \cdots \;\; z_p(k)]^T$. As usual in fuzzy framework, the membership function α_k is a state-dependent time-varying parameter that is measurable or possible to be estimated in real time and verifies the unitary simplex:

$$\Xi = \left\{\alpha_k \in \mathbb{R}^N; \sum_{i=1}^{N}\alpha_{k(i)} = 1, \alpha_{k(i)} \geq 0, i = 1, \ldots, N\right\}. \tag{6}$$

Therefore, matrices in (5) can be rewritten as

$$[A(\alpha_k) \;\; A_d(\alpha_k) \;\; B(\alpha_k)] = \sum_{i=1}^{N}\alpha_{k(i)}[A_i \;\; A_{di} \;\; B_i], \quad \alpha_k \in \Xi, \tag{7}$$

which characterizes exactly the nonlinear system (1) inside the region of validity $\mathscr{V}_0$. See [8, 30] for further discussions about the use of T-S models to represent exactly nonlinear systems (1) inside a prespecified region $\mathscr{V}_0$. The initial condition that assures existence and uniqueness of solutions for (5) is given by a sequence $\varphi_{\overline{d},0}$, where $\varphi_{\overline{d},0} = \{\phi_{\overline{d},0}, x_0\}$ with $\varphi_{\overline{d},0} \in \mathscr{D}_{\overline{d}}$, $\phi_{\overline{d},0}(j) = x_j$, $j \in [-\overline{d}, -1]$, and $\overline{d}$ sufficiently large. It is necessary that $\varphi_{\overline{d},0} \in \mathscr{V}_0$ such that this initial condition sequence leads the states of nonlinear system to be controlled (1) to satisfy $x_k \in \mathscr{V}_0$.

We suppose that it is possible to access to the value of d_k on real time. The control law is defined as follows:

$$u_k = K(\alpha_k)x_k + K_d(\alpha_k)x_{k-d_k}. \tag{8}$$

If the value of d_k is unknown, it is enough to assume that $K_d(\alpha_k) = \mathbf{0}$ $\forall k$ in (8). Note that the matrices of the controller are dependent on the membership functions and, likewise the matrices of the fuzzy system (5), they are defined as follows:

$$[K(\alpha_k) \;\; K_d(\alpha_k)] = \sum_{i=1}^{N}\alpha_{k(i)}[K_i \;\; K_{di}], \quad \alpha_k \in \Xi, \tag{9}$$

where $K_i \in \mathbb{R}^{m\times n}$ and $K_{d_i} \in \mathbb{R}^{m\times n}$.

Using the fuzzy formulation (5)–(9), we have the fuzzy closed-loop system with a time-varying delay in the state

$$x_{k+1} = \widehat{A}(\alpha_k)x_k + \widehat{A}_d(\alpha_k)x_{k-d_k}, \tag{10}$$

where, by construction,

$$\begin{aligned}\widehat{A}(\alpha_k) &= A(\alpha_k) + B(\alpha_k)K(\alpha_k)\\ &= \sum_{i=1}^{N}\sum_{j=i}^{N}\mu_{ij}\alpha_{k(i)}\alpha_{k(j)}\frac{(A_i + B_iK_j + A_j + B_jK_i)}{2},\end{aligned} \tag{11}$$

$$\begin{aligned}\widehat{A}_d(\alpha_k) &= A_d(\alpha_k) + B(\alpha_k)K_d(\alpha_k)\\ &= \sum_{i=1}^{N}\sum_{j=i}^{N}\mu_{ij}\alpha_{k(i)}\alpha_{k(j)}\frac{A_{di} + B_iK_{dj} + A_{dj} + B_jK_{di}}{2},\end{aligned} \tag{12}$$

with

$$\mu_{ij} = 2, \quad \text{if } i \neq j, \text{ otherwise } \mu_{ij} = 1. \tag{13}$$

It is worth pointing out that, by construction, the fuzzy system is supposed to represent exactly the dynamics of the closed-loop system only inside the region of validity $\mathscr{V}_0$, where it is guaranteed that the representation of the nonlinear system by fuzzy T-S model is valid. Thus a main purpose in this work is to determine the regions of stability related to the sequences of initial states $\varphi_{\overline{d},0} = \{\phi_{\overline{d},0}, x_0\} \in \mathscr{D}_{\overline{d}}$, for which the corresponding trajectories are asymptotically stable and do not leave $\mathscr{V}_0$. Thus, we define $\Upsilon_\varphi \triangleq \mathscr{B}(r) \times \mathscr{C}_x = \{(\phi_{\overline{d},0}, x_0) \mid \phi_{\overline{d},0} \in \mathscr{B}(r) \text{ and } x_0 \in \mathscr{C}_x\}$ with

$$\mathscr{C}_x = \{x_0 \in \mathscr{D}_{\overline{d}}; V_1(x_0, \alpha_0) \leq c(\phi_{\overline{d},0})\}, \tag{14}$$

$$\mathscr{B}(r) = \{\phi_{\overline{d},0} \in \mathscr{D}_{\overline{d}}; \|\phi_{\overline{d},0}\|_{\overline{d}} \leq r\}, \tag{15}$$

where $V_1(x_0, \alpha_0)$ is a parameter dependent quadratic form, $c(\phi_{\overline{d},0})$ is a function on $\mathbb{R}^+$ with the sequence $\phi_{\overline{d},0}$ as argument, and $0 \leq r \in \mathbb{R}^+$. The set Υ_φ is fundamental to determine possible initial sequences whose corresponding trajectories are asymptotically stable and remain in $\mathscr{V}_0$. The characterization of Υ_φ is new and it is a contribution of this paper.

Problem 1. Determine the gains K_i and K_{di} of the controller (8)-(9) and characterize the regions $\mathscr{C}_x \subseteq \mathscr{V}_0$ and $\mathscr{B}(r)$, such that $\Upsilon_\varphi = \mathscr{B}(r) \times \mathscr{C}_x$ is a safe set of initial conditions, whose corresponding trajectories of the closed-loop system remain in $\mathscr{V}_0$ and converge asymptotically to the origin.

3. Mains Results

Consider the following fuzzy Lyapunov-Krasovskii (L-K) candidate function, $V(x_k,\alpha_k):\mathbb{R}^n\times\Xi\rightarrow\mathbb{R}^+$:

$$V(x_k,\alpha_k)=\sum_{i=1}^{3}V_i(x_k,\alpha_k)>0, \tag{16}$$

with

$$\begin{aligned}V_1(x_k,\alpha_k)&=x_k^TQ^{-1}(\alpha_k)x_k,\\ V_2(x_k,\alpha_k)&=\sum_{i=k-d_k}^{k-1}x_i^TR^{-1}(\alpha_i)x_i,\\ V_3(x_k,\alpha_k)&=\sum_{\ell=2-\delta}^{1}\sum_{i=k+\ell-1}^{k-1}x_i^TR^{-1}(\alpha_i)x_i,\end{aligned} \tag{17}$$

where $Q(\alpha_k)=\sum_{i=1}^{N}\alpha_{k(i)}Q_i$, $\mathbf{0}<Q_i^T=Q_i\in\mathbb{R}^{n\times n}$, and $R(\alpha_k)=\sum_{i=1}^{N}\alpha_{k(i)}R_i$, $\mathbf{0}<R_i^T=R_i\in\mathbb{R}^{n\times n}$. This L-K function is nonlinear and composed of three therms. Note that the Lyapunov matrices are dependent of the membership function.

In this L-K function we can associate level sets defined as follows.

Definition 2. Consider the fuzzy L-K function given by (16). For all scalar $c>0$, we define the level set $\mathscr{L}_{V_1}(c)$ from the intersection of ellipsoidal sets associate the matrices $Q_i>\mathbf{0}$, $i=1,\ldots,N$, as follows:

$$\begin{aligned}\mathscr{L}_{V_1}(c)&=\left\{\mathscr{E}\left(Q_i^{-1},c\right),\forall\alpha_k\in\Xi\right\}\\ &=\bigcap_{\alpha_k\in\Xi}\mathscr{E}\left(Q^{-1}(\alpha_k),c\right)\triangleq\bigcap_{i\in\{1,\ldots,N\}}\mathscr{E}\left(Q_i^{-1},c\right),\end{aligned} \tag{18}$$

where $\mathscr{E}(Q_i^{-1},c)$, for $i=1,\ldots,N$; denote the ellipsoidal sets defined as follows:

$$\mathscr{E}\left(Q_i^{-1},c\right)=\left\{x_k\in\mathbb{R}^n;x_k^TQ_i^{-1}x_k\le c\right\}. \tag{19}$$

The equivalence given in (18) has been proved in [31, Lemma 4].

The Definition 2 is used for the characterization of the set of initial conditions $\mathscr{C}_x$ and the set where corresponding trajectories remain confined, through of the determination of values for c. In Definition 2, if $c=1$, we use the simplified notation $\mathscr{L}_{V_1}\triangleq\mathscr{L}_{V_1}(1)$ and $\mathscr{E}(Q_i^{-1})\triangleq\mathscr{E}(Q_i^{-1},1)$.

We consider the decomposition of the initial conditions sequence $\Upsilon_\varphi=(\phi_{\overline{d},0},x_0)$ and therewith we treat separately x_0 and $\phi_{\overline{d},0}$. Firstly, consider the following lemma.

Lemma 3. *Let $R(\alpha_k)=\sum_{i=1}^{N}\alpha_{k(i)}R_i$ and $\mathbf{0}<R_i^T=R_i$; then*

$$\lambda_{\max}\left(R^{-1}(\alpha_k)\right)\le\max_i\left(\lambda_{\max}\left(R_i^{-1}\right)\right), \tag{20}$$

for $i=1,\ldots,N$ and $\forall\alpha_k\in\Xi$.

Proof. Because of the positive definiteness of R_i, we have

$$0<\lambda_{\max}\left(R_i^{-1}\right)\le\max_i\left(\lambda_{\max}\left(R_i^{-1}\right)\right). \tag{21}$$

Consider that $\max_i(\lambda_{\max}(R_i^{-1}))=\tilde{\lambda}$, where $\tilde{\lambda}$ is a positive scalar. Thus, $\lambda_{\max}(R_i^{-1})\le\tilde{\lambda}\Leftrightarrow R_i^{-1}\le\tilde{\lambda}\mathbf{I}$ that by Schur's complement equals

$$\begin{bmatrix}\tilde{\lambda}\mathbf{I} & \mathbf{I}\\ \mathbf{I} & R_i\end{bmatrix}\ge\mathbf{0}. \tag{22}$$

Multiplying the inequality (22) by $\alpha_{k(i)}$ and summing up for $i=1,\ldots,N$, knowing that $\sum_{i=1}^{N}\alpha_{k(i)}R_i=R(\alpha_k)$, and applying Schur's complement in the results, we have $R^{-1}(\alpha_k)\le\tilde{\lambda}\mathbf{I}\Leftrightarrow\lambda_{\max}(R^{-1}(\alpha_k))\le\tilde{\lambda}$. Therefore, we proof Lemma 3. □

Through the terms $V_2(x_k,\alpha_k)$ and $V_3(x_k,\alpha_k)$ of the L-K function (16) and using the Lemma 3, we have:

$$\begin{aligned}&\sum_{i=k-d_k}^{k-1}x_i^TR^{-1}(\alpha_i)x_i+\sum_{\ell=2-\delta}^{1}\sum_{i=k+\ell-1}^{k-1}x_i^TR^{-1}(\alpha_i)x_i\\ &\quad\le\sum_{i=-d_k}^{-1}\phi_{\overline{d},0}(i)^TR^{-1}(\alpha_i)\phi_{\overline{d},0}(i)\\ &\qquad+\sum_{\ell=2-\delta}^{1}\sum_{i=\ell-1}^{1}\phi_{\overline{d},0}(i)^TR^{-1}(\alpha_i)\phi_{\overline{d},0}(i)\\ &\quad\le\rho\left\|\phi_{\overline{d},0}\right\|_{\overline{d}}^2,\end{aligned} \tag{23}$$

where, for $i=1,\ldots,N$,

$$\rho=\max_{i=1,\ldots,N}\left(\lambda_{\max}\left(R_i^{-1}\right)\right)\left(\overline{d}+\frac{\delta^2-\delta}{2}\right). \tag{24}$$

By considering Definition 2 and assuming $c(\phi_{\overline{d},0})=1-\rho\|\phi_{\overline{d},0}\|_{\overline{d}}^2$, with $\rho\in\mathbb{R}^+$ given by (24), the set $\mathscr{C}_x$ in (14) can be defined as

$$\begin{aligned}\mathscr{C}_x&\triangleq\mathscr{L}_{V_1}\left(1-\rho\left\|\phi_{\overline{d},0}\right\|_{\overline{d}}^2\right)\\ &=\left\{x_0\in\mathbb{R}^n;x_0^TQ^{-1}(\alpha_0)x_0\le1-\rho\left\|\phi_{\overline{d},0}\right\|_{\overline{d}}^2\right\}.\end{aligned} \tag{25}$$

Moreover, for the set $\mathscr{B}(r)$ in (15), r is limited by $0 \le r \le \rho^{-1/2}$.

The following lemma allows connecting the sets $\mathscr{C}_x$ and $\mathscr{B}(r)$ in terms of the confinement of trajectories in $\mathscr{L}_{V_1}$ and local asymptotic stability.

Lemma 4. *Let $V(x_k, \alpha_k) > 0$ be given by (16). If $\Delta V(x_k, \alpha_k) = V(x_{k+1}, \alpha_{k+1}) - V(x_k, \alpha_k) < 0$, then*

$$V(x_k, \alpha_k) < V(x_0, \alpha_0) \le x_0^T Q^{-1}(\alpha_0) x_0 + \rho \left\| \phi_{\bar{d},0} \right\|_{\bar{d}}^2. \quad (26)$$

Therefore, $\forall x_0 \in \mathscr{C}_x \triangleq \mathscr{L}_{V_1}(1 - \rho \| \phi_{\bar{d},0} \|_{\bar{d}}^2)$ and for all $\phi_{\bar{d},0} \in \mathscr{B}(r)$ it ensures that $x_k \in \mathscr{L}_{V_1}$, $\forall k \ge 0$ and $\lim_{k \to \infty} x_k = 0$.

Proof. Through (16) and considering $\Delta V(x_k, \alpha_k) < 0$, $\forall k$, and $\forall \alpha_k \in \Xi$, it verifies that

$$x_k^T Q^{-1}(\alpha_k) x_k \le V(x_k, \alpha_k) < V(x_0, \alpha_0). \quad (27)$$

Moreover, $V(x_0, \alpha_0) = x_0^T Q^{-1}(\alpha_0) x_0 + \sum_{i=-d_k}^{-1} \phi_0^T(i) R^{-1}(\alpha_i) \phi_0(i) + \sum_{\ell=2-\bar{d}}^{1-\underline{d}} \sum_{i=\ell-1}^{-1} \phi_0^T(i) \times R^{-1}(\alpha_i) \phi_0(i)$. By using (23)-(24), we have

$$V(x_0, \alpha_0) \le x_0^T Q^{-1}(\alpha_0) x_0 + \rho \left\| \phi_{\bar{d},0} \right\|_{\bar{d}}^2. \quad (28)$$

From (27) and (28), it can be verified that if $x_0^T Q^{-1}(\alpha_0) x_0 \le 1 - \rho \| \phi_{\bar{d},0} \|_{\bar{d}}^2$, then $x_k^T Q^{-1}(\alpha_k) x_k \le 1$. Therefore, the local asymptotic stability of the state delayed closed-loop system is ensured. □

Based on the L-K function (16), Definition 2, and Lemma 4, we present in the following theorem the convex conditions for synthesis of T-S fuzzy controllers for local stabilization of nonlinear discrete-time system with time-varying delay and constraints on the states.

Theorem 5. *Suppose that there exist symmetric definite positive matrices $Q_i \in \mathbb{R}^{n \times n}$ and $R_i \in \mathbb{R}^{n \times n}$, $i = 1, \dots, N$, and matrices $U \in \mathbb{R}^{n \times n}$, $H \in \mathbb{R}^{n \times n}$, $Y_i \in \mathbb{R}^{m \times n}$, and $Y_{di} \in \mathbb{R}^{m \times n}$ verifying the following LMIs, for $\forall i, \ell, q = 1, \dots, N$, $j = i, \dots, N$, and $\forall r = 1, \dots, \kappa$:*

$$\begin{bmatrix} -Q_q & \dfrac{A_i U + B_i Y_j + A_j U + B_j Y_i}{2} & \dfrac{A_{di} H + B_i Y_{dj} + A_{dj} H + B_j Y_{di}}{2} & \mathbf{0} \\ \star & \dfrac{Q_i + Q_j}{2} - U^T - U & \mathbf{0} & U^T \\ \star & \star & R_\ell - H^T - H & \mathbf{0} \\ \star & \star & \star & -\dfrac{R_i + R_j}{2(1+\delta)} \end{bmatrix} < \mathbf{0}, \quad (29)$$

$$\begin{bmatrix} -Q_i & Q_i L_{(r)}^T \\ \star & -\eta_{(r)}^2 \end{bmatrix} \le \mathbf{0}. \quad (30)$$

Then, the controller matrices (8)-(9) obtained with

$$K_i = Y_i U^{-1}, \qquad K_{di} = Y_{di} H^{-1} \quad (31)$$

are such that the origin of the nonlinear system (1) in closed-loop by control law (8)–(9) is asymptotically stable for any set of initial conditions $\Upsilon_\varphi = \mathscr{B}(r) \times \mathscr{C}_x$, with $0 \le r \le \rho^{-1/2}$ and ρ given by (24), ensuring that the respective trajectories remain in $\mathscr{L}_{V_1} \subseteq \mathscr{V}_0$.

Proof. Firstly we show that the feasibility of (29) assures the asymptotic stability of the T-S fuzzy model (3) in closed-loop by control law (8)-(9) if the control gains are given by (9) with (31). If additionally (30) is verified, we show that it is assured that any set of initial conditions belonging to $\Upsilon_\varphi = \mathscr{B}(r) \times \mathscr{C}_x$ and ρ given by (24) contains only initial conditions ensuring that the respective trajectories of the T-S fuzzy model (3) remain in $\mathscr{L}_{V_1} \subseteq \mathscr{V}_0$. Therefore, we can guarantee the local asymptotic stability of the nonlinear system (1) in closed-loop by control (8)-(9) with the gains given by (31).

If (29) is verified, then we have assured the positivity of R_i and Q_i, $i = 1, \dots, N$, and by consequence $V(x_k, \alpha_k)$ verifies (16). Besides, the regularity of U and H is assured by blocks (2, 2) and (3, 3). By replacing Y_i and Y_{di} by $K_i U$ and $K_{di} H$, respectively, multiplying the resulting inequality successively by $\alpha_{k(i)}$, $\alpha_{k(j)}$, $\alpha_{k+1(q)}$, and $\alpha_{k-d_k(\ell)}$, and summing up on $i = 1, \dots, N$, $j = i, \dots, N$, $q = 1, \dots, N$, and $\ell = 1, \dots, N$, we get

$$\Theta_k \equiv \begin{bmatrix} -Q(\alpha_k^+) & \widehat{A}(\alpha_k) U & \widehat{A}_d(\alpha_k) H & \mathbf{0} \\ \star & Q(\alpha_k) - U - U^T & \mathbf{0} & U^T \\ \star & \star & R(\alpha_k^-) - H - H^T & \mathbf{0} \\ \star & \star & \star & -\dfrac{R(\alpha_k)}{1+\delta} \end{bmatrix} < \mathbf{0}, \quad (32)$$

where $\widehat{A}(\alpha_k)$ and $\widehat{A}_d(\alpha_k)$ are given in (11) and (12), respectively. Note that the positive definite matrices $Q(\alpha_k)$ and $R(\alpha_k)$ can be written as

$$F(\alpha_k) = \left(\sum_{j=1}^{N}\alpha_j\right)F(\alpha_k) = \sum_{i=1}^{N}\sum_{j=i}^{N}\mu_{ij}\alpha_{k(i)}\alpha_{k(j)}\frac{F_i+F_j}{2}, \quad (33)$$

where $F(\alpha_k)$ can be replaced by $Q(\alpha_k)$ and $R(\alpha_k)$, with μ_{ij} given in (13), $Q(\alpha_k^+) = \sum_{q=1}^{N}\alpha_{k(q)}^+ \times Q_q$, $R(\alpha_k^-) = \sum_{\ell=1}^{N}\alpha_{k(\ell)}^- R_\ell$, and the shorthands $\alpha_k^+ \equiv \alpha_{k+1}$ and $\alpha_k^- \equiv \alpha_{k-d_k}$.

Because the inequality $-M^T G^{-1} M \le G - M^T - M$ holds for all pairs of square matrices M and $G = G^T > \mathbf{0}$ (see [32] for details), we can apply it on blocks $(2,2)$ and $(3,3)$ of Θ_k to obtain $\overline{\Theta}_k \le \Theta_k$, where

$$\overline{\Theta}_k \equiv \begin{bmatrix} -Q(\alpha_k^+) & \widehat{A}(\alpha_k)U & \widehat{A}_d(\alpha_k)H & \mathbf{0} \\ \star & -U^T Q^{-1}(\alpha_k)U & \mathbf{0} & U^T \\ \star & \star & -H^T R^{-1}(\alpha_k^-)H & \mathbf{0} \\ \star & \star & \star & -\dfrac{R(\alpha_k)}{1+\delta} \end{bmatrix} < \mathbf{0}. \quad (34)$$

Then, applying Schur's complement on (34) we get

$$\widetilde{\Pi}_k = \begin{bmatrix} -Q(\alpha_k^+) & \widehat{A}(\alpha_k)U & \widehat{A}_d(\alpha_k)H \\ \star & U^T\left[(1+\delta)R^{-1}(\alpha_k) - Q^{-1}(\alpha_k)\right]U & \mathbf{0} \\ \star & \star & -H^T R^{-1}(\alpha_k^-)H \end{bmatrix} < \mathbf{0}. \quad (35)$$

Then, taking into account the regularity of U and H, let us consider the congruence transformation $\Pi_k = \mathcal{T}^T\widetilde{\Pi}_k\mathcal{T}$ with $\mathcal{T} = \operatorname{diag}\{\mathbf{I}, U^{-1}, H^{-1}\}$. This allows us to obtain

$$\Pi_k = \begin{bmatrix} -Q(\alpha_k^+) & \widehat{A}(\alpha_k) & \widehat{A}_d(\alpha_k) \\ \star & (1+\delta)R^{-1}(\alpha_k) - Q^{-1}(\alpha_k) & \mathbf{0} \\ \star & \star & -R^{-1}(\alpha_k^-) \end{bmatrix} < \mathbf{0}. \quad (36)$$

Again, by using Schur's complement on this last inequality, we obtain

$$\begin{bmatrix} (1+\delta)R^{-1}(\alpha_k) - Q^{-1}(\alpha_k) & \mathbf{0} \\ \mathbf{0} & -R^{-1}(\alpha_k^-) \end{bmatrix} + \begin{bmatrix} \widehat{A}^T(\alpha_k) \\ \widehat{A}_d^T(\alpha_k) \end{bmatrix} Q^{-1}(\alpha_k^+) \times \left[\widehat{A}(\alpha_k) \;\; \widehat{A}_d(\alpha_k)\right] < \mathbf{0}. \quad (37)$$

Pre- and postmultiplying (37) by $\widetilde{X}_k^T = \left[x_k^T \;\; x_{k-d_k}^T\right]$ and its transpose, respectively, and from (10)–(12), we can replace $\widehat{A}(\alpha_k)x_k + \widehat{A}_d(\alpha_k)x_{k-d_k}$ by x_{k+1}, getting

$$\Omega_k \equiv x_{k+1}^T Q^{-1}(\alpha_k^+)x_{k+1} + x_k^T\left[(1+\delta)R^{-1}(\alpha_k) - Q^{-1}(\alpha_k)\right] \times x_k - x_{k-d_k}^T R^{-1}(\alpha_k^-)x_{k-d_k} < 0. \quad (38)$$

On the other hand, following [33], we can obtain from (16)

$$\Delta V(x_k, \alpha_k) \le \Omega_k < 0. \quad (39)$$

Then, we can conclude that the feasibility of (29) assures the negativity of $\Delta V(x_k, \alpha_k)$ which with the positivity of $V(x_k, \alpha_k)$ and the Lyapunov-Krasovskii's theorem (see [33] for details) assure the stability of T-S fuzzy model (3) in closed-loop by control law (8)-(9).

Now we assume that (29) is verified and additionally (30) is satisfied. Then, we multiply (30) by $\alpha_{k(i)}$ and sum up on $i = 1, \ldots, N$, getting

$$\Lambda = \begin{bmatrix} -Q(\alpha_k) & Q(\alpha_k)L_{(\ell)}^T \\ \star & -\eta_{(\ell)}^2 \end{bmatrix} \le \mathbf{0}. \quad (40)$$

By using the congruence transformation $\mathcal{F}^T\Lambda\mathcal{F} = \widetilde{\Lambda}$ with $\mathcal{F} = \operatorname{diag}\{Q^{-1}(\alpha_k), 1\}$, we get

$$\widetilde{\Lambda} = \begin{bmatrix} -Q^{-1}(\alpha_k) & L_{(\ell)}^T \\ \star & -\eta_{(\ell)}^2 \end{bmatrix} \le \mathbf{0}. \quad (41)$$

Applying Schur's complement in $\widetilde{\Lambda}$, we have

$$L_{(\ell)}^T\eta_{(\ell)}^{-2}L_{(\ell)} - Q^{-1}(\alpha_k) \le \mathbf{0}. \quad (42)$$

In this last inequality we can pre- and postmultiply by x_k^T and x_k, respectively, and using the S-procedure, we have that

$$x_k^T L_{(\ell)}^T\eta_{(\ell)}^{-2}L_{(\ell)}x_k \le 1, \quad \forall x_k \in \mathcal{L}_{V_1} = x_k^T Q^{-1}(\alpha_k)x_k \le 1. \quad (43)$$

Then, by considering (43), we prove that $\mathscr{L}_{V_1} \subseteq \mathscr{V}_0$ and any trajectory that starts in $\Upsilon_\varphi = \mathscr{C}_x \times \mathscr{B}(r)$ remains $\mathscr{L}_{V_1}$ and thus the local stability of nonlinear system (1) in closed-loop by control law (8)-(9) is assured. □

Thus, Theorem 5 is stated as a convex feasibility problem that can be used to synthesize fuzzy gains $K(\alpha_k)$ and $K_d(\alpha_k)$ assuring that the trajectories of the closed-loop system do not leave the model region of validity, once the initial conditions belong to the set Υ_φ with $\mathscr{B}(r)$ and $\mathscr{C}_x$ given in (15) and (25), respectively.

Note that if the delay is time-invariant but unknown, it is enough to assume $\delta = 0$ and $K_{di} = \mathbf{0}$ in (29); that is, $Y_{di} = \mathbf{0}$, for $i = 1, \ldots, N$. Thus, the uncertain time-invariant delay case is encompassed by Theorem 5. The quadratic stability approach, that is, the use of crisp matrices Q and R instead of the fuzzy matrices $Q(\alpha_k)$ and $R(\alpha_k)$, can be obtained as a special case of Theorem 5. Such an approach can be obtained by imposing $Q_1 = \cdots = Q_N = Q$ and $R_1 = \cdots = R_N = R$ and solving (29) only for i and j. Naturally, this procedure reduces the numerical complexity of the conditions but it may lead to more conservative results.

In the next section, this theorem is exploited in a convex optimization problem to maximize the size of the region of initial conditions.

3.1. Optimization Problem. The objective is to determine the control law defined by (8), with the set inclusion $\mathscr{C}_x \subseteq \mathscr{L}_{V_1} \subseteq \mathscr{V}_0$ and $\mathscr{C}_x$ as large as possible. Thus, to obtain the largest set $\mathscr{C}_x$, we can proceed as follows.

(1) Minimizing ρ to have $\mathscr{C}_x = \mathscr{L}_{V_1}(1 - \rho\|\phi_{\overline{d},0}\|^2_{\overline{d}})$ as big as $\mathscr{L}_{V_1}$. According to Lemma 3 and (24), we need to minimize $\lambda_{\max}(R^{-1}(\alpha_i))$. Therefore, $R^{-1}(\alpha_i) \le W$, where $\mathbf{0} < W = W^T$, and we have

$$\begin{bmatrix} W & \mathbf{I} \\ \mathbf{I} & R_i \end{bmatrix} \ge \mathbf{0}, \quad i = 1, \ldots, N. \tag{44}$$

(2) Optimizing the size of $\mathscr{L}_{V_1}$ to have $\mathscr{L}_{V_1}$ as big as $\mathscr{V}_0$. Therewith, we consider an ellipse included in level set $\mathscr{L}_{V_1}$:

$$\mathscr{E}(W) = \left\{x \in \mathbb{R}^n; x^T W x \le 1\right\} \subset \mathscr{L}_{V_1}. \tag{45}$$

Therefore, this inclusion is equivalent to

$$\begin{bmatrix} W & \mathbf{I} \\ \mathbf{I} & Q_i \end{bmatrix} \ge \mathbf{0}, \quad i = 1, \ldots, N. \tag{46}$$

Thus, a convex optimization problem is proposed as follows:

$$\begin{aligned} &\min \quad \operatorname{trace}(W) \\ &\text{subject to} \quad (29), (30), (44), \text{ and } (46). \end{aligned} \tag{47}$$

Remark 6. Note that the computation of ρ depends on the value of the maximum delay. This means that the larger the delay is, the smaller the region $\mathscr{C}_x$ is, although the region $\mathscr{L}_{V_1}$ remains the same, because the conditions of stabilization (29) and of inclusion $\mathscr{L}_{V_1} \subseteq \mathscr{V}_0$ (30) are independent of the maximum delay. Note also that if $r = 0$, the radius of the ball $\mathscr{B}(r)$, the region $\mathscr{C}_x$ is the own region $\mathscr{L}_{V_1}$. Then, in this case it does not matter the value of the maximum delay for computation of $\mathscr{C}_x$.

4. Numerical Examples

We present two examples, one physically motivated and other purely academic, to demonstrate the relevance of taking into account the region of validity of T-S fuzzy models. In these examples we make some comparisons with other approaches found in the literature.

4.1. Physically Motivated Example. Consider the following nonlinear equations that represent a magnetic suspension system investigated in [34, 35]:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) \\ \dot{x}_2(t) &= \frac{g\mu(\mu x_1(t) + 2\mu y_0 + 2)x_1(t)}{(1 + \mu(x_1(t) + y_0))^2} x_1(t) \\ &\quad - \frac{K_m}{m} x_2(t) + \frac{\lambda\mu}{2m(1 + \mu(x_1(t) + y_0))^2} u(t), \end{aligned} \tag{48}$$

where x_1 and x_2 are the ball position and vertical velocity, respectively, and $y_0 = 0.05$ m the desired controlled position. The physical parameters are $m = 0.068$ Kg the mass of the suspended ball, $g = 9.8\,\text{m s}^{-2}$ the gravity acceleration, $K_m = 0.001\,\text{N s m}^{-1}$ the viscous friction coefficient, $\lambda = 0.46$ H the inductance, and $\mu = 2\,\text{m}^{-1}$ the variation of the inductance. Motivated by this system we consider the following discretized version of model (48), with a discretization period of T seconds and where a delay in state x_2 is included to match, for example, practical sensor dynamics in the velocity measure:

$$\begin{aligned} x_{1,k+1} &= x_{1,k} + cTx_{2,k} + (1-c)Tx_{2,k-d_k} \\ x_{2,k+1} &= \frac{Tg\mu(\mu x_{1,k} + 2\mu y_0 + 2)x_{1,k}}{(1 + \mu(x_{1,k} + y_0))^2} x_{1,k} \\ &\quad + c\left(1 - \frac{TK_m}{m}\right)x_{2,k} + (1-c)\left(1 - \frac{TK_m}{m}\right)x_{2,k-d_k} \\ &\quad + \frac{T\lambda\mu}{2m(1 + \mu(x_{1,k} + y_0))^2} u_k, \end{aligned} \tag{49}$$

where the time-variant delay is assumed $d_k \in [1, 5]$ and the parameter $c = 0.7$ weights the delay effects. The physical structure of the assembling imposes that $0 \le \overline{x}_{1,k} \le 0.1$, where $\overline{x}_{1,k} = x_{1,k} + y_0$. Just to follow this notation, we call $\overline{x}_{2,k} = x_{2,k}$. Thus the operational region for this system can be modeled by (4) with $L = [1 \;\; 0]$ and $\eta = 0.05$. Considering such a region, an exact T-S model as given in (5)–(7) can be obtained with $i = 1, \ldots, 4$, $p = 1, 2$, $B_1 = B_2 = [0 \quad 0.6765]^T$, $B_3 = B_4 = [0 \quad 0.4698]^T$, and

$$A_1 = A_3 = \begin{bmatrix} 1 & 0.07 \\ 0.1565 & 0.699 \end{bmatrix}, \quad A_2 = A_4 = \begin{bmatrix} 1 & 0.07 \\ -0.2058 & 0.699 \end{bmatrix}, \quad A_{di} = \begin{bmatrix} 0 & 0.03 \\ 0 & 0.2996 \end{bmatrix}. \tag{50}$$

The fuzzy sets M_{ip} are computed as

$$\begin{aligned} M_{11}(z_1(\overline{x}_{1,k})) = M_{31}(z_1(\overline{x}_{1,k})) &= \frac{z_1(\overline{x}_{1,k}) - a_2}{a_1 - a_2}, \\ M_{21}(z_1(\overline{x}_{1,k})) = M_{41}(z_1(\overline{x}_{1,k})) &= \frac{a_1 - z_1(\overline{x}_{1,k})}{a_1 - a_2}, \\ M_{12}(z_2(\overline{x}_{1,k})) = M_{22}(z_2(\overline{x}_{1,k})) &= \frac{z_2(\overline{x}_{1,k}) - b_2}{b_1 - b_2}, \\ M_{32}(z_2(\overline{x}_{1,k})) = M_{42}(z_2(\overline{x}_{1,k})) &= \frac{b_1 - z_2(\overline{x}_{1,k})}{b_1 - b_2}, \end{aligned} \tag{51}$$

where

$$\begin{aligned} z_1(\overline{x}_{1,k}) &= \frac{Tg\mu(\mu\overline{x}_{1,k} + 2\mu y_0 + 2)\overline{x}_{1,k}}{(1 + \mu(\overline{x}_{1,k} + y_0))^2}, \\ a_1 &= \max(z_1(\overline{x}_{1,k})), \qquad a_2 = \min(z_1(\overline{x}_{1,k})), \\ z_2(\overline{x}_{1,k}) &= \frac{T\lambda\mu}{2m(1 + \mu(\overline{x}_{1,k} + y_0))^2}, \\ b_1 &= \max(z_2(\overline{x}_{1,k})), \\ b_2 &= \min(z_2(\overline{x}_{1,k})). \end{aligned} \tag{52}$$

From these fuzzy sets it is possible to compute the membership function α_k.

Solving the optimization problem (47), we obtain the following gains for the control law (8):

$$\begin{aligned} K_1 &= -[1.6118 \quad 1.1049], & K_{d1} &= [0 \quad -0.4752], \\ K_2 &= -[0.9862 \quad 1.0933], & K_{d2} &= [0 \quad -0.4686], \\ K_3 &= -[2.1914 \quad 1.5743], & K_{d3} &= [0 \quad -0.6747], \\ K_4 &= -[1.5459 \quad 1.574], & K_{d4} &= [0 \quad -0.6776]. \end{aligned} \tag{53}$$

Considering $\|\phi_{5,0}\|_5^2 = 0$; that is, all the delayed states equal zero, we have the sets $\mathscr{C}_x = \mathscr{L}_{V_1}$ and $\mathscr{B}(r) = \{0\}$. The set $\mathscr{C}_x = \mathscr{L}_{V_1}$ (dashed line) and the stable trajectories for two initial conditions $\varphi_{5,0} = \{\phi_{5,0}, \overline{x}_0\} \in \mathscr{D}_5$ with $\|\phi_{5,0}\|_5^2 = 0$ and $\overline{x}_0^i = [x_{1,0} \quad \overline{x}_{2,0}]^T$, for $i = 1, 2$, (×marks) are shown in Figure 1, where we have $\overline{x}_0^1 = [0.01624 \quad 0.1535]^T$ and $\overline{x}_0^2 = [-0.04199 \quad -0.05754]^T$. In these two simulations, the nonlinear system (49) was employed in closed-loop with control law (8) and the gains shown in the previous paragraph, and the time-varying delay was assumed as $d_k = \texttt{round}(3 + 2\cos(k))$. As expected, the two trajectories started in $\mathscr{C}_x$ converge asymptotically to origin, that is, $[y_0, 0]$ in the physical structure.

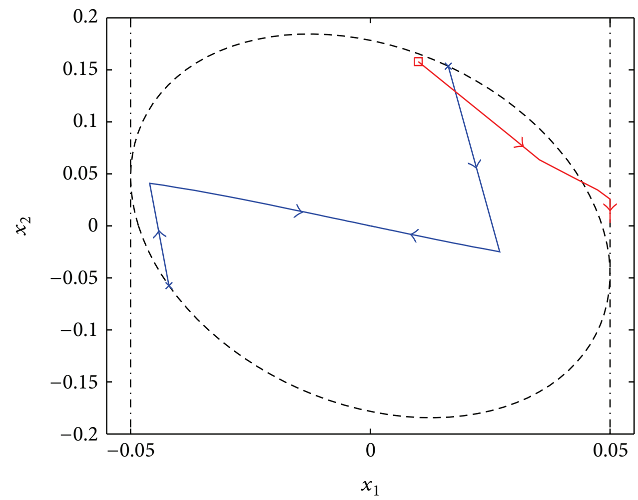

Figure 1: The set $\mathscr{C}_x = \mathscr{L}_{V_1}$ (dashed line), two stable trajectories starting on × marks, and a divergent trajectory starting on □ mark.

Additionally, we considered the condition shown in [24, Theorem 2], where the region of validity is not taken into account to calculate the gains for the control law (8). We obtain the following gains: $K_1 = [-0.2705 \quad 0.2059]$, $K_{d1} = -[0.0014 \quad 0.4554]$, $K_2 = [0.3079 \quad 0.2059]$, $K_{d2} = [0.0006 \quad -0.4549]$, $K_3 = [-0.3016 \quad 0.2939]$, $K_{d3} = [0.0005 \quad -0.6434]$, $K_4 = [0.3553 \quad 0.2939]$, and $K_{d4} = -[0.0019 \quad 0.6438]$ considering the constant $\epsilon = 1.5$. Applying these gains in the control law (8) in the nonlinear system (49) and considering the initial condition $\varphi_{5,0}$ with $\|\phi_{5,0}\|_5^2 = 0$ and $\overline{x}_0 = [0.01 \quad 0.1579]^T$, we obtain the divergent trajectory shown in the Figure 1 marked by □, that is, bounded by the structure limitation of the physical system.

4.2. Academic Example. Consider the following T-S fuzzy model described by (5)–(7) with $i = 1, \ldots, 4$, $p = 1, 2$:

$$\begin{aligned} A_1 &= \begin{bmatrix} -2.4 & 0.44 \\ -2 & 1.15 \end{bmatrix}, & A_2 &= \begin{bmatrix} 2.4 & 0.44 \\ -2 & 1.15 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} -2.4 & 0.25 \\ -2 & 0.65 \end{bmatrix}, & A_4 &= \begin{bmatrix} 2.4 & 0.25 \\ -2 & 0.65 \end{bmatrix}, \\ A_{d1} = A_{d3} &= \begin{bmatrix} 0.08 & -0.16 \\ 0 & -0.08 \end{bmatrix}, \\ A_{d2} = A_{d4} &= \begin{bmatrix} -0.08 & 0.16 \\ 0 & 0.08 \end{bmatrix}, \\ B_i &= \begin{bmatrix} 1 \\ 0.8 \end{bmatrix}, \end{aligned} \tag{54}$$

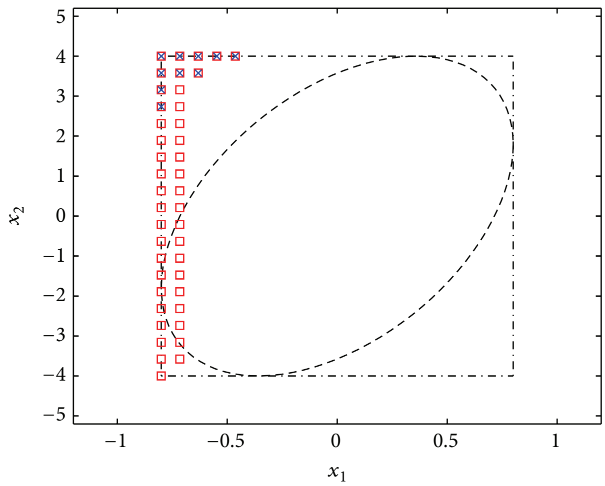

Figure 2: The sets $\mathscr{C}_x = \mathscr{L}_{V_1}$ and $\mathscr{V}_0$ and initial conditions that generate unstable trajectories.

with $d_k \in [1, 10]$, and the operational region for this system is fixed by (4) with $L = \mathbf{I}$ and $\eta = [0.8 \quad 4]^T$ and the fuzzy sets are given by $M_{11}(x_{1,k}) = M_{31}(x_{1,k}) = (1/2)(1 + 1.25x_{1,k})$ and $M_{21}(x_{1,k}) = M_{41}(x_{1,k}) = (1/2)(1 - 1.25x_{1,k})$. Besides, if $|x_{2,k}| < 3$ then $M_{12}(x_{2,k}) = M_{22}(x_{2,k}) = 0$ and $M_{32}(x_{2,k}) = M_{42}(x_{2,k}) = 1$, otherwise $M_{12}(x_{2,k}) = M_{22}(x_{2,k}) = (x_{2,k}^2 - 9)/7$ and $M_{32}(x_{2,k}) = M_{42}(x_{2,k}) = (16 - x_{2,k}^2)/7$.

In this example, we consider $\|\phi_{10,0}\|_{10}^2 = 0$ which yields the sets $\mathscr{C}_x = \mathscr{L}_{V_1}$ and $\mathscr{B}(r) = \{0\}$. We compare the controller designed by using the condition presented in [24, Theorem 2] and $\epsilon = 2$ with the one obtained by solving optimization problem (47). For each of these designs we tested the controller with initial x_0 belonging to a grid on the valid region of the space $x_{1,k} \times x_{2,k}$. For each divergent trajectory we marked the respective starting point with □ for the design based on [24, Theorem 2] and with × for the design based on our proposal. These marks are shown in Figure 2 with the region of validity (dash-dotted line) and the region of stability estimated through (25) (dashed line). Note that the number of unstable points using [24, Theorem 2] is about 430% greater than the number of unstable ones using the design proposed here. For the simulation we considered d_k =`round`$(5.5 + 4.5\cos(k))$. It is clear that our proposal leads to a wider region of stability for this system. Besides, Figure 2 can give an idea of the lower conservatism of the estimative of such a region in our approach.

5. Conclusions

Convex conditions for the synthesis of state-feedback Takagi-Sugeno (T-S) fuzzy gains are proposed in this paper for local stabilization of nonlinear discrete-time systems with time-varying delay and restriction in the states. These conditions are formulated to ensure that the trajectories of the controlled system evolves only inside the region of validity of the T-S model used in the controller design step. Beside this, the characterization of the region of stability is developed by splitting the initial vectors to two parts: one composed by initial states in the sampling instant zero and another part encompassing the sequence of delayed state vectors. Therewith, the calculated region of initial conditions is characterized by a ellipsoidal set that contains the state vector in the sampling instant zero and another region encompassing the delayed state vectors. The size of these regions is depending on the norm of the delayed states as well as on the size of the delay. We proposed a convex optimization problem to maximize the region of stability. The stabilization condition was developed from a fuzzy Lyapunov-Krasovskii function candidate. Two examples with nonlinear systems are given to illustrate the proposal and also to compare with recent conditions where the region of validity and the local stability are not taken into account during the controller design phase.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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