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New iteration process for pseudocontractive mappings with convergence analysis

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Abstract

In this study, we introduce a three step iteration process and prove a convergence result for a countable family of pseudocontractive mappings. Our results improve and generalize most of the results that have been proved for this important class of nonlinear mappings.

MSC: 47H09; 47H10

Keywords: strong convergence; pseudocontractive mapping; iteration process

1 Introduction

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let K be nonempty, closed, and convex subset of H. We recall that a mapping $T: K \to K$ is called strongly pseudocontractive, if for some $\beta \in (0,1)$,

$$\langle Tx - Ty, x - y\rangle \leq \beta\|x - y\|^2, \tag{1.1}$$

holds, for all $x, y \in K$, while T is called a pseudocontractive mapping if (1.1) holds for $\beta = 1$. Equivalently T is called pseudocontractive if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \big\|(I - T)x - (I - T)y\big\|^2, \quad \forall x, y \in K. \tag{1.2}$$

The mapping T is called Lipschitz if there exists a $L \geq 0$ such that $\|Tx - Ty\| \leq L\|x - y\|$, for all $x, y \in K$. The mapping T is called nonexpansive if $L = 1$ and is called contractive if $L < 1$. Every nonexpansive mapping is pseudocontractive. The converse is not true. For example $Tx = 1 - x^{\frac{2}{3}}$, $0 \leq x \leq 1$, is a continuous pseudocontractive mapping which is not a nonexpansive mapping.

Let $T: K \to K$ be a mapping and $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0,1]$. For arbitrary chosen $x_0 \in K$, construct a sequence $\{x_n\}$, where x_n is defined iteratively for each positive integer $n \geq 0$ by

$$x_{n+1} = Tx_n, \tag{1.3}$$

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n Tx_n, \tag{1.4}$$

$$\left.\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Ty_n, \\ y_n &= (1-\beta_n)x_n + \beta_n Tx_n. \end{aligned}\right\} \tag{1.5}$$

The sequences $\{x_n\}$ generated by (1.3), (1.4), and (1.5) are called Picard, Mann [1], and Ishikawa [2] iteration sequences, respectively.

In 1955, Krasnoselskii [3] showed that the Picard iteration scheme (1.3) for a nonexpansive mapping T may fail to converge to a fixed point of T even if T has a unique fixed point, but the Mann sequence (1.4) for $\alpha_n = \frac{1}{2}$, $\forall n \geq 0$, converges strongly to the fixed point of T.

In 1974, Ishikawa [2], introduced the iteration scheme (1.5) and proved the following theorem.

Theorem 1.1 *If K be a compact convex subset of a Hilbert space H, $T: K \to K$ is a Lipschitzian pseudocontractive mapping and x_0 is any point of K, then the sequence $\{x_n\}$ defined iteratively by* (1.5) *converges strongly to a fixed point of T, where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions*: $0 \leq \alpha_n \leq \beta_n < 1$, $\lim_{n\to\infty} \beta_n = 0$, *and* $\sum_{n\geq 0} \alpha_n \beta_n = \infty$.

Since its publication, in 1974, till 2000, it was not known whether or not the Mann iteration process with the simpler iteration (1.4) would converge under the setting of Theorem 1.1 to a fixed point of T if the mapping T is pseudocontractive and continuous (or even Lipschitzian with constant $L > 1$).

Hicks and Kubicek [4] gave an example of pseudocontractive mapping for which Mann iteration process fails. Borwein and Borwein ([5], Proposition 8) gave an example of a Lipschitzian mapping with a unique fixed point for which the Mann iteration fails to converge. In the example of Borwein and Borwein the mapping was not pseudocontractive, while in the example of Hicks and Kubicek the mapping was not continuous (and hence not Lipschitzian).

The problem for a continuous pseudocontractive mapping still remained open. This question was finally settled by Chidume and Mutangadura [6], by constructing an example of a Lipschitz pseudocontractive mapping with a unique fixed point for which every nontrivial Mann sequence fails to converge.

Example 1.2 Let H be the real Hilbert space $\mathbb{R}^2$ under the usual Euclidean inner product. If $x = (a, b) \in H$, define $x^\perp \in H$ to be $(b, -a)$. Let $K := \{x \in H : \|x\| \leq 1\}$ and set

$$K_1 := \left\{x \in H : \|x\| \leq \frac{1}{2}\right\}, \qquad K_2 := \left\{x \in H : \frac{1}{2} \leq \|x\| \leq 1\right\}.$$

Define $T: K \to K$ as follows:

$$Tx = \begin{cases} x + x^\perp, & \text{if } x \in K_1, \\ \frac{x}{\|x\|} - x + x^\perp, & \text{if } x \in K_2. \end{cases} \tag{1.6}$$

Then:

1. T is Lipschitz and pseudocontractive.
2. The origin is the unique fixed point of T.
3. No Mann sequence converges to the fixed point zero.
4. No Mann sequence converges to any $x \neq 0$.

The Ishikawa iterative method has been studied extensively by various authors. But it was still an open question whether or not this method can be employed to approximate fixed points of Lipschitz pseudocontractive mappings without the compactness assumption on K or T (see, *e.g.*, [7–9]).

In [10], Schu introduced an iteration scheme and proved the following.

Theorem 1.3 *Let H be a Hilbert space, K a nonempty, closed, bounded, and convex subset of H, $w \in K$, $T: K \to K$ pseudocontractive and Lipschitzian with $L > 0$, $\{\lambda_n\}_{n\in\mathbb{N}} \subset (0,1)$ with $\lim_{n\to\infty} \lambda_n = 1$, $\{\alpha_n\}_{n\in\mathbb{N}} \subset (0,1)$ with $\lim_{n\to\infty} \alpha_n = 0$, such that $(\{\alpha_n\},\{\mu_n\})$ has certain properties, $((1-\mu_n)(1-\lambda_n)^{-1})$ is bounded and $\lim_{n\to\infty} \alpha_n^{-1}(1-\mu_n) = 0$, where $k_n := (1+\alpha_n^2(1+L^2))^{1/2}$ and $\mu_n := \lambda_n k_n^{-1}$ for all $n \in \mathbb{N}$. For arbitrary $z_0 \in K$, define for all $n \in \mathbb{N}$,*

$$\left.\begin{aligned} z_{n+1} &= (1-\mu_{n+1})w + \mu_{n+1}y_n, \\ y_n &= (1-\alpha_n)z_n + \alpha_n Tz_n. \end{aligned}\right\} \tag{1.7}$$

Then $\{z_n\}$ converges strongly to the unique fixed point of T closest to w.

Theorem 1.3 has some advantages over Theorem 1.1. First, the recursion formula (1.7) is slightly simpler than (1.5). More importantly, no compactness assumption is imposed on K. However, the choices of α_n and μ_n in Theorem 1.3 are not as simple as those of α_n and β_n in Theorem 1.3 (where one can choose $\alpha_n = \beta_n = \frac{1}{\sqrt{n+1}}$ for all positive integers n).

Zhou [11] established the hybrid Ishikawa algorithm for Lipschitz pseudocontractive mappings without the compactness assumption. In 2009, Yao *et al.* [12] introduced the hybrid Mann algorithm for a L-Lipschitz pseudocontractive mapping, which was generalized by Tang *et al.* [13] to the hybrid Ishikawa iterative process. The schemes given by Zhou [11], Yao *et al.* [12] and Tang *et al.* [13] are not easy to compute, since for each $n \geq 1$, it involves the computation of the intersection of C_n and Q_n.

Finding a point in the intersection of fixed point sets of a family of nonexpansive mappings is a task that occurs frequently in various areas of mathematical sciences and engineering. For example, the well-known convex feasibility problem reduces to finding a point in the intersection of fixed point sets of a family of nonexpansive mappings; see, *e.g.*, [14, 15].

In 2011, Zegeye *et al.* [16] extended the result of Tang *et al.* [13] to the Ishikawa iterative process (not hybrid) for a finite family of Lipschitz pseudocontractive mappings as follows:

$$\left.\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T_n x_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_n y_n, \end{aligned}\right\} \tag{1.8}$$

and they proved a strong convergence theorem under some conditions.

Cheng *et al.* [17] generalized algorithm (1.8) to a three step iterative process for a countable family of Lipschitz pseudocontractive mappings as follows:

$$\left.\begin{aligned} z_n &= (1-\gamma_n)x_n + \gamma_n T_n x_n, \\ y_n &= (1-\beta_n)x_n + \beta_n T_n z_n, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T_n y_n. \end{aligned}\right\} \tag{1.9}$$

The iteration (1.9) was initially introduced by Noor [18] for a single mapping.

Recently, Zegeye and Shahzad [19] proved the strong convergence of a three step iteration process for a common fixed point of two Lipschitz pseudocontractive mappings.

In this paper, our purpose is to introduce a new three step iteration process for a countable family of pseudocontractive mappings. The results obtained in this paper improve and extend the results of Zhou [11], Tang *et al.* [13], Zegeye *et al.* [16] and Cheng *et al.* [17] and some other results in this direction.

2 Preliminaries

In the sequel, we also need the following definitions and lemma.

Let H be a real Hilbert space, and $\phi: H \times H \to \mathbb{R}$ the function given by

$$\phi(x,y) := \|x-y\|^2 = \|x\|^2 - 2\langle x,y\rangle + \|y\|^2, \quad \text{for } x,y \in H.$$

It is obvious from the definition of the function ϕ that

$$\big(\|x\| - \|y\|\big)^2 \le \phi(x,y) \le \big(\|x\| + \|y\|\big)^2, \quad \text{for } x,y \in H.$$

The function ϕ also has the following property:

$$\phi(y,x) = \phi(y,z) + \phi(z,x) + 2\langle z-y, x-z\rangle, \quad \text{for } x,y,z \in H. \tag{2.1}$$

A countable family of mappings $\{T_n\}_{n=1}^{\infty}: K \to H$ is called uniformly Lipschitz with Lipschitz constant $L_n > 0$, $n \ge 1$, if there exists $0 < L = \sup_{n\ge 1} L_n$ such that

$$\|T_n x - T_n y\| \le L\|x-y\|, \quad \forall x,y \in K, n \ge 1.$$

A countable family of mappings $\{T_n\}_{n=1}^{\infty}: K \to H$ is called uniformly closed if $x_n \to x^*$ and $\|x_n - T_n x_n\| \to 0$ imply $x^* \in \bigcap_{n=1}^{\infty} F(T_n)$.

Lemma 2.1 *Let H be a real Hilbert space. Then, for all $x_i \in H$ and $\alpha_i \in [0,1]$ for $i = 1,2,\ldots,n$ such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$, the following inequality holds*:

$$\|\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n\|^2 = \sum_{i=1}^{n} \alpha_i \|x_i\|^2 - \sum_{1\le i,j\le n} \alpha_i\alpha_j \|x_i - x_j\|^2.$$

Remark 2.2 We now give recall an example [17] of a countable family of uniformly closed and uniformly Lipschitz pseudocontractive mappings with the interior of the common fixed points nonempty.

Suppose that $X = \mathbb{R}$ and $K = [-1,1] \subset \mathbb{R}$. Let $\{T_n\}_{n=1}^{\infty}: K \to K$ be given by

$$T_n x = \begin{cases} x, & x \in [-1,0), \\ (\frac{1}{2^n} + \frac{1}{2}), & x \in [0,1]. \end{cases}$$

Then we observe that $F = \bigcap_{n=1}^{\infty} F(T_n) = [-1,0]$, and hence the interior of the common fixed points is nonempty. Also, it is clear that $\{T_n\}_{n=1}^{\infty}$ is a countable family of uniformly closed and uniformly Lipschitz pseudocontractive mappings with Lipschitz constant $L = \sup_{n\ge 1} L_n = 2$.

3 Main results

Theorem 3.1 *Let K be a nonempty, closed, and convex subset of a real Hilbert space H. Let $\{T_n\}_{n=1}^{\infty}: K \to K$ be a countable family of uniformly closed and Lipschitz pseudocontractive mappings with Lipschitzian constants L_n. Assume that the interior of $\mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1 \in K$ by the following algorithm*:

$$\left.\begin{aligned} &z_n = \gamma_n x_n + (1-\gamma_n)T_n x_n,\\ &y_n = \beta_n x_n + (1-\beta_n)T_n z_n,\\ &x_{n+1} = \alpha_n x_n + (1-\alpha_n)[\theta_n x_n + \delta_n T_n y_n + \sigma_n T_n z_n], \end{aligned}\right\} \tag{3.1}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\theta_n\}$, $\{\delta_n\}$, and $\{\sigma_n\}$ are in $(0,1)$ satisfying the following conditions:

(i) $\theta_n + \delta_n + \sigma_n = 1$,

(ii) $(1-\alpha_n) \le (1-\theta_n) \le (1-\beta_n) \le (1-\gamma_n)$, $\forall n \ge 1$,

(iii) $\liminf_{n\to\infty}(1-\alpha_n) = \alpha > 0$,

(iv) $\sup_{n\ge 1}(1-\gamma_n) \le \gamma$ *with* $\gamma^3L^4 + 2\gamma^2L^3 + \gamma^2L^2 + \gamma L^2 + 2\gamma < 1$, *where* $L = \sup_{n\ge 1} L_n$.

Then $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$.

Proof Suppose that $p \in \mathcal{F}$. Using (3.1) and (1.2), we have

$$\begin{aligned} \|z_n - p\|^2 &= \gamma_n\|x_n - p\|^2 + (1-\gamma_n)\|T_n x_n - p\|^2 - \gamma_n(1-\gamma_n)\|T_n x_n - x_n\|^2\\ &\le \gamma_n\|x_n - p\|^2 + (1-\gamma_n)\|x_n - p\|^2 + (1-\gamma_n)\|T_n x_n - x_n\|^2\\ &\quad - \gamma_n(1-\gamma_n)\|T_n x_n - x_n\|^2\\ &= \|x_n - p\|^2 + (1-\gamma_n)^2\|T_n x_n - x_n\|^2, \end{aligned} \tag{3.2}$$

$$\begin{aligned} \|y_n - p\|^2 &= \beta_n\|x_n - p\|^2 + (1-\beta_n)\|T_n z_n - p\|^2 - \beta_n(1-\beta_n)\|T_n z_n - x_n\|^2\\ &\le \beta_n\|x_n - p\|^2 + (1-\beta_n)\|z_n - p\|^2 + (1-\beta_n)\|T_n z_n - z_n\|^2\\ &\quad - \beta_n(1-\beta_n)\|T_n z_n - x_n\|^2. \end{aligned} \tag{3.3}$$

Using Lemma 2.1 and (3.1), we have

$$\begin{aligned} \|x_{n+1} - p\|^2 &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\bigl[\theta_n\|x_n - p\|^2 + \delta_n\|T_n y_n - p\|^2\\ &\quad + \sigma_n\|T_n z_n - p\|^2 - \theta_n\delta_n\|T_n y_n - x_n\|^2 - \theta_n\sigma_n\|T_n z_n - x_n\|^2\bigr]\\ &\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\theta_n\|x_n - p\|^2 + (1-\alpha_n)\bigl[\delta_n\|y_n - p\|^2\\ &\quad + \delta_n\|T_n y_n - y_n\|^2\bigr] + (1-\alpha_n)\bigl[\sigma_n\|z_n - p\|^2 + \sigma_n\|T_n z_n - z_n\|^2\bigr]\\ &\quad - \theta_n\delta_n(1-\alpha_n)\|T_n y_n - x_n\|^2 - \theta_n\sigma_n(1-\alpha_n)\|T_n z_n - x_n\|^2, \end{aligned} \tag{3.4}$$

$$\begin{aligned} \|T_n z_n - z_n\|^2 &= \gamma_n\|T_n z_n - x_n\|^2 + (1-\gamma_n)\|T_n z_n - T_n x_n\|^2\\ &\quad - \gamma_n(1-\gamma_n)\|T_n x_n - x_n\|^2\\ &\le \gamma_n\|T_n z_n - x_n\|^2 + (1-\gamma_n)^3L^2\|T_n x_n - x_n\|^2\\ &\quad - \gamma_n(1-\gamma_n)\|T_n x_n - x_n\|^2 \end{aligned}$$

$$\begin{aligned} &= \gamma_n \| T_n z_n - x_n \|^2 \\ &\quad + (1-\gamma_n)\big[(1-\gamma_n)^2 L^2 + (1-\gamma_n) - 1\big] \| T_n x_n - x_n \|^2, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \| T_n y_n - y_n \|^2 &= \beta_n \| T_n y_n - x_n \|^2 + (1-\beta_n) \| T_n y_n - T_n z_n \|^2 \\ &\quad - \beta_n(1-\beta_n) \| T_n z_n - x_n \|^2 \\ &\le \beta_n \| T_n y_n - x_n \|^2 + (1-\beta_n) L^2 \| y_n - z_n \|^2 \\ &\quad - \beta_n(1-\beta_n) \| T_n z_n - x_n \|^2 \\ &\le \beta_n \| T_n y_n - x_n \|^2 + (1-\beta_n) L^2 \| y_n - z_n \|^2, \end{aligned} \tag{3.6}$$

$$\begin{aligned} \| y_n - z_n \| &= \big\| \beta_n x_n + (1-\beta_n) T_n z_n - \gamma_n x_n - (1-\gamma_n) T_n x_n \big\| \\ &= \big\| (1-\gamma_n) x_n - (1-\beta_n) x_n - \big\{(1-\gamma_n) - (1-\beta_n)\big\} T_n x_n \\ &\quad + (1-\beta_n)(T_n z_n - T_n x_n) \big\| \\ &\le \big\{(1-\gamma_n) - (1-\beta_n)\big\} \| T_n x_n - x_n \| + (1-\beta_n) L \| z_n - x_n \| \\ &= \big\{(1-\gamma_n) - (1-\beta_n) + (1-\beta_n)(1-\gamma_n) L\big\} \| T_n x_n - x_n \|. \end{aligned} \tag{3.7}$$

Substituting (3.2) and (3.5) in (3.3), we obtain

$$\begin{aligned} \| y_n - p \|^2 &\le \| x_n - p \|^2 + (1-\beta_n)(1-\gamma_n)^2 \| T_n x_n - x_n \|^2 + (1-\beta_n)\big[\gamma_n \| T_n z_n - x_n \|^2 \\ &\quad + (1-\gamma_n)\big[(1-\gamma_n)^2 L^2 + (1-\gamma_n) - 1\big] \| T_n x_n - x_n \|^2\big] \\ &\quad - \beta_n(1-\beta_n) \| T_n z_n - x_n \|^2 \\ &\le \| x_n - p \|^2 - (1-\beta_n)\big[(1-\gamma_n) - (1-\beta_n)\big] \| T_n z_n - x_n \|^2 \\ &\quad + (1-\beta_n)(1-\gamma_n)\big[(1-\gamma_n)^2 L^2 + 2(1-\gamma_n) - 1\big] \| T_n x_n - x_n \|^2. \end{aligned} \tag{3.8}$$

From (3.7) and (3.6), we have

$$\begin{aligned} \| T_n y_n - y_n \|^2 &\le \beta_n \| T_n y_n - x_n \|^2 + (1-\beta_n) L^2 \big\{(1-\gamma_n) - (1-\beta_n) \\ &\quad + (1-\beta_n)(1-\gamma_n) L\big\}^2 \| T_n x_n - x_n \|^2. \end{aligned} \tag{3.9}$$

Substituting (3.2), (3.5), (3.8), and (3.9) in (3.4), we get

$$\begin{aligned} &\| x_{n+1} - p \|^2 \\ &\quad \le \| x_n - p \|^2 - (1-\alpha_n)\big[(1-\gamma_n)\big\{\delta_n(1-\beta_n) + \sigma_n\big\}\big\{1 - 2(1-\gamma_n) - (1-\gamma_n)^2 L^2\big\} \\ &\qquad - \delta_n(1-\beta_n) L^2 \big\{(1-\gamma_n) - (1-\beta_n) + (1-\beta_n)(1-\gamma_n) L\big\}^2\big] \| T_n x_n - x_n \|^2 \\ &\qquad + (1-\alpha_n)\sigma_n\big[(1-\theta_n) - (1-\gamma_n)\big] \| T_n z_n - x_n \|^2 \\ &\qquad + (1-\alpha_n)\delta_n\big[(1-\theta_n) - (1-\beta_n)\big] \| T_n y_n - x_n \|^2. \end{aligned}$$

From condition (ii), we have

$$\begin{aligned} &\| x_{n+1} - p \|^2 \\ &\quad \le \| x_n - p \|^2 - (1-\alpha_n)\big[(1-\gamma_n)\big\{\delta_n(1-\beta_n) + \sigma_n\big\}\big\{1 - 2(1-\gamma_n) - (1-\gamma_n)^2 L^2\big\} \end{aligned}$$

$$
\begin{aligned}
&- \delta_n(1-\beta_n)L^2\{(1-\gamma_n)-(1-\beta_n)+(1-\beta_n)(1-\gamma_n)L\}^2\big]\|T_nx_n-x_n\|^2 \\
&\le \|x_n-p\|^2-(1-\alpha_n)\big[(1-\gamma_n)\{\delta_n(1-\beta_n)+\sigma_n(1-\beta_n)\}\{1-2(1-\gamma_n) \\
&- (1-\gamma_n)^2L^2\}-\delta_n(1-\beta_n)L^2\{(1-\gamma_n)-(1-\beta_n) \\
&+ (1-\beta_n)(1-\gamma_n)L\}^2\big]\|T_nx_n-x_n\|^2 \\
&\le \|x_n-p\|^2-(1-\alpha_n)\big[(1-\gamma_n)(1-\beta_n)(1-\theta_n)\{1-2(1-\gamma_n)-(1-\gamma_n)^2L^2\} \\
&- \delta_n(1-\beta_n)L^2\{(1-\gamma_n)-(1-\beta_n)+(1-\beta_n)(1-\gamma_n)L\}^2\big]\|T_nx_n-x_n\|^2 \\
&\le \|x_n-p\|^2-(1-\alpha_n)\big[(1-\gamma_n)(1-\beta_n)(1-\theta_n)\{1-2(1-\gamma_n)-(1-\gamma_n)^2L^2\} \\
&- (1-\theta_n)(1-\beta_n)L^2\{(1-\gamma_n)-(1-\beta_n)+(1-\beta_n)(1-\gamma_n)L\}^2\big]\|T_nx_n-x_n\|^2 \\
&\le \|x_n-p\|^2-(1-\alpha_n)(1-\beta_n)(1-\theta_n)\big[(1-\gamma_n)\{1-2(1-\gamma_n)-(1-\gamma_n)^2L^2\} \\
&- L^2\{(1-\gamma_n)-(1-\beta_n)+(1-\beta_n)(1-\gamma_n)L\}^2\big]\|T_nx_n-x_n\|^2. \qquad (3.10)
\end{aligned}
$$

From condition (iv), we have

$$
\begin{aligned}
&\gamma^3L^4+2\gamma^2L^3+\gamma^2L^2+\gamma L^2+2\gamma<1, \\
&(1-\gamma_n)^3L^4+2(1-\gamma_n)^2L^3+(1-\gamma_n)^2L^2+(1-\gamma_n)L^2+2(1-\gamma_n)<1, \\
&(1-\gamma_n)^2L^2+2(1-\gamma_n)-1+(1-\gamma_n)^3L^4+2(1-\gamma_n)^2L^3+(1-\gamma_n)L^2<0, \\
&\big[(1-\gamma_n)^2L^2+2(1-\gamma_n)-1\big]+(1-\gamma_n)L^2\big[(1-\gamma_n)^2L^2+2(1-\gamma_n)L+1\big]<0, \\
&\big[(1-\gamma_n)^2L^2+2(1-\gamma_n)-1\big]+(1-\gamma_n)L^2\big[1+(1-\gamma_n)L\big]^2<0, \\
&(1-\gamma_n)\big[(1-\gamma_n)^2L^2+2(1-\gamma_n)-1\big]+(1-\gamma_n)^2L^2\big[1+(1-\gamma_n)L\big]^2<0, \\
&(1-\gamma_n)\big[(1-\gamma_n)^2L^2+2(1-\gamma_n)-1\big]+L^2\big[(1-\gamma_n)+(1-\gamma_n)^2L\big]^2<0, \\
&(1-\gamma_n)\big[(1-\gamma_n)^2L^2+2(1-\gamma_n)-1\big]+L^2\big[(1-\gamma_n)-(1-\beta_n)+(1-\beta_n)(1-\gamma_n)L\big]^2<0, \\
&(1-\gamma_n)\big[1-2(1-\gamma_n)-(1-\gamma_n)^2L^2\big]-L^2\big[(1-\gamma_n)-(1-\beta_n)+(1-\beta_n)(1-\gamma_n)L\big]^2>0.
\end{aligned}
$$

Then

$$\|x_{n+1}-p\|^2\le\|x_n-p\|^2. \qquad (3.11)$$

It is obvious that if $\lim_{n\to\infty}\|x_n-p\|$ exists, then $\{\|x_n-p\|\}$ is bounded. This implies that $\{x_n\}$, $\{T_nx_n\}$, $\{z_n\}$, $\{T_nz_n\}$, $\{y_n\}$, and $\{T_ny_n\}$ are also bounded.

Furthermore, from (2.1), we have

$$\phi(p,x_n)=\phi(p,x_{n+1})+\phi(x_{n+1},x_n)+2\langle x_{n+1}-p,x_n-x_{n+1}\rangle.$$

This implies that

$$\langle x_{n+1}-p,x_n-x_{n+1}\rangle+\frac{1}{2}\phi(x_{n+1},x_n)=\frac{1}{2}\big(\phi(p,x_n)-\phi(p,x_{n+1})\big). \qquad (3.12)$$

Moreover, since the interior of $\mathcal{F}$ is nonempty, there exist $p^*\in\mathcal{F}$ and $r>0$ such that $p^*+rh\in\mathcal{F}$ whenever $\|h\|\le1$. Thus, from the fact that $\phi(x,y)=\|x-y\|^2$, and (3.11) and

(3.12), we get

$$\begin{aligned} 0 &\leq \langle x_{n+1} - (p^* + rh), x_n - x_{n+1} \rangle + \frac{1}{2}\phi(x_{n+1}, x_n) \\ &= \frac{1}{2}\bigl(\phi\bigl((p^* + rh), x_n\bigr) - \phi\bigl((p^* + rh), x_{n+1}\bigr)\bigr). \end{aligned} \tag{3.13}$$

Then from (3.12) and (3.13), we obtain

$$\begin{aligned} r\langle h, x_n - x_{n+1} \rangle &\leq \langle x_{n+1} - p^*, x_n - x_{n+1} \rangle + \frac{1}{2}\phi(x_{n+1}, x_n) \\ &= \frac{1}{2}\bigl(\phi\bigl(p^*, x_n\bigr) - \phi\bigl(p^*, x_{n+1}\bigr)\bigr), \end{aligned}$$

and hence

$$\langle h, x_n - x_{n+1} \rangle \leq \frac{1}{2r}\bigl(\phi\bigl(p^*, x_n\bigr) - \phi\bigl(p^*, x_{n+1}\bigr)\bigr).$$

Since h with $\|h\| \leq 1$ is arbitrary, we have

$$\|x_n - x_{n+1}\| \leq \frac{1}{2r}\bigl(\phi\bigl(p^*, x_n\bigr) - \phi\bigl(p^*, x_{n+1}\bigr)\bigr).$$

So, if $n > m$, then we get

$$\begin{aligned} \|x_m - x_n\| &= \|x_m - x_{m+1} + x_{m+1} - \cdots - x_{n-1} + x_{n-1} - x_n\| \\ &\leq \sum_{i=m}^{n-1} \|x_i - x_{i+1}\| \\ &\leq \frac{1}{2r}\sum_{i=m}^{n-1}\bigl(\phi\bigl(p^*, x_i\bigr) - \phi\bigl(p^*, x_{i+1}\bigr)\bigr) \\ &\leq \frac{1}{2r}\bigl(\phi\bigl(p^*, x_m\bigr) - \phi\bigl(p^*, x_n\bigr)\bigr). \end{aligned}$$

But we know that $\{\phi(p^*, x_n)\}$ converges. Therefore, we see that $\{x_n\}$ is a Cauchy sequence. Since K is closed subset of H, there exists $x^* \in K$ such that

$$x_n \to x^*. \tag{3.14}$$

Furthermore, (3.10) and conditions (ii), (iii), and (iv), we get

$$\begin{aligned} &\alpha^4\bigl[\bigl\{1 - \gamma^2L^2 - 2\gamma\bigr\} - \gamma L^2\{\gamma L + 1\}^2\bigr]\sum\|T_nx_n - x_n\|^2 \\ &\quad\leq \sum(1 - \alpha_n)(1 - \beta_n)(1 - \theta_n)(1 - \gamma_n)\bigl[\bigl\{1 - (1 - \gamma_n)^2L^2 - 2(1 - \gamma_n)\bigr\} \\ &\qquad - (1 - \gamma_n)L^2\bigl\{(1 - \gamma_n)L + 1\bigr\}^2\bigr]\|T_nx_n - x_n\|^2 \\ &\quad= \sum(1 - \alpha_n)(1 - \beta_n)(1 - \theta_n)\bigl[(1 - \gamma_n)\bigl\{1 - (1 - \gamma_n)^2L^2 - 2(1 - \gamma_n)\bigr\} \\ &\qquad - L^2\bigl\{(1 - \gamma_n) + (1 - \gamma_n)^2L\bigr\}^2\bigr]\|T_nx_n - x_n\|^2 \end{aligned}$$

$$\leq \sum (1-\alpha_n)(1-\beta_n)(1-\theta_n)\big[(1-\gamma_n)\big\{1-(1-\gamma_n)^2L^2-2(1-\gamma_n)\big\}$$
$$-L^2\big\{(1-\gamma_n)-(1-\beta_n)+(1-\gamma_n)^2L\big\}^2\big]\|T_nx_n-x_n\|^2$$
$$\leq \|x_n-p\|^2-\|x_{n+1}-p\|^2<\infty,$$

from which it follows that

$$\lim_{n\to\infty}\|T_nx_n-x_n\|=0. \tag{3.15}$$

Since $\{T_n\}_{n=1}^{\infty}$ is uniformly closed, from (3.14) and (3.15), we obtain $x^* \in \mathcal{F} = \bigcap_{n=1}^{\infty} F(T_n)$. This completes the proof. □

We get the following result from Theorem 3.1.

Theorem 3.2 *Let K and H be as in Theorem* 3.1. *Let $T: K \to K$ be a uniformly closed and Lipschitz pseudocontractive mapping with Lipschitzian constant L. Assume that the interior of $F(T)$ is nonempty. Let $\{x_n\}$ be a sequence generated from an arbitrary $x_1 \in K$ by the following algorithm*:

$$\left.\begin{aligned} z_n &= \gamma_n x_n + (1-\gamma_n)Tx_n,\\ y_n &= \beta_n x_n + (1-\beta_n)Tz_n,\\ x_{n+1} &= \alpha_n x_n + (1-\alpha_n)[\theta_n x_n + \delta_n Ty_n + \sigma_n Tz_n], \end{aligned}\right\} \tag{3.16}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\theta_n\}$, $\{\delta_n\}$, and $\{\sigma_n\}$ are in $(0,1)$ satisfying the conditions (i)-(iv) *of Theorem* 3.1. *Then $\{x_n\}$ converges strongly to $x^* \in \mathcal{F}$.*

Remark 3.3 Our results improve and generalize the corresponding results of Zhou [11], Yao *et al.* [12], Tang *et al.* [13], Zegeye *et al.* [16] and Cheng *et al.* [17] and many others.

4 Numerical examples

In this section, using Example 1.2, we numerically demonstrate the convergence of the algorithm defined in this paper and compare its behavior with the Ishikawa and Noor iterations.

Consider $T_n = T$ for all $n \in \mathbb{N}$, where T is given by (1.6). Set the control conditions

$$\alpha_n = 1-\frac{1}{\sqrt{n+1}},\qquad \beta_n=\frac{1}{\sqrt{n+1}}-\frac{1}{(n+1)^2},\qquad \gamma_n=\frac{1}{\sqrt{2n+9}},$$
$$\theta_n = 1-\frac{1}{\sqrt{n+1}},\qquad \sigma_n=\frac{1}{100}(1-\theta_n),\qquad \delta_n = 1-\theta_n-\sigma_n,$$

for all $n \in \mathbb{N}$.

We examine the behavior of the above iterations for the different initial points (see Table 1).

The details for different cases are as below:

Case I. Using the initial point $x_1 = (0.6, 0.8)$, a Matlab program leads to the evaluation illustrated in Table 2 and Figure 1.

Table 1 Initial point and its norm

Case	Initial point	$\|x\|$
Case I	(0.6, 0.8)	1.0000000000
Case II	(0.5, 0.7)	0.8602325267
Case III	(0.2, 0.7)	0.7280109889
Case IV	(0.4, 0.3)	0.5000000000
Case V	(0.2, 0.3)	0.3605551275

Table 2 Comparative results, case I

Iterate	Ishikawa	Noor	New iter.
x_1	(0.6000000000, 0.8000000000)	(0.6000000000, 0.8000000000)	(0.6000000000, 0.8000000000)
x_2	(0.5539771951, 0.3823475759)	(0.5845576135, 0.4251282975)	(0.4179365588, 0.2127647459)
x_{50}	(–0.4450369330, –0.4976206090)	(–0.3239420295, –0.5825226873)	(0.0018082804, 0.0014649100)
x_{100}	(0.6964587601, 0.0133562179)	(0.5821514537, 0.3809434572)	(0.0003286008, –0.0002556144)
x_{500}	(–0.6777455360, –0.3368753388)	(0.3971613402, –0.6438475994)	(–0.0000043324, 0.0000033756)
$x_{1,000}$	(0.5456751982, –0.5586691004)	(0.0954007431, 0.7748693478)	(0.0000003429, 0.0000006995)
$x_{2,000}$	(–0.1037052181, –0.7971860220)	(0.3048527823, 0.7437145034)	(0.0000001035, –0.0000000270)
$x_{3,000}$	(0.7752046594, –0.2571690139)	(–0.8026212027, 0.1507284165)	(0.0000000162, –0.0000000289)
$x_{4,000}$	(–0.6052498145, 0.5614732843)	(0.7824477126, –0.2631121035)	(–0.0000000008, –0.0000000143)
$x_{5,000}$	(0.4877239037, –0.6743717613)	(–0.7969068795, 0.2397588381)	(–0.0000000036, –0.0000000066)
$x_{10,000}$	(–0.7837825951, 0.3340666307)	(0.4705344309, 0.7102474078)	(–0.0000000009, 0.0000000003)
$x_{15,000}$	(0.0995508262, –0.8570695454)	(–0.8011620666, –0.3202712792)	(–0.0000000001, 0.0000000003)
$x_{20,000}$	(0.5437374559, –0.6793797373)	(–0.2633715133, –0.8293418385)	(0.0000000000, 0.0000000001)
$x_{25,000}$	(0.1948437754, –0.8537277083)	(–0.4224421378, –0.7670260001)	(0.0000000000, 0.0000000001)
$x_{25,662}$	(–0.1202342253, –0.8680256967)	(–0.6498446973, –0.5878762853)	(0.0000000000, 0.0000000001)
$x_{25,663}$	(–0.8762826038, –0.0074383873)	(–0.6762590030, 0.5572899348)	(0.0000000000, 0.0000000000)
$x_{30,000}$	(–0.1288396162, 0.8705678507)	(0.3187366255, 0.8202867023)	(0.0000000000, 0.0000000000)

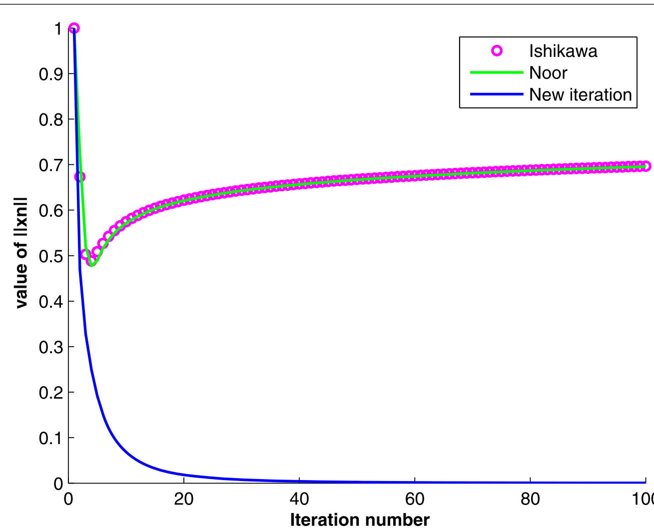

Figure 1 Convergence behavior of first 100 iterates for Case I.

Case II. For the initial point $x_1 = (0.5, 0.7)$, we make the observations as in Table 3 and Figure 2.

Case III. Using the initial point $x_1 = (0.2, 0.7)$, we make the observations as in Table 4 and Figure 3.

Case IV. Using the initial point $x_1 = (0.4, 0.3)$, we make the observations as in Table 5 and Figure 4.

Table 3 Comparative results, case II

Iterate	Ishikawa	Noor	New iter.
x_1	(0.5000000000, 0.7000000000)	(0.5000000000, 0.7000000000)	(0.5000000000, 0.7000000000)
x_2	(0.5077039037, 0.3414830422)	(0.5222783000, 0.3691361889)	(0.3936091367, 0.1328495346)
x_{50}	(−0.4522218355, −0.4911003500)	(−0.3578594849, −0.5623234911)	(0.0019722548, 0.0010101314)
x_{100}	(0.6965793132, 0.0032331971)	(0.6036824557, 0.3458116905)	(0.0002560188, −0.0003026363)
x_{500}	(−0.6825697572, −0.3269901101)	(0.3583486889, −0.6662304996)	(−0.0000033744, 0.0000039952)
$x_{1,000}$	(0.5374984520, −0.5665403780)	(0.1411060451, 0.7678625476)	(0.0000004568, 0.0000005844)
$x_{2,000}$	(−0.1152797418, −0.7955946876)	(0.3483463482, 0.7243626875)	(0.0000000911, −0.0000000455)
$x_{3,000}$	(0.7713853627, −0.2684078761)	(−0.7922903051, 0.1979796028)	(0.0000000095, −0.0000000301)
$x_{4,000}$	(−0.5970260234, 0.5702100615)	(0.7654990306, −0.3089719017)	(−0.0000000036, −0.0000000132)
$x_{5,000}$	(0.4778717750, −0.6813886155)	(−0.7813153619, 0.2865155847)	(−0.0000000046, −0.0000000054)
$x_{10,000}$	(−0.7788448421, 0.3454220346)	(0.5117561426, 0.6811458591)	(−0.0000000008, 0.0000000005)
$x_{15,000}$	(0.0870845517, −0.8584257997)	(−0.8187171280, −0.2722803949)	(−0.0000000001, 0.0000000003)
$x_{20,000}$	(0.5338066309, −0.6872101056)	(−0.3120070105, −0.8122955525)	(0.0000000000, 0.0000000001)
$x_{24,000}$	(−0.1437915180, 0.8627856757)	(0.5322398911, 0.6940936228)	(0.0000000000, 0.0000000001)
$x_{24,478}$	(0.8473016080, 0.2190814433)	(0.7489567757, −0.4527152205)	(0.0000000000, 0.0000000001)
$x_{24,479}$	(0.3414098540, −0.8058273678)	(−0.3375080849, −0.8074511489)	(0.0000000000, 0.0000000000)
$x_{25,000}$	(0.1824160039, −0.8564692045)	(−0.4671095169, −0.7406719542)	(0.0000000000, 0.0000000000)

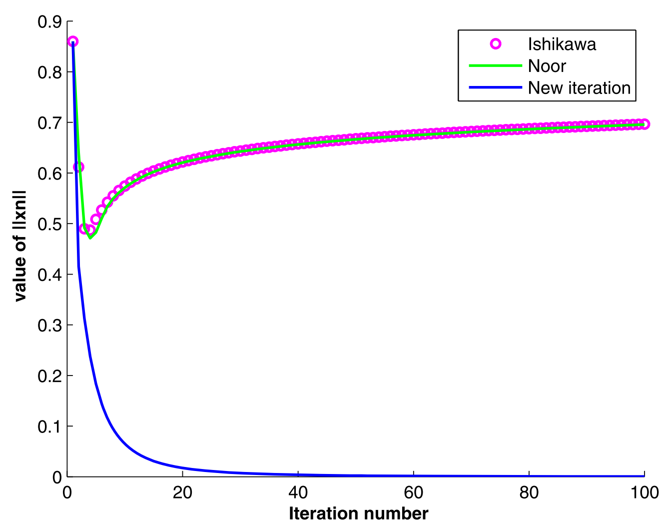

Figure 2 Convergence behavior of first 100 iterates for Case II.

Table 4 Comparative results, case III

Iterate	Ishikawa	Noor	New iter.
x_1	(0.2000000000, 0.7000000000)	(0.2000000000, 0.7000000000)	(0.2000000000, 0.7000000000)
x_2	(0.3450926068, 0.4336578892)	(0.3440379880, 0.4453754852)	(0.3353767960, 0.1812746972)
x_{50}	(−0.2825203374, −0.6048689124)	(−0.1985930762, −0.6362640252)	(0.0016680265, 0.0012183686)
x_{100}	(0.6629813795, 0.2137495813)	(0.4924269913, 0.4914609759)	(0.0002799554, −0.0002411817)
x_{500}	(−0.5517134650, −0.5181083458)	(0.5198970951, −0.5495305923)	(−0.0000036908, 0.0000031847)
$x_{1,000}$	(0.6836678095, −0.3774531919)	(−0.0643145529, 0.7780664793)	(0.0000003350, 0.0000006048)
$x_{2,000}$	(0.1307318905, −0.7932020541)	(0.1470941858, 0.7901960402)	(0.0000000905, −0.0000000285)
$x_{3,000}$	(0.8164372572, −0.0225470783)	(−0.8164988720, −0.0157937523)	(0.0000000131, −0.0000000263)
$x_{4,000}$	(−0.7415175937, 0.3629479923)	(0.8196224517, −0.0983434723)	(−0.0000000014, −0.0000000127)
$x_{5,000}$	(0.6615671585, −0.5049562094)	(−0.8290255718, 0.0725360346)	(−0.0000000035, −0.0000000057)
$x_{10,000}$	(−0.8468389535, 0.0936987565)	(0.3161194504, 0.7911526553)	(−0.0000000008, 0.0000000003)
$x_{15,000}$	(0.3426217981, −0.7918894343)	(−0.7192022147, −0.4766366788)	(−0.0000000001, 0.0000000002)
$x_{20,000}$	(0.7166432454, −0.4935886013)	(−0.0890525429, −0.8655877100)	(0.0000000000, 0.0000000001)
$x_{24,767}$	(−0.6197996742, −0.6182758717)	(−0.8746592516, 0.0368619317)	(0.0000000000, 0.0000000001)
$x_{24,768}$	(−0.7025624816, 0.5223259212)	(−0.0919736849, 0.8705918501)	(0.0000000000, 0.0000000000)
$x_{25,000}$	(0.4328972371, −0.7611931941)	(−0.2574770981, −0.8369538748)	(0.0000000000, 0.0000000000)

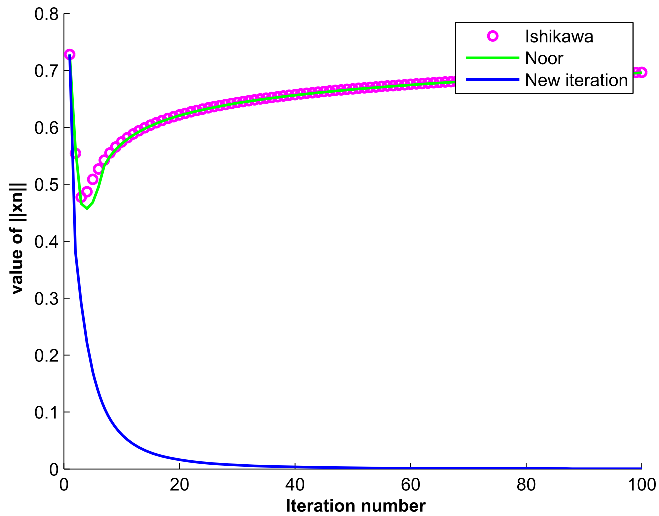

Figure 3 Convergence behavior of first 100 iterates for Case III.

Table 5 Comparative results, case IV

Iterate	Ishikawa	Noor	New iter.
x_1	(0.4000000000, 0.3000000000)	(0.4000000000, 0.3000000000)	(0.4000000000, 0.3000000000)
x_2	(0.4460001486, 0.0829124405)	(0.4319456624, 0.0683790444)	(0.3104879326, −0.1461827853)
x_{50}	(−0.6024706350, −0.2875991590)	(−0.5884980583, −0.3129555160)	(0.0017737757, −0.0005578461)
x_{100}	(0.6477778238, −0.2561583185)	(0.6957057872, 0.0034190954)	(−0.0000252209, −0.0003316770)
x_{500}	(−0.7552263573, −0.0495696962)	(−0.0164948791, −0.7563099763)	(0.0000003362, 0.0000043754)
$x_{1,000}$	(0.2881348085, −0.7258449682)	(0.5011778842, 0.5986188573)	(0.0000006172, 0.0000000803)
$x_{2,000}$	(−0.4030019869, −0.6955930739)	(0.6600714458, 0.4586416551)	(0.0000000277, −0.0000000808)
$x_{3,000}$	(0.6161509491, −0.5361307431)	(−0.5918479368, 0.5627040699)	(−0.0000000121, −0.0000000235)
$x_{4,000}$	(−0.3420238652, 0.7513982050)	(0.5138396999, −0.6460813917)	(−0.0000000099, −0.0000000058)
$x_{5,000}$	(0.1900596641, −0.8102648966)	(−0.5386684962, 0.6343351844)	(−0.0000000060, −0.0000000005)
$x_{10,000}$	(−0.5944221496, 0.6103916596)	(0.7809654309, 0.3405099509)	(−0.0000000002, 0.0000000008)
$x_{15,000}$	(−0.2385405946, −0.8292026034)	(−0.8465824288, 0.1665308994)	(0.0000000001, 0.0000000002)
$x_{20,000}$	(0.2398133400, −0.8364788165)	(−0.6717832259, −0.5530639534)	(0.0000000001, 0.0000000000)
$x_{25,481}$	(−0.2593778379, −0.8368678394)	(−0.8118123305, −0.3294798816)	(0.0000000001, 0.0000000000)
$x_{25,482}$	(−0.8657493245, 0.1345526508)	(−0.4443892718, 0.7550603245)	(0.0000000000, 0.0000000000)
$x_{30,000}$	(0.2167190982, 0.8529483348)	(0.7133385040, 0.5153750955)	(0.0000000000, 0.0000000000)

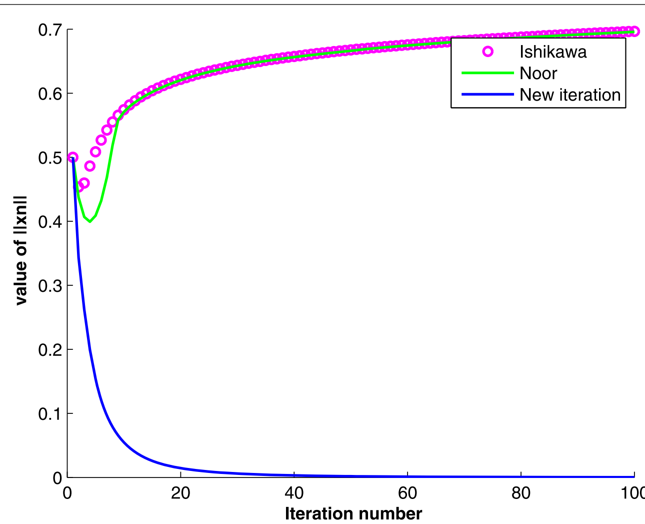

Figure 4 Comparison of first 100 iterates for Case IV.

Table 6 Comparative results, case V

Iterate	Ishikawa	Noor	New iter.
x_1	(0.2000000000, 0.3000000000)	(0.2000000000, 0.3000000000)	(0.2000000000, 0.3000000000)
x_2	(0.3012563133, 0.1744796180)	(0.2851093585, 0.1502591858)	(0.3021399173, –0.0487361054)
x_{50}	(–0.4169844503, –0.5213521942)	(–0.4295257007, –0.5096849930)	(0.0016577246, –0.0000407865)
x_{100}	(0.6946690134, 0.0516541858)	(0.6443413556, 0.2623784481)	(0.0000601568, –0.0002904776)
x_{500}	(–0.6581845589, –0.3736537059)	(0.2664781341, –0.7080015998)	(–0.0000007906, 0.0000038327)
$x_{1,000}$	(0.5755858813, –0.5278006053)	(0.2420603177, 0.7422470011)	(0.0000005092, 0.0000002210)
$x_{2,000}$	(–0.0596880659, –0.8016842650)	(0.4416662024, 0.6715485441)	(0.0000000437, –0.0000000625)
$x_{3,000}$	(0.7881795628, –0.2141287987)	(–0.7588869631, 0.3016793462)	(–0.0000000045, –0.0000000231)
$x_{4,000}$	(–0.6352245439, 0.5273228287)	(0.7175599474, –0.4081177818)	(–0.0000000071, –0.0000000074)
$x_{5,000}$	(0.5240881644, –0.6465164148)	(–0.7362246871, 0.3879666032)	(–0.0000000050, –0.0000000019)
$x_{10,000}$	(–0.8009753063, 0.2904379947)	(0.5978692867, 0.6069648647)	(–0.0000000004, 0.0000000006)
$x_{15,000}$	(0.1465548210, –0.8502942180)	(–0.8476747755, –0.1608782900)	(0.0000000001, 0.0000000002)
$x_{20,000}$	(0.5802924064, –0.6484350172)	(–0.4173549517, –0.7635360394)	(0.0000000001, 0.0000000001)
$x_{23,860}$	(0.8666005213, –0.1175952413)	(0.3460420046, –0.8031500630)	(0.0000000001, 0.0000000000)
$x_{23,861}$	(0.0120033293, –0.8744614434)	(–0.7430451870, –0.4611732164)	(0.0000000000, 0.0000000000)
$x_{25,000}$	(0.2415195695, –0.8417145563)	(–0.5615434561, –0.6719041535)	(0.0000000000, 0.0000000000)

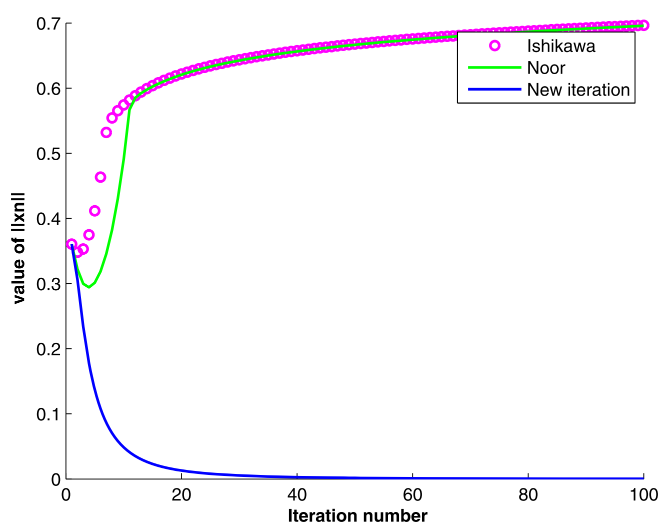

Figure 5 Convergence behavior of first 100 iterates for Case V.

Case V. We use the initial point $x_1 = (0.2, 0.3)$ and obtain Table 6 and Figure 5.

As discussed in Example 1.2, $x = (0, 0)$ with $\|x\| = 0$ is the only fixed point of T. From the tables and the graphs it can be seen that the sequence $\{x_n\}$ generated by the new iteration converging toward 0, while for the Ishikawa and Noor iteration $\|x_n\|$ tends toward 1 in all the above cases.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors contributed equally to this work. All authors read and approved the final manuscript.

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