Lazarev *Boundary Value Problems* (2015) 2015:180
DOI 10.1186/s13661-015-0437-y

Boundary Value Problems
a SpringerOpen Journal

RESEARCH

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Existence of an optimal size of a delaminated rigid inclusion embedded in the Kirchhoff-Love plate

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Abstract

We consider equilibrium problems for an inhomogeneous plate with a crack situated at the inclusion-matrix interface. The matrix of the plate is assumed to be elastic. The boundary condition on the crack curve are given in the form of inequalities and describes the mutual nonpenetration of the crack faces. We analyze the dependence of solutions on the size of rigid inclusion. The existence of the solution to the optimal control problem is proved. For that problem the cost functional characterizes the deviation of the displacement vector from a given function, while the size parameter of rigid inclusion is chosen as the control function.

MSC: Primary 74G55; 49J40; secondary 49J30

Keywords: crack; plate; rigid inclusion; nonpenetration condition; variational inequality

1 Introduction

It is well known that the presence of inclusions as well as of cracks in an elastic body can cause a high stress concentration. The mechanical and geometric properties of inclusions are responsible for crack initiation and propagation. Problems for different models of elastic bodies containing rigid inclusions and cracks with both linear and nonlinear boundary conditions have been under active study; see [1–9]. Using the variational methods, various problems for bodies with rigid inclusions have been successfully formulated and investigated; see for example [1, 10–14]. In particular, a framework for two-dimensional elasticity problems with nonlinear Signorini-type conditions on a part of boundary of a thin delaminated rigid inclusion is proposed in [1]. The three-dimensional case is considered in [15]. Reference [16] is devoted to the analysis of the shapes of cracks and thin rigid inclusions in elastic bodies. The formula for the shape derivative of the energy functional is obtained for the equilibrium problem for an elastic body with a delaminated thin rigid inclusion [12]. For a Kirchhoff-Love plate containing a thin rigid inclusion the cases both with and without delamination of inclusion are considered [17]. In that work, for the plate without delamination of inclusions it is established that by passing to the limit in the equilibrium problems for volume inclusions embedded in an elastic plate as the size of the inclusions tends to zero, we obtain the equilibrium problem for the plate with a thin inclusion.

In this paper, we investigate equilibrium problems for the Kirchhoff-Love plate with a rigid inclusion. We consider volume inclusions defined by three-dimensional domains and thin inclusions defined by cylindrical surfaces. For all cases, we suppose that the crack is situated at the inclusion interface. The present study investigates the effect of varying the inclusion size. We formulate an optimal control problem with the cost functional characterizing the deviation of the displacement vector from a given function. The control functions depend on the size parameter of the rigid inclusion. We prove the existence of an optimal inclusion size.

Additionally, we establish a qualitative connection between the equilibrium problems for the Kirchhoff-Love plate with delaminated thin rigid inclusions and delaminated volume inclusions. In particular, we prove the strong convergence of the solutions for problems with volume inclusions to the solution of the problem for thin inclusion as the size parameter of the volume inclusion tends to zero. It should be noted that we impose boundary conditions of inequality type on the crack. Investigations on mathematical modeling of the crack theory with nonlinear conditions for nonpenetration of the opposite crack faces presented as a system of equalities and inequalities, with relevant bibliography can be found in [16, 18–24]. Other models of deformable solids can be found in [25, 26].

2 Equilibrium problems for an elastic plate containing a rigid inclusion

Let us formulate an equilibrium problem for an elastic plate containing a volume rigid inclusion. We consider the case of the delaminated inclusion, when the crack passes through the inclusion interface. Let $\Omega \subset \mathbf{R}^2$ be a bounded domain with a smooth boundary Γ. Suppose that a smooth unclosed curve γ lies strictly inside Ω, *i.e.* $\bar{\gamma} \subset \Omega$. We require that the curve γ can be extended up to the outer boundary Γ in such a way that Ω is divided into two subdomains Ω_1, Ω_2 with Lipschitz boundaries. The latter condition is sufficient to fulfill the Korn and Poincaré inequalities in the domain $\Omega_\gamma = \Omega \setminus \bar{\gamma}$ [18]. We consider the family of simply connected domains $\omega_t \subset \Omega$, $t \in (0, t_0]$, with the following properties:

(a) the boundaries $\partial\omega_t$ are smooth such that $\partial\omega_t \in C^{1,1}$;

(b) $\omega_t \subset \omega_{t'}$, $\overline{\omega}_{t'} \subset \Omega$ for all $t, t' \in (0, t_0]$, $t < t'$;

(c) for any fixed $\hat{t} \in (0, t_0)$ and any neighborhood $\mathcal{O}$ of the set $\overline{\omega}_t$ there exists $t_{\mathcal{O}} > \hat{t}$ such that $\omega_t \subset \mathcal{O}$ for all $t \in [\hat{t}, t_{\mathcal{O}}]$;

(d) for any neighborhood $\mathcal{O}$ of the curve γ there exists $t_{\mathcal{O}} > 0$ such that $\omega_t \subset \mathcal{O}$ for all $t \in (0, t_{\mathcal{O}}]$;

(e) $\gamma \subset \partial\omega_t$ for all $t \in (0, t_0]$;

(f) $\bigcup_{t<t'} \omega_t = \omega_{t'}$ for all $t' \in (0, t_0]$.

As an example, the family of domains ω_t ($\overline{\omega}_t \subset \Omega$), $t \in (0, t_0]$ with boundaries $\partial\omega_t = \gamma \cup \gamma_t \cup \gamma_t^1 \cup \gamma_t^2$, with

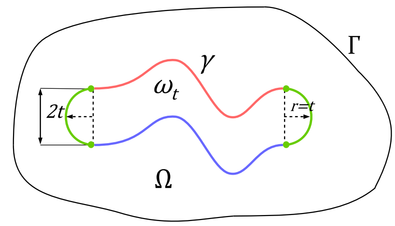
$$\gamma = \{(x_1, x_2) \mid -a < x_1 < 0, x_2 = g(x_1)\},$$

$$\gamma_t = \{(x_1, x_2) \mid -a < x_1 < 0, x_2 = g(x_1) - 2t\}, \quad g \in C^{1,1}[-t_0, 1 + t_0],$$

and with semicircles γ_t^1, γ_t^2 satisfy the properties (a)-(f) (see Figure 1). For this example the domain thickness along the Ox_2 axis is equal to $2t$.

For simplicity, suppose the plate has a uniform thickness $2h = 2$. Let us assign a three-dimensional Cartesian space $\{x_1, x_2, z\}$ with the set $\{\Omega_\gamma\} \times \{0\} \subset \mathbf{R}^3$ corresponding to the middle plane of the plate. The curve γ defines a crack (a cut) in the plate. This means that

Figure 1 Example of the domains ω_t.



the cylindrical surface of the through crack may be defined by the relations $x = (x_1, x_2) \in \gamma$, $-1 \le z \le 1$ where $|z|$ is the distance to the middle plane. For fixed $t \in (0, t_0]$ the volume rigid inclusion is specified by the set $\omega_t \times [-1, 1]$, *i.e.* the boundary of the rigid inclusion is defined by the cylindrical surface $\partial\omega_t \times [-1, 1]$. The elastic part of the plate corresponds to the domain $\Omega\backslash\overline{\omega}_t$. Depending on the direction of the normal $\nu = (\nu_1, \nu_2)$ to γ we will speak about a positive face γ^+ or a negative face γ^- of the curve γ. The jump $[q]$ of the function q on the curve γ is found by the formula $[q] = q|_{\gamma^+} - q|_{\gamma^-}$.

Denote by $\chi = \chi(x) = (W, w)$ the displacement vector of the mid-surface points ($x \in \Omega_\gamma$), by $W = (w_1, w_2)$ the displacements in the plane $\{x_1, x_2\}$, and by w the displacements along the axis z. The strain and integrated stress tensors are denoted by $\varepsilon_{ij} = \varepsilon_{ij}(W)$, $\sigma_{ij} = \sigma_{ij}(W)$, respectively [18]:

$$\varepsilon_{ij}(W) = \frac{1}{2}\left(\frac{\partial w_j}{\partial x_i} + \frac{\partial w_i}{\partial x_j}\right), \qquad \sigma_{ij}(W) = a_{ijkl}\varepsilon_{kl}(W), \quad i,j = 1,2,$$

where $\{a_{ijkl}\}$ is the given elasticity tensor, assumed as usual to be symmetric and positive definite:

$$c_{ijkl} = c_{klij} = c_{jikl}, \quad i,j,k,l = 1,2, c_{ijkl} \in L^\infty(\Omega_\gamma),$$
$$c_{ijkl}\xi_{ij}\xi_{kl} \ge c_0|\xi|^2 \quad \forall\xi, \xi_{ij} = \xi_{ji}, i,j = 1,2, c_0 = \text{const} > 0.$$

The summation convention over repeated indices is used in the sequel. Next we denote the bending moments by the formulas [18]

$$m_{ij}(w) = d_{ijkl}w_{,kl}, \quad i,j = 1,2 \left(w_{,kl} = \frac{\partial^2 w}{\partial x_k\,\partial x_l}\right),$$

where the tensor $\{d_{ijkl}\}$ has the same properties as the tensor $\{a_{ijkl}\}$. Let $B(\cdot,\cdot)$ be a bilinear form defined by the equality

$$B(\chi, \bar{\chi}) = \int_{\Omega_\gamma} \{\sigma_{ij}(W)\varepsilon_{ij}(\bar{W}) + m_{ij}(w)\bar{w}_{,ij}\}\,dx, \tag{1}$$

where $\chi = (W, w)$, $\bar{\chi} = (\bar{W}, \bar{w})$. The potential energy functional of the plate has the following representation [18]:

$$\Pi(\chi) = \frac{1}{2}B(\chi, \chi) - \int_{\Omega_\gamma} F\chi\,dx, \quad \chi = (W, w),$$

where the vector $F = (f_1, f_2, f_3) \in L_2(\Omega)^3$ describes the body forces [18]. Introduce the Sobolev spaces

$$H^{1,0}(\Omega_\gamma) = \left\{ v \in H^1(\Omega_\gamma) \mid v = 0 \text{ on } \Gamma \right\},$$
$$H^{2,0}(\Omega_\gamma) = \left\{ v \in H^2(\Omega_\gamma) \,\middle|\, v = \frac{\partial v}{\partial n} = 0 \text{ on } \Gamma \right\},$$
$$H(\Omega_\gamma) = H^{1,0}(\Omega_\gamma)^2 \times H^{2,0}(\Omega_\gamma).$$

Note that the inequality

$$B(\chi, \chi) \geq c\|\chi\|^2 \quad \forall \chi \in H(\Omega_\gamma) \ \left(\|\chi\| = \|\chi\|_{H(\Omega_\gamma)}\right) \tag{2}$$

with a constant $c > 0$ independent of χ holds for the bilinear form $B(\cdot, \cdot)$ [18].

Remark 1 The inequality (2) yields the equivalence of the standard norm and the seminorm determined by the bilinear form $B(\cdot, \cdot)$ in the space $H(\Omega_\gamma)$.

Due to the presence of a rigid inclusion in the plate, restrictions of the functions describing displacements χ to the domain ω_t satisfy a special kind of relations. We introduce the space which allows us to characterize the properties of the volume rigid inclusion [1]

$$R(\omega_t) = \left\{ \zeta(x) = (\rho, l) \mid \rho(x) = b(x_2, -x_1) + (c_1, c_2); l(x) = a_0 + a_1 x_1 + a_2 x_2, x \in \omega_t \right\}, \tag{3}$$

where $b, c_1, c_2, a_0, a_1, a_2 \in \mathbf{R}$. The condition of mutual nonpenetration of opposite faces of the crack is given by [1, 18]

$$[W]\nu \geq \left| \left[\frac{\partial w}{\partial \nu} \right] \right| \quad \text{on } \gamma \ (W\nu = w_i \nu_i).$$

The variational formulation describing the equilibrium state for the elastic plate with the volume delaminated rigid inclusion can be formulated as follows:

$$\text{find} \quad \xi_t = (U_t, u_t) \in K_t \quad \text{such that} \quad \Pi(\xi_t) = \inf_{\chi \in K_t} \Pi(\chi), \tag{4}$$

where $K_t = \{\chi = (W, w) \in H(\Omega_\gamma) \mid [W]\nu \geq |[\frac{\partial w}{\partial \nu}]| \text{ on } \gamma; \chi|_{\omega_t} \in R(\omega_t)\}$. The problem (4) is known to have a unique solution $\xi_t \in K_t$, which satisfies the variational inequality [27]

$$\xi_t \in K_t, \quad B(\xi_t, \chi - \xi_t) \geq \int_{\Omega_\gamma} F(\chi - \xi_t)\, dx \quad \forall \chi \in K_t. \tag{5}$$

In parallel with the equilibrium problem for a plate with a volume rigid inclusion we consider also the equilibrium problem for an elastic plate with a delaminated thin inclusion. Here we assume that the thin rigid inclusion is described by the cylindrical surface $x = (x_1, x_2) \in \gamma$, $-1 \leq z \leq 1$, while the crack is still located on the positive side (γ^+) of the inclusion's surface. Let us first introduce some notation:

$$R(\gamma) = \left\{ \zeta(x) = (\rho, l) \mid \rho(x) = b(x_2, -x_1) + (c_1, c_2); \right.$$
$$\left. l(x) = a_0 + a_1 x_1 + a_2 x_2, x \in \gamma \right\}, \tag{6}$$

where $b, c_1, c_2, a_0, a_1, a_2 \in \mathbf{R}$;

$$K_0 = \left\{ \chi = (W, w) \in H(\Omega_\gamma) \,\middle|\, [W]\nu \geq \left| \left[\frac{\partial w}{\partial \nu} \right] \right| \text{ on } \gamma; \chi|_{\gamma^-} \in R(\gamma) \right\}.$$

Consider a variational formulation of the problem. We want to find a function $\xi_0 = (U_0, u_0) \in K_0$, such that

$$\Pi(\xi_0) = \inf_{\chi \in K_0} \Pi(\chi). \tag{7}$$

It is well known that the problem (7) has a unique solution which satisfies the variational inequality [28]

$$\xi_0 \in K_0, \quad B(\xi_0, \chi - \xi_0) \geq \int_{\Omega_\gamma} F(\chi - \xi_0)\, dx \quad \forall \chi \in K_0. \tag{8}$$

3 An optimal control problem

Consider the cost functional

$$J(t) = \| \xi_t - \xi^* \|_{H(\Omega_\gamma)}, \quad t \in [0, t_0],$$

where ξ^* is a prescribed element, ξ_t is a solution of the problem (4) for $t > 0$, and ξ_0 is a solution of the problem (7). We have to find a solution of the maximization problem

$$\sup_{t \in [0, t_0]} J(t). \tag{9}$$

The following assertion holds.

Theorem 1 *There exists a solution of the optimal control problem* (9).

Proof Let $\{t_n\}$ be a maximizing sequence. By the boundedness of the segment $[0, t_0]$, we can extract a convergent subsequence $\{t_{n_k}\} \subset \{t_n\}$ such that

$$t_{n_k} \to t^* \quad \text{as } k \to \infty, t^* \in [0, t_0].$$

Without loss of generality we assume that $t_{n_k} \neq t^*$ for sufficiently large k. Otherwise there would exist a sequence $\{t_{n_l}\}$ such that $t_{n_l} \equiv t^*$, and therefore $J(t^*)$ is a solution of (9). Consider the case of the subsequence $\{t_{n_k}\}$ satisfying $t_{n_k} \neq t^*$ for sufficiently large k. Now we take into account Lemma 2, proved below: the solutions ξ_k of (4) corresponding to the parameters t_{n_k} converge to the solution ξ_{t^*} strongly in $H(\Omega_\gamma)$ as $k \to \infty$. This allows us to obtain convergence,

$$J(t_{n_k}) \to J(t^*).$$

This means that

$$J(t^*) = \sup_{t \in [0, t_0]} J(t).$$

The theorem is proved. □

Before proceeding we first prove the following lemma.

Lemma 1 *Let $t^* \in [0, t_0]$ be a fixed real number and let $\{t_n\} \subset [t^*, t_0]$ be a sequence of real numbers converging to t^* as $n \to \infty$. Then for an arbitrary function $\eta = (V, v) \in K_{t^*}$ there exist a subsequence $\{t_k\} = \{t_{n_k}\} \subset \{t_n\}$ and a sequence of functions $\{\eta_k\}$ such that $\eta_k = (V_k, v_k) \in K_{t_k}$, $k \in \mathbf{N}$ and $\eta_k \to \eta$ weakly in $H(\Omega_\gamma)$ as $k \to \infty$.*

Proof If there exists a subsequence $\{t_{n_k}\}$ such that $t_{n_k} = t^*$, then the assertion of this lemma holds for the sequence $\eta_k \equiv \eta$, $k \in \mathbf{N}$. Therefore, below we assume that $t_n > t^*$ for sufficiently large n. Denote by $\zeta^* = (\rho^*, l^*)$ with $\rho^* = b^*(x_2, -x_1) + (c_1^*, c_2^*)$, $l^*(x) = a_0^* + a_1^* x_1 + a_2^* x_2$ the function describing the structure of η in ω_{t^*} for the case $t^* > 0$. If $t^* = 0$ and the function η has the specified structure on γ^-, then we adopt the same notation, *i.e.* $\zeta^* = \eta$ on γ^-. We extend the definition of $\zeta^* = (\rho^*, l^*)$ to the whole domain Ω by the equalities

$$\zeta^* = (\rho^*, l^*), \quad \text{where } \begin{cases} \rho^* = b^*(x_2, -x_1) + (c_1^*, c_2^*), & x \in \Omega, \\ l^*(x) = a_0^* + a_1^* x_1 + a_2^* x_2, & x \in \Omega. \end{cases}$$

Fix an arbitrary value $t \in (0, t_0]$ and consider the following family of auxiliary problems:

$$\text{find} \quad \eta_t \in K_t' \quad \text{such that} \quad p(\eta_t) = \inf_{\chi \in K_t'} p(\chi), \tag{10}$$

where $p(\chi) = B(\chi - \eta, \chi - \eta)$,

$$K_t' = \left\{ \chi = (W, w) \in H(\Omega_\gamma) \;\middle|\; \chi = \eta, \frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \text{ on } \gamma^\pm, \chi|_{\omega_t} = \zeta^* \right\}.$$

It is easy to see that the functional $p(\chi)$ is coercive and weakly lower semicontinuous on the space $H(\Omega_\gamma)$. It can be verified that the set K_t' is convex and closed in $H(\Omega_\gamma)$. These properties guarantee the existence of solution of the problem (10). Besides, the solution η_t is unique [18, 29].

Since the functional $p(\chi)$ is convex and differentiable on $H(\Omega_\gamma)$, the problem (10) can be written in the equivalent form

$$\eta_t \in K_t', \quad B(\eta_t - \eta, \chi - \eta_t) \geq 0 \quad \forall \chi \in K_t'. \tag{11}$$

By the property (b), it is evident that the solution η_{t_0} of (11) for $t = t_0$ belongs to the set K_t' with $t' \in (0, t_0]$. Substituting η_{t_0} as the test functions into (11), we get

$$B(\eta_t - \eta, \eta_{t_0}) + B(\eta, \eta_t) \geq B(\eta_t, \eta_t) \quad \forall t \in (0, t_0].$$

Using the inequality (2) we obtain from this relation the following uniform upper bound:

$$\|\eta_t\| \leq c \quad \forall t \in (0, t_0].$$

Therefore, we can extract from the sequence $\{\eta_{t_n}\}$ a subsequence $\{\eta_k\}$, which is defined by equalities $\eta_k = \eta_{t_{n_k}}$, $k \in \mathbf{N}$ (henceforth we define a sequence $\{t_k\}$ by the equality $t_k = t_{n_k}$) and $\{\eta_k\}$ weakly converges to some function $\tilde{\eta}$ in $H(\Omega_\gamma)$.

Show that $\tilde{\eta} = \eta$. To this end we must distinguish two cases for t^*, namely $t^* > 0$ and $t^* = 0$. Let us first assume that $t^* > 0$. Then by construction $(\eta_k - \eta) \in H_0^1(\Omega_\gamma \backslash \overline{\omega}_{t^*})^2 \times H_0^2(\Omega_\gamma \backslash \overline{\omega}_{t^*})$. Consequently, in virtue of the weak closeness of $H_0^1(\Omega_\gamma \backslash \overline{\omega}_{t^*})^2 \times H_0^2(\Omega_\gamma \backslash \overline{\omega}_{t^*})$ we have $(\tilde{\eta} - \eta) \in H_0^1(\Omega_\gamma \backslash \overline{\omega}_{t^*})^2 \times H_0^2(\Omega_\gamma \backslash \overline{\omega}_{t^*})$. We consider now the functions of the form $\chi_k^\pm = \eta_k \pm \phi$, where ϕ is the function $\overline{\phi} \in C_0^\infty(\Omega_\gamma \backslash \overline{\omega}_{t^*})^3$ extended by zero to the whole domain Ω_γ. Bearing in mind the property (c), observe that $\chi_k^\pm \in K'_{t_k}$ for sufficiently large k. We next substitute the elements of these sequences $\{\chi_k^+\}$ and $\{\chi_k^-\}$ as test functions into inequalities (11) corresponding to t_k. As a result, we obtain

$$\eta_k \in K'_{t_k}, \quad B(\eta_k - \eta, \phi) = 0. \tag{12}$$

Fix the function ϕ. Passing to the limit in (12), we deduce

$$B(\tilde{\eta} - \eta, \phi) = 0 \quad \forall \phi \in C_0^\infty(\Omega_\gamma \backslash \overline{\omega}_{t^*})^3.$$

Hence, by the density of $C_0^\infty(\Omega_\gamma \backslash \overline{\omega}_{t^*})$ both in $H_0^1(\Omega_\gamma \backslash \overline{\omega}_{t^*})$ and $H_0^2(\Omega_\gamma \backslash \overline{\omega}_{t^*})$ (see [30]), we infer that $\tilde{\eta} - \eta = 0$ in $H_0^1(\Omega_\gamma \backslash \overline{\omega}_{t^*})^2 \times H_0^2(\Omega_\gamma \backslash \overline{\omega}_{t^*})$. Finally, by construction, the equality $\tilde{\eta} = \eta$ is fulfilled in ω_{t^*}. Therefore, $\tilde{\eta} = \eta$ in $H(\Omega_\gamma)$ and there is a sequence $\{\eta_k\}$ such that $\eta_k \in K_{t_k}$, $k \in \mathbf{N}$, and $\eta_k \to \eta$ weakly in $H(\Omega_\gamma)$ as $n \to \infty$.

Let us consider the second case. Suppose that $t^* = 0$. By construction, we have $(\eta_k - \eta) \in H_0^1(\Omega_\gamma)^2 \times H_0^2(\Omega_\gamma)$, and consequently, the relation $(\tilde{\eta} - \eta) \in H_0^1(\Omega_\gamma)^2 \times H_0^2(\Omega_\gamma)$ is fulfilled. We now consider functions of the form $\chi_k^\pm = \eta_k \pm \phi$, where $\phi \in C_0^\infty(\Omega_\gamma)^3$. Observe that the property (d) yields for sufficiently large k the inclusion $\chi_k^\pm \in K'_{t_k}$. Substituting these functions in (11) corresponding to t_k, yields the equality

$$\eta_k \in K'_{t_k}, \quad B(\eta_k - \eta, \phi) = 0. \tag{13}$$

We fix the function ϕ in (13) and pass to the limit as $k \to \infty$. As a result, we get

$$B(\tilde{\eta} - \eta, \phi) = 0 \quad \forall \phi \in C_0^\infty(\Omega_\gamma)^3. \tag{14}$$

The density of $C_0^\infty(\Omega_\gamma)$ in $H_0^n(\Omega_\gamma)$ ($n \in \mathbf{N}$), see [30], allows us to obtain from (14) the equality $\tilde{\eta} - \eta = 0$ in $H_0^1(\Omega_\gamma)^2 \times H_0^2(\Omega_\gamma)$. It remains to observe that $\tilde{\eta} = \eta$ on $\gamma^\pm$ by the construction. Thus, $\tilde{\eta} = \eta$ in $H(\Omega_\gamma)$ and there exists a subsequence of functions $\{\eta_k\}$ such that $\eta_k \in K_{t_k}$ and $\eta_k \to \eta$ weakly in $H(\Omega_\gamma)$. The lemma is proved. □

Now we can prove the following statement.

Lemma 2 *Let $t^* \in [0, t_0]$ be a fixed real number. Then $\xi_t \to \xi_{t^*}$ strongly in $H(\Omega_\gamma)$ as $t \to t^*$, where ξ_t is the solution of* (4) *corresponding to $t \in (0, t_0]$, while ξ_{t^*} is the solution corresponding to* (4) *for $t^* > 0$ and to the problem* (7) *for $t^* = 0$.*

Proof We will prove it by contradiction. Let us assume that there exist a number $\epsilon_0 > 0$ and a sequence $\{t_n\} \subset (0, t_0]$ such that $t_n \to t^*$, $\|\xi_n - \xi_{t^*}\| \geq \epsilon_0$, where $\xi_n = \xi_{t_n}$, $n \in \mathbf{N}$ are the solutions of (4) corresponding to t_n.

Since $\chi^0 \equiv 0 \in K_t$ for all $t \in [0, t_0]$, we can substitute $\chi = \chi^0$ in (5) for all $t \in (0, t_0]$ and in (8) for $t = 0$. This provides

$$\xi_t \in K_t, \quad B(\xi_t, \xi_t) \leq \int_{\Omega_\gamma} F \xi_t \, dx \quad \forall t \in [0, t_0].$$

From this, using (2) we can deduce that for all $t \in [0, t_0]$ the following estimate holds:

$$\|\xi_t\| \le c,$$

with some constant $c > 0$ independent of t. Consequently, replacing ξ_n with a subsequence if necessary, we can assume that ξ_n converges to some $\tilde{\xi}$ weakly in $H(\Omega_\gamma)$.

Now we show that $\tilde{\xi} \in K_{t^*}$. Indeed, we have $\xi_n|_{\omega_{t_n}} = \zeta_n \in R(\omega_{t_n})$. In accordance with the Sobolev embedding theorem [31], we obtain

$$\xi_n|_{\omega_{t^*}} \to \tilde{\xi}|_{\omega_{t^*}} \quad \text{strongly in } L_2(\omega_{t^*})^3 \text{ as } n \to \infty, \tag{15}$$

$$U_n|_\gamma \to \tilde{U}|_\gamma \quad \text{strongly in } L_2(\gamma)^2 \text{ as } n \to \infty, \tag{16}$$

$$\left.\frac{\partial u_n}{\partial \nu}\right|_\gamma \to \left.\frac{\partial \tilde{u}}{\partial \nu}\right|_\gamma \quad \text{strongly in } L_2(\gamma) \text{ as } n \to \infty. \tag{17}$$

Choosing a subsequence, if necessary, we assume as $n \to \infty$ that $\xi_n \to \tilde{\xi}$ a.e. in ω_{t^*}. This allows us to conclude that each of the numerical sequences $\{b_n\}$, $\{c_{1n}\}$, $\{c_{2n}\}$, $\{a_{0n}\}$, $\{a_{1n}\}$, $\{a_{2n}\}$ defining the structure of ζ_n in domains ω_{t_n} is bounded in absolute value. Thus, we can extract subsequences (retaining notation) such that

$$b_n \to b, \qquad a_{0n} \to a_0, \qquad c_{in} \to c_i, \qquad a_{in} \to a_i, \quad i = 1, 2 \text{ as } n \to \infty.$$

Further we must distinguish two different cases: $t^* = 0$ and $t^* > 0$. In the first case for the sequence $\{\xi_n\}$ corresponding to the specified convergent number sequences $\{b_n\}$, $\{c_{1n}\}$, $\{c_{2n}\}$, $\{a_{0n}\}$, $\{a_{1n}\}$, $\{a_{2n}\}$ we have

$$U_n|_{\gamma^-} \to b(x_2, -x_1) + (c_1, c_2) \quad \text{strongly in } L_2(\gamma)^2 \text{ as } n \to \infty,$$

$$u_n|_{\gamma^-} \to a_0 + a_1 x_1 + a_2 x_2 \quad \text{strongly in } L_2(\gamma) \text{ as } n \to \infty.$$

The last two relations with (15) lead to the equality

$$\tilde{U}|_{\gamma^-} = b(x_2, -x_1) + (c_1, c_2), \qquad \tilde{u}|_{\gamma^-} = a_0 + a_1 x_1 + a_2 x_2 \quad \text{a.e. on } \gamma.$$

This means $\tilde{\xi}|_{\gamma^-} \in R(\gamma)$.

Consider the second case. If there exists a subsequence $\{t_k\} \subset \{t_n\}$ such that $t_k \ge t^*$ for all $k \in \mathbf{N}$, then we can easily obtain the following convergences:

$$U_k|_{\omega_{t^*}} \to b(x_2, -x_1) + (c_1, c_2) \quad \text{strongly in } L_2(\omega_{t^*})^2 \text{ as } k \to \infty, \tag{18}$$

$$u_k|_{\omega_{t^*}} \to a_0 + a_1 x_1 + a_2 x_2 \quad \text{strongly in } L_2(\omega_{t^*}) \text{ as } k \to \infty. \tag{19}$$

Therefore, from (18), (19), and (15) we obtain $\tilde{\xi}|_{\omega_{t^*}} \in R(\omega_{t^*})$.

Suppose that there exists a subsequence $\{t_k\} \subset \{t_n\}$ that satisfies $t_k < t^*$ for all $k \in \mathbf{N}$ and $t_k \to t^*$ as $k \to \infty$. In this case for an arbitrary fixed $k' \in \mathbf{N}$ and the corresponding value $t' = t_{k'}$, by the property (b) we have

$$U_k \to b(x_2, -x_1) + (c_1, c_2) \quad \text{strongly in } L_2(\omega_{t'})^2 \text{ as } k \to \infty,$$

$$u_k \to a_0 + a_1 x_1 + a_2 x_2 \quad \text{strongly in } L_2(\omega_{t'}) \text{ as } k \to \infty.$$

It is possible to define a function $l = a_0 + a_1x_1 + a_2x_2$ in ω_{t^*}. In view of the absolute continuity of the Lebesgue integral and the properties (b) and (f), for any $\epsilon > 0$ we can choose a number $k' \in \mathbf{N}$ large enough such that

$$\|l\|_{L^2(\omega_{t^*}\setminus\omega_{t'})} < \sqrt{\epsilon}, \qquad \|\tilde{u}\|_{L^2(\omega_{t^*}\setminus\omega_{t'})} < \sqrt{\epsilon}.$$

Further, using the triangle inequality, it follows from this that

$$\begin{aligned}\|u_k - l\|_{L^2(\omega_{t^*}\setminus\omega_{t'})} &\le \|u_k\|_{L^2(\omega_{t^*}\setminus\omega_{t'})} + \|l\|_{L^2(\omega_{t^*}\setminus\omega_{t'})}\\ &\le \|\tilde{u}\|_{L^2(\omega_{t^*}\setminus\omega_{t'})} + \|u_k - \tilde{u}\|_{L^2(\omega_{t^*}\setminus\omega_{t'})} + \|l\|_{L^2(\omega_{t^*}\setminus\omega_{t'})}\\ &< 2\sqrt{\epsilon} + \|u_k - \tilde{u}\|_{L^2(\omega_{t^*})}.\end{aligned}$$

Therefore we have

$$\begin{aligned}\|u_k - l\|^2_{L^2(\omega_{t^*})} &= \|u_k - l\|^2_{L^2(\omega_{t^*}\setminus\omega_{t'})} + \|u_k - l\|^2_{L^2(\omega_{t'})}\\ &< \bigl(2\sqrt{\epsilon} + \|u_k - \tilde{u}\|_{L^2(\omega_{t^*})}\bigr)^2 + \|u_k - l\|^2_{L^2(\omega_{t'})}.\end{aligned} \tag{20}$$

We can see that for all sufficiently large numbers k we have the following estimates:

$$\|u_k - \tilde{u}\|_{L^2(\omega_{t^*})} < \sqrt{\epsilon}, \qquad \|u_k - l\|_{L^2(\omega_{t'})} < \sqrt{\epsilon}$$

and the right-hand side of (20) is less than 10ϵ. Therefore, $u_k \to l$ strongly in $L^2(\omega_{t^*})$. Consequently, taking into account (15) we get $\tilde{u}|_{\omega_{t^*}} = l$ in ω_{t^*}.

It can be proved analogously that $\tilde{U}|_{\omega_{t^*}} = b(x_2, -x_1) + (c_1, c_2)$ a.e. in ω_{t^*}. Thus, we conclude that $\tilde{\xi}|_{\omega_{t^*}} \in R(\omega_{t^*})$. Therefore, in all possible cases we have $\tilde{\xi}|_{\omega_{t^*}} \in R(\omega_{t^*})$.

It remains to show that $\tilde{\xi}$ satisfies the inequality $[\tilde{U}\nu] \ge |[\frac{\partial\tilde{u}}{\partial\nu}]|$ on γ. In view of (16), (17), we can extract subsequences once again and obtain the convergences $U_n|_\gamma \to \tilde{U}|_\gamma$, $\frac{\partial u_n}{\partial\nu}|_\gamma \to \frac{\partial\tilde{u}}{\partial\nu}|_\gamma$ a.e. on both γ^+ and γ^-. Now we pass to the limit in the following inequalities as $n \to \infty$:

$$[U_n]\nu \ge \left|\left[\frac{\partial u_n}{\partial\nu}\right]\right| \quad \text{on } \gamma.$$

This leads to $[\tilde{U}]\nu \ge |[\frac{\partial\tilde{u}}{\partial\nu}]|$ on γ and $\tilde{\xi} \in K_{t^*}$.

Observe that, as $t_n \to t^*$, there must exist either a subsequence $\{t_{n_l}\}$ such that $t_{n_l} \le t^*$ for all $l \in \mathbf{N}$ or, if that is not the case, a subsequence $\{t_{n_m}\}$, $t_{n_m} > t^*$ for all $m \in \mathbf{N}$.

For the first case we have the subsequence $\{t_{n_l}\} \subset (0, t_0]$ with the property $t_{n_l} \le t^*$ for all $l \in \mathbf{N}$. This implies that $t^* > 0$. For convenience, we denote this subsequence by $\{t_n\}$. Since $t_n \le t^*$, by the property (b) the arbitrary test function $\chi \in K_{t^*}$ also belongs to the set K_{t_n}. This property allows us to pass to the limit as $n \to \infty$ in the following inequalities with the test functions $\chi \in K_{t^*}$:

$$\xi_n \in K_{t_n}, \quad B(\xi_n, \chi - \xi_n) \ge \int_{\Omega_\gamma} F(\chi - \xi_n)\,dx, \quad t_n \in \bigl(0, t^*\bigr].$$

Taking into account the weak convergence of ξ_n to $\tilde{\xi}$, the variational inequality in the limit takes the form

$$B(\tilde{\xi}, \chi - \tilde{\xi}) \geq \int_{\Omega_\gamma} F(\chi - \tilde{\xi})\, dx \quad \forall \chi \in K_{t^*}.$$

This means that $\tilde{\xi} = \xi_{t^*}$. To complete the proof for the first case we must establish the strong convergence $\xi_n \to \xi_{t^*}$. By substituting $\chi = 2\xi_t$ and $\chi = 0$ into the variational inequalities (5) for $t \in (0, t_0]$, we get

$$\xi_t \in K_t, \quad B(\xi_t, \xi_t) = \int_{\Omega_\gamma} F\xi_t\, dx \quad \forall t \in (0, t_0]. \tag{21}$$

In view of (5) this means that the relation

$$\xi_t \in K_t, \quad B(\xi_t, \chi) \geq \int_{\Omega_\gamma} F\chi\, dx \quad \forall \chi \in K_t \tag{22}$$

holds for all $t \in (0, t_0]$. Hence, by the weak convergence $\xi_n \to \xi_{t^*}$ in $H(\Omega_\gamma)$ as $n \to \infty$, we deduce

$$\lim_{n\to\infty} B(\xi_n, \xi_n) = \lim_{n\to\infty} \int_{\Omega_\gamma} F\xi_n\, dx = \int_{\Omega_\gamma} F\xi_{t^*}\, dx = B(\xi_{t^*}, \xi_{t^*}).$$

Since we have the equivalence of norms (see Remark 1), one can see that $\xi_n \to \xi_{t^*}$ strongly in $H(\Omega_\gamma)$ as $n \to \infty$. Thus in the first case we get a contradiction to the assumption: $\|\xi_n - \xi_{t^*}\| \geq \epsilon$ for all $n \in \mathbf{N}$.

Consider the second case, *i.e.* we suppose that elements of the subsequence $\{t_{n_m}\}$ satisfy $t_{n_m} > t^*$ for all $m \in \mathbf{N}$. For convenience we keep the same notation for the subsequence. In doing so we have $t_n \to t^*$ and $t_n > t^*$. Taking into account the results at the beginning of the proof, we have $\xi_n \to \tilde{\xi}$ weakly in $H(\Omega_\gamma)$ as $n \to \infty$. For instance we will prove that $\xi_n \to \tilde{\xi}$ strongly in $H(\Omega_\gamma)$ as $n \to \infty$. In view of the weak convergence $\xi_n \to \tilde{\xi}$ in $H(\Omega_\gamma)$ as $n \to \infty$, from (21) we deduce

$$\lim_{n\to\infty} B(\xi_n, \xi_n) = \int_{\Omega_\gamma} F\tilde{\xi}\, dx. \tag{23}$$

Next, substituting $\chi = \xi_{t'} \in K_{t'} \subset K_t$, for arbitrary fixed numbers $t, t' \in (0, t_0]$ such that $t' \geq t$, in (22) as the test function, we arrive at the inequality

$$B(\xi_t, \xi_{t'}) \geq \int_{\Omega_\gamma} F\xi_{t'}\, dx.$$

Therefore, we conclude that for all t_n and t_m satisfying $t_n \leq t_m$ the following inequality is fulfilled:

$$B(\xi_n, \xi_m) \geq \int_{\Omega_\gamma} F\xi_m\, dx. \tag{24}$$

Fix an arbitrary value m in (24) and pass to the limit in the last relation as $n \to \infty$. As a result we have

$$B(\tilde{\xi}, \xi_m) \geq \int_{\Omega_\gamma} F\xi_m \, dx. \tag{25}$$

Passing to the limit in (25) as $m \to \infty$, we find

$$B(\tilde{\xi}, \tilde{\xi}) \geq \int_{\Omega_\gamma} F\tilde{\xi} \, dx.$$

This inequality, (23), and the weak lower semicontinuity of the bilinear form $B(\cdot,\cdot)$ yield the following chain of relations:

$$B(\tilde{\xi}, \tilde{\xi}) \geq \int_{\Omega_\gamma} F\tilde{\xi} \, dx = \lim_{n\to\infty} B(\xi_n, \xi_n) \geq B(\tilde{\xi}, \tilde{\xi}).$$

This means that

$$B(\tilde{\xi}, \tilde{\xi}) = \lim_{n\to\infty} B(\xi_n, \xi_n).$$

Again, by the equivalence of norms (see Remark 1), we deduce that $\xi_n \to \tilde{\xi}$ strongly in $H(\Omega_\gamma)$ as $n \to \infty$.

From Lemma 1, for any $\eta \in K_{t^*}$ there exist a subsequence $\{t_k\} = \{t_{n_k}\} \subset \{t_n\}$ and a sequence of functions $\{\eta_k\}$ such that $\eta_k \in K_{t_k}$ and $\eta_k \to \eta$ weakly in $H(\Omega_\gamma)$ as $k \to \infty$.

The properties established above for the convergent sequences $\{\eta_k\}$ and $\{\xi_n\}$ allow us to pass to the limit as $k \to \infty$ in the following inequalities derived from (5) for t_k and with test functions η_k:

$$B(\xi_k, \eta_k - \xi_k) \geq \int_{\Omega_\gamma} F(\eta_k - \xi_k) \, dx.$$

As a result, we have

$$B(\tilde{\xi}, \eta - \tilde{\xi}) \geq \int_{\Omega_\gamma} F(\eta - \tilde{\xi}) \, dx \quad \forall \eta \in K_{t^*}.$$

The unique solvability of this variational inequality implies that $\tilde{\xi} = \xi_{t^*}$. Therefore, in either case there exists a subsequence $\{t_{n_k}\} \subset \{t_n\}$ such that $t_k \to t^*$, $\xi_k \to \xi_{t^*}$ strongly in $H(\Omega_\gamma)$, which is a contradiction. The lemma is proved. □

4 Conclusion

The existence of the solution to the optimal control problem (9) is proved. For that problem the cost functional $J(t)$ characterizes the deviation of the displacement vector from a given function ξ^*, while the size parameter t of the rigid inclusion is chosen as the control function.

Lemmas 1 and 2 establish a qualitative connection between the equilibrium problems for plates with rigid delaminated inclusions of varying thickness. In particular it is shown that as the thickness of volume rigid inclusion tends to zero, the solutions of the equilibrium problems converge to the solution of the equilibrium problem for a plate containing a thin rigid delaminated inclusion.

The obtained results can be used to investigate some mathematical and mechanical problems concerning inhomogeneous solids with rigid inclusions.

Competing interests
The author declares to have no competing interests.

Received: 26 June 2015 Accepted: 10 September 2015 Published online: 06 October 2015

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