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Convergence rates in the law of large numbers for long-range dependent linear processes

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Abstract
Baum and Katz (Trans. Am. Math. Soc. 120:108-123, 1965) obtained convergence rates in the Marcinkiewicz-Zygmund law of large numbers. Their result has already been extended to the short-range dependent linear processes by many authors. In this paper, we extend the result of Baum and Katz to the long-range dependent linear processes. As a corollary, we obtain convergence rates in the Marcinkiewicz-Zygmund law of large numbers for short-range dependent linear processes.

MSC: 60F15

Keywords: linear process; convergence rate; Marcinkiewicz-Zygmund law of large numbers

1 Introduction

There are many literature works concerning the convergence rates in the Marcinkiewicz-Zygmund law of large numbers. One can refer to Alf [2], Alsmeyer [3], Baum and Katz [1], Heyde and Rohatgi [4], Hu and Weber [5], Rohatgi [6], and so on.

Baum and Katz [1] obtained the following convergence rates in the Marcinkiewicz-Zygmund law of large numbers.

Theorem 1.1 (Baum and Katz [1]) *Let $r \ge 1$, $1 \le p < 2$ and $\{X, X_n, n \ge 1\}$ be a sequence of independent and identically distributed (i.i.d.) random variables. Then $EX = 0$ and $E|X|^{rp} < \infty$ imply*

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{k=1}^{n} X_k\right| > n^{1/p}\varepsilon\right) < \infty \quad \textit{for all } \varepsilon > 0.$$

When $r = 2$, the cases of $p = 1$ and $1 \le p < 2$ have already been proved by Hsu and Robbins [7] and Katz [8], respectively.

Let $\{\zeta_i, i \in \mathbb{Z}\}$ be a sequence of i.i.d. random variables and $\{a_i, i \in \mathbb{Z}\}$ be a sequence of real numbers. Here and in the following, $\mathbb{Z}$ denotes the set of all integers. Then $\{X_n, n \ge 1\}$ is called a linear process or an infinite order moving average process if X_n is defined by

$$X_n = \sum_{i=-\infty}^{\infty} a_{i+n}\zeta_i \quad \text{for } n \geq 1. \tag{1.1}$$

If $\sum_{i=-\infty}^{\infty} |a_i| < \infty$, then $\{X_n, n \geq 1\}$ has short memory or is short-range dependent. If $\sum_{i=-\infty}^{\infty} |a_i| = \infty$, then $\{X_n, n \geq 1\}$ has long memory or is long-range dependent (see Chapter 3 in Giraitis et al. [9]).

In the short-range dependent case, Koopmans [10] showed that if ζ_0 has the moment generating function, then the strong law of large numbers for the linear process holds with exponential convergence rate. Hanson and Koopmans [11] generalized this result to a class of linear processes of independent but non-identically distributed random variables $\{\zeta_i, i \in \mathbb{Z}\}$ and to arbitrary subsequences of $\{X_n, n \geq 1\}$. Li et al. [12] extended Katz [8] theorem to the setting of short-range dependent linear processes.

Theorem 1.2 (Li et al. [12]) *Let $1 \leq p < 2$. Let $\{a_i, i \in \mathbb{Z}\}$ be an absolutely summable sequence of real numbers. Suppose that $\{X_n, n \geq 1\}$ is the linear process of a sequence $\{\zeta_i, i \in \mathbb{Z}\}$ of i.i.d. random variables with mean zero and $E|\zeta_0|^{2p} < \infty$. Then*

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{k=1}^{n} X_k\right| > n^{1/p}\varepsilon\right) < \infty \quad \textit{for all } \varepsilon > 0.$$

Note that Theorem 1.2 corresponds to Theorem 1.1 with $r = 2$. Zhang [13] extended Theorem 1.1 with $r > 1$ to the short-range dependent linear process of a sequence of identically distributed φ-mixing random variables. Since independent random variables are also φ-mixing, it follows by Zhang [13] theorem that Theorem 1.2 also holds for $r > 1$.

In this paper, we obtain convergence rates in the Marcinkiewicz-Zygmund law of large numbers for long-range dependent linear processes of i.i.d. random variables. For convenience of notation, let

$$W_n(t) = \left(\sum_{i=-\infty}^{\infty} |\omega_{ni}|^t\right)^{1/t} \quad \text{for } n \geq 1 \text{ and } t > 0,$$

where $\omega_{ni} = \sum_{k=1}^{n} a_{i+k}$. In the long-range dependent case, Characiejus and Račkauskas [14] obtained the convergence rate in the Marcinkiewicz-Zygmund law of large numbers for the linear process $\{Y_n, n \geq 1\}$ which is slightly different from (1.1) and defined by

$$Y_n = \sum_{i=0}^{\infty} a_i \zeta_{n-i} \quad \text{for } n \geq 1, \tag{1.2}$$

where $a_i = 0$ if $i < 0$.

Theorem 1.3 (Characiejus and Račkauskas [14]) *Let $\{Y_n, n \geq 1\}$ be defined as above and $1 < p < 2$. Let $\{a_i, i \in \mathbb{Z}\}$ be a sequence of real numbers such that*

$$\sum_{i=-\infty}^{\infty} |a_i|^p < \infty,$$

where $a_i = 0$ *if* $i < 0$. *Assume that*

$$W_n(q)/W_n(p) = O\bigl(n^{1/q-1/p}\bigr) \quad \textit{for some } q \in (p,2].$$

If $E\zeta_0 = 0$ *and* $E[|\zeta_0|^p \log(1 + |\zeta_0|)] < \infty$, *then*

$$\sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{k=1}^{n} Y_k\right| > W_n(p)\varepsilon\right) < \infty \quad \textit{for all } \varepsilon > 0. \tag{1.3}$$

The above theorem shows a convergence rate in the Marcinkiewicz-Zygmund weak law of large numbers with the norming sequence $W_n(p)$.

We now compare Theorem 1.3 with Theorem 1.1. Since Theorem 1.3 deals with only the case $r = 1$, it is interesting to prove that Theorem 1.3 holds for the case $r > 1$. When $r = 1$, Theorem 1.1 requires a finite pth moment condition, but Theorem 1.3 requires more than finite pth moment. To apply Theorem 1.3, it is necessary to estimate $W_n(p)$. If $\{a_i, i \in \mathbb{Z}\}$ is an absolutely summable sequence, then we have, by the result of Burton and Dehling [15] (see also Lemma 2.4), that for any $t > 0$

$$\frac{1}{n} W_n^t(t) \to \sum_{i=-\infty}^{\infty} a_i,$$

and hence (1.3) holds with $W_n(p)$ replaced by $n^{1/p}$. However, for the long-range dependent case, it is not easy to estimate $W_n(t)$.

In this paper, we extend Theorem 1.1 to the long-range dependent linear processes. As a corollary, we obtain a long-range dependent setting of Theorem 1.2. Further, we propose a method to estimate $W_n(t)$ for the long-range dependent case.

Throughout this paper, C denotes a positive constant which may vary at each occurrence. For events A and B, $I(A)$ denotes the indicator function of the event A, and $I(A, B) = I(A \cap B)$.

2 Convergence of long-range dependent linear processes

In this section, we extend Theorem 1.1 to the long-range dependent linear processes. To prove the main results, we need the following lemmas. The first one is the von Bahr-Esseen inequality (see von Bahr and Esseen [16]). The second is known as Fuk-Nagaev inequality (see Corollary 1.8 in Nagaev [17]).

Lemma 2.1 *Let* $\{\zeta_i, i \geq 1\}$ *be a sequence of independent random variables with* $E\zeta_i = 0$ *and* $E|\zeta_i|^t < \infty$ *for some* $1 \leq t \leq 2$. *Then, for all* $n \geq 1$,

$$E\left|\sum_{i=1}^{n} \zeta_i\right|^t \leq C_t \sum_{i=1}^{n} E|\zeta_i|^t,$$

where $C_t > 0$ *is a positive constant depending only on* t.

Lemma 2.2 *Let $\{\zeta_i, i \geq 1\}$ be a sequence of independent random variables with $E\zeta_i = 0$. Then, for any $t \geq 2$ and $x > 0$,*

$$P\left(\left|\sum_{i=1}^{n} \zeta_i\right| > x\right) \leq (1 + 2/t)^t x^{-t} \sum_{i=1}^{n} E|\zeta_i|^t + 2\exp\left\{-\frac{2x^2}{(t+2)^2 e^t \sum_{i=1}^{n} \operatorname{Var}(\zeta_i)}\right\}.$$

The following lemma is well known and can be easily proved by using a standard method.

Lemma 2.3 *Let $p > 0$ and ζ be a random variable. Then the following statements hold.*

(i) *If $0 < \theta < p$, then $\sum_{n=1}^{\infty} n^{-\theta/p} E|\zeta|^{\theta} I(|\zeta| > n^{1/p}) \leq CE|\zeta|^p$.*
(ii) *If $p < q$, then $\sum_{n=1}^{\infty} n^{-q/p} E|\zeta|^{q} I(|\zeta| \leq n^{1/p}) \leq CE|\zeta|^p$.*
(iii) *If $r > 1$, then $\sum_{n=1}^{\infty} n^{r-2} E|\zeta|^{p} I(|\zeta| > n^{1/p}) \leq CE|\zeta|^{rp}$.*
(iv) *If $rp < q$, then $\sum_{n=1}^{\infty} n^{r-1-q/p} E|\zeta|^{q} I(|\zeta| \leq n^{1/p}) \leq CE|\zeta|^{rp}$.*

The following lemma is useful to estimate $W_n(t)$ when the sequence $\{a_i, i \in \mathbb{Z}\}$ is absolutely summable. However, it is not applicable to the long-range dependent case.

Lemma 2.4 (Burton and Dehling [15]) *Let $\sum_{i=-\infty}^{\infty} a_i$ be an absolutely convergent series of real numbers with $a = \sum_{i=-\infty}^{\infty} a_i$. Then, for any $t > 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=-\infty}^{\infty} |\omega_{ni}|^t = |a|^t,$$

where $\omega_{ni} = \sum_{k=1}^{n} a_{i+k}$.

We now state and prove our main results. The first theorem treats the case $r > 1$.

Theorem 2.1 *Let $r > 1$ and $1 \leq p < 2$. Let $\{a_i, i \in \mathbb{Z}\}$ be a sequence of real numbers with*

$$\sum_{i=-\infty}^{\infty} |a_i|^p < \infty.$$

Suppose that $\{X_n, n \geq 1\}$ is the linear process of a sequence $\{\zeta_i, i \in \mathbb{Z}\}$ of i.i.d. random variables with mean zero and $E|\zeta_0|^{rp} < \infty$. Furthermore, assume that one of the following conditions holds.

(1) *If $1 < rp < 2$, then*

$$W_n(q)/W_n(p) = O\left(n^{1/q-1/p}\right) \quad \textit{for some } q \in (rp, 2).$$

(2) *If $rp \geq 2$, then*

$$W_n(q)/W_n(p) = O\left(n^{1/q-1/p}\right) \quad \textit{for some } q > rp$$

and

$$W_n(s)/W_n(p) = o\left((\log n)^{-1/s}\right) \quad \textit{for some } s \in (p, 2].$$

Then

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{k=1}^{n} X_k\right| > W_n(p)\varepsilon\right) < \infty \quad \textit{for all } \varepsilon > 0.$$

Proof (1) For each $n \geq 1$, we have

$$\begin{aligned}\sum_{k=1}^{n} X_k &= \sum_{i=-\infty}^{\infty} \sum_{k=1}^{n} a_{i+k}\zeta_i = \sum_{i=-\infty}^{\infty} \omega_{ni}\zeta_i \\ &= \sum_{i=-\infty}^{\infty} \omega_{ni}\left[\zeta_i I\left(|\zeta_i| > n^{1/p}\right) - E\zeta_i I\left(|\zeta_i| > n^{1/p}\right)\right] \\ &\quad + \sum_{i=-\infty}^{\infty} \omega_{ni}\left[\zeta_i I\left(|\zeta_i| \leq n^{1/p}\right) - E\zeta_i I\left(|\zeta_i| \leq n^{1/p}\right)\right] \\ &:= S'_n + S''_n\end{aligned}$$

and hence,

$$\begin{aligned}&\sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{k=1}^{n} X_k\right| > W_n(p)\varepsilon\right) \\ &\quad \leq \sum_{n=1}^{\infty} n^{r-2} P\left(\left|S'_n\right| > W_n(p)\varepsilon/2\right) + \sum_{n=1}^{\infty} n^{r-2} P\left(\left|S''_n\right| > W_n(p)\varepsilon/2\right).\end{aligned} \tag{2.1}$$

By the Markov inequality, Lemmas 2.1 and 2.3, we have

$$\begin{aligned}\sum_{n=1}^{\infty} n^{r-2} P\left(\left|S'_n\right| > W_n(p)\varepsilon/2\right) &\leq \sum_{n=1}^{\infty} n^{r-2} \frac{2^p E|S'_n|^p}{\varepsilon^p W_n^p(p)} \\ &\leq C\sum_{n=1}^{\infty} n^{r-2} \frac{\sum_{i=-\infty}^{\infty} |\omega_{ni}|^p E|\zeta_0|^p I(|\zeta_0| > n^{1/p})}{\sum_{i=-\infty}^{\infty} |\omega_{ni}|^p} \\ &= C\sum_{n=1}^{\infty} n^{r-2} E|\zeta_0|^p I\left(|\zeta_0| > n^{1/p}\right) \\ &\leq CE|\zeta_0|^{rp} < \infty.\end{aligned}$$

Thus the first series on the right-hand side of (2.1) converges.

Similarly, by the Markov inequality, Lemmas 2.1 and 2.3, we have

$$\begin{aligned}\sum_{n=1}^{\infty} n^{r-2} P\left(\left|S''_n\right| > W_n(p)\varepsilon/2\right) &\leq \sum_{n=1}^{\infty} n^{r-2} \frac{2^q E|S''_n|^q}{\varepsilon^q W_n^q(p)} \\ &\leq C\sum_{n=1}^{\infty} n^{r-2} \frac{\sum_{i=-\infty}^{\infty} |\omega_{ni}|^q E|\zeta_0|^q I(|\zeta_0| \leq n^{1/p})}{W_n^q(p)} \\ &= C\sum_{n=1}^{\infty} n^{r-2} \left(\frac{W_n(q)}{W_n(p)}\right)^q E|\zeta_0|^q I\left(|\zeta_0| \leq n^{1/p}\right)\end{aligned}$$

$$
\begin{aligned}
&\leq C\sum_{n=1}^{\infty} n^{r-2}\big(n^{1/q-1/p}\big)^q E|\zeta_0|^q I\big(|\zeta_0|\leq n^{1/p}\big)\\
&= C\sum_{n=1}^{\infty} n^{r-1-q/p} E|\zeta_0|^q I\big(|\zeta_0|\leq n^{1/p}\big)\\
&\leq CE|\zeta_0|^{rp} < \infty.
\end{aligned}
$$

Hence the second series on the right-hand side of (2.1) also converges.

(2) For each $n \geq 1$, we have

$$
\begin{aligned}
\sum_{k=1}^{n} X_k &= \sum_{i=-\infty}^{\infty}\sum_{k=1}^{n} a_{i+k}\zeta_i = \sum_{i=-\infty}^{\infty}\omega_{ni}\zeta_i\\
&= \sum_{i=-\infty}^{\infty}\big[\omega_{ni}\zeta_i I\big(|\omega_{ni}\zeta_i| > W_n(p)\big) - E\omega_{ni}\zeta_i I\big(|\omega_{ni}\zeta_i| > W_n(p)\big)\big]\\
&\quad + \sum_{i=-\infty}^{\infty}\big[\omega_{ni}\zeta_i I\big(|\omega_{ni}\zeta_i| \leq W_n(p)\big) - E\omega_{ni}\zeta_i I\big(|\omega_{ni}\zeta_i| \leq W_n(p)\big)\big]\\
&:= T_n' + T_n''
\end{aligned}
$$

and hence,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{k=1}^{n} X_k\right| > W_n(p)\varepsilon\right)\\
&\quad \leq \sum_{n=1}^{\infty} n^{r-2} P\big(\big|T_n'\big| > W_n(p)\varepsilon/2\big) + \sum_{n=1}^{\infty} n^{r-2} P\big(\big|T_n''\big| > W_n(p)\varepsilon/2\big).
\end{aligned} \tag{2.2}
$$

By the Markov inequality, Lemmas 2.1 and 2.3, we have

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} P\big(\big|T_n'\big| > W_n(p)\varepsilon/2\big)\\
&\quad \leq \sum_{n=1}^{\infty} n^{r-2}\frac{2^p E|T_n'|^p}{\varepsilon^p W_n^p(p)}\\
&\quad \leq C\sum_{n=1}^{\infty} n^{r-2}\frac{\sum_{i=-\infty}^{\infty} E|\omega_{ni}\zeta_i|^p I(|\omega_{ni}\zeta_i| > W_n(p))}{W_n^p(p)}\\
&\quad = C\sum_{n=1}^{\infty} n^{r-2}\frac{\sum_{i=-\infty}^{\infty} E|\omega_{ni}\zeta_i|^p I(|\omega_{ni}\zeta_i| > W_n(p), |\zeta_i| > n^{1/p})}{W_n^p(p)}\\
&\qquad + C\sum_{n=1}^{\infty} n^{r-2}\frac{\sum_{i=-\infty}^{\infty} E|\omega_{ni}\zeta_i|^p I(|\omega_{ni}\zeta_i| > W_n(p), |\zeta_i| \leq n^{1/p})}{W_n^p(p)}\\
&\quad \leq C\sum_{n=1}^{\infty} n^{r-2}\frac{\sum_{i=-\infty}^{\infty} |\omega_{ni}|^p E|\zeta_i|^p I(|\zeta_i| > n^{1/p})}{W_n^p(p)}\\
&\qquad + C\sum_{n=1}^{\infty} n^{r-2}\frac{\sum_{i=-\infty}^{\infty} E[|\omega_{ni}\zeta_i|^{p-q}|\omega_{ni}\zeta_i|^q I(|\omega_{ni}\zeta_i| > W_n(p), |\zeta_i| \leq n^{1/p})]}{W_n^p(p)}
\end{aligned}
$$

$$\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{r-2}\frac{\sum_{i=-\infty}^{\infty}|\omega_{ni}|^{p}E|\zeta_0|^{p}I(|\zeta_0|>n^{1/p})}{W_n^{p}(p)} \\
&\quad + C\sum_{n=1}^{\infty} n^{r-2}\frac{(W_n(p))^{p-q}\sum_{i=-\infty}^{\infty}|\omega_{ni}|^{q}E|\zeta_0|^{q}I(|\zeta_0|\le n^{1/p})}{W_n^{p}(p)} \\
&= C\sum_{n=1}^{\infty} n^{r-2}E|\zeta_0|^{p}I\bigl(|\zeta_0|>n^{1/p}\bigr) \\
&\quad + C\sum_{n=1}^{\infty} n^{r-2}\left(\frac{W_n(q)}{W_n(p)}\right)^{q}E|\zeta_0|^{q}I\bigl(|\zeta_0|\le n^{1/p}\bigr) \\
&\le C\sum_{n=1}^{\infty} n^{r-2}E|\zeta_0|^{p}I\bigl(|\zeta_0|>n^{1/p}\bigr) + C\sum_{n=1}^{\infty} n^{r-1-q/p}E|\zeta_0|^{q}I\bigl(|\zeta_0|\le n^{1/p}\bigr) \\
&\le CE|\zeta_0|^{rp}<\infty.
\end{aligned}$$

Thus the first series on the right-hand side of (2.2) converges.

We next prove that the second series on the right-hand side of (2.2) converges. We have by Lemma 2.2 that for $t>2$,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2}P\bigl(\bigl|T_n''\bigr|>W_n(p)\varepsilon/2\bigr) \\
&\quad\le C\sum_{n=1}^{\infty} n^{r-2}\frac{\sum_{i=-\infty}^{\infty}E|\omega_{ni}\zeta_i|^{t}I(|\omega_{ni}\zeta_i|\le W_n(p))}{W_n^{t}(p)} \\
&\qquad + C\sum_{n=1}^{\infty} n^{r-2}\exp\left\{-\frac{\varepsilon^2 W_n^2(p)}{2(t+2)^2e^t\sum_{i=-\infty}^{\infty}\operatorname{Var}(\omega_{ni}\zeta_iI(|\omega_{ni}\zeta_i|\le W_n(p)))}\right\}. \qquad (2.3)
\end{aligned}$$

Hence it is enough to show that two series on the right-hand side of (2.3) converge.

If we take $t>q$, then we have by Lemma 2.3 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2}\frac{\sum_{i=-\infty}^{\infty}E|\omega_{ni}\zeta_i|^{t}I(|\omega_{ni}\zeta_i|\le W_n(p))}{W_n^{t}(p)} \\
&\quad= \sum_{n=1}^{\infty} n^{r-2}\frac{\sum_{i=-\infty}^{\infty}E|\omega_{ni}\zeta_i|^{t}I(|\omega_{ni}\zeta_i|\le W_n(p),|\zeta_i|>n^{1/p})}{W_n^{t}(p)} \\
&\qquad + \sum_{n=1}^{\infty} n^{r-2}\frac{\sum_{i=-\infty}^{\infty}E|\omega_{ni}\zeta_i|^{t}I(|\omega_{ni}\zeta_i|\le W_n(p),|\zeta_i|\le n^{1/p})}{W_n^{t}(p)} \\
&\quad= \sum_{n=1}^{\infty} n^{r-2}\frac{\sum_{i=-\infty}^{\infty}E[|\omega_{ni}\zeta_i|^{t-p}|\omega_{ni}\zeta_i|^{p}I(|\omega_{ni}\zeta_i|\le W_n(p),|\zeta_i|>n^{1/p})]}{W_n^{t}(p)} \\
&\qquad + \sum_{n=1}^{\infty} n^{r-2}\frac{\sum_{i=-\infty}^{\infty}E[|\omega_{ni}\zeta_i|^{t-q}|\omega_{ni}\zeta_i|^{q}I(|\omega_{ni}\zeta_i|\le W_n(p),|\zeta_i|\le n^{1/p})]}{W_n^{t}(p)} \\
&\quad\le \sum_{n=1}^{\infty} n^{r-2}\frac{(W_n(p))^{t-p}\sum_{i=-\infty}^{\infty}|\omega_{ni}|^{p}E|\zeta_0|^{p}I(|\zeta_0|>n^{1/p})}{W_n^{t}(p)}
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{n=1}^{\infty} n^{r-2}\frac{(W_n(p))^{t-q}\sum_{i=-\infty}^{\infty}|\omega_{ni}|^q E|\zeta_0|^q I(|\zeta_0|\le n^{1/p})}{W_n^t(p)}\\
&=\sum_{n=1}^{\infty} n^{r-2}E|\zeta_0|^p I\big(|\zeta_0|>n^{1/p}\big)+\sum_{n=1}^{\infty} n^{r-2}\left(\frac{W_n(q)}{W_n(p)}\right)^q E|\zeta_0|^q I\big(|\zeta_0|\le n^{1/p}\big)\\
&\le\sum_{n=1}^{\infty} n^{r-2}E|\zeta_0|^p I\big(|\zeta_0|>n^{1/p}\big)+\sum_{n=1}^{\infty} n^{r-1-q/p}E|\zeta_0|^q I\big(|\zeta_0|\le n^{1/p}\big)\\
&\le CE|\zeta_0|^{rp}<\infty.
\end{aligned}
$$

Hence the first series on the right-hand side of (2.3) converges.

Finally, we show that the second series on the right-hand side of (2.3) converges. Since $p<s\le 2$, we have that

$$
\begin{aligned}
&\frac{\sum_{i=-\infty}^{\infty}\operatorname{Var}(\omega_{ni}\zeta_i I(|\omega_{ni}\zeta_i|\le W_n(p)))}{W_n^2(p)}\\
&\quad\le\frac{\sum_{i=-\infty}^{\infty}E|\omega_{ni}\zeta_i|^2 I(|\omega_{ni}\zeta_i|\le W_n(p))}{W_n^2(p)}\\
&\quad=\frac{\sum_{i=-\infty}^{\infty}E|\omega_{ni}\zeta_i|^{s+2-s} I(|\omega_{ni}\zeta_i|\le W_n(p))}{W_n^2(p)}\\
&\quad\le\frac{(W_n(p))^{2-s}\sum_{i=-\infty}^{\infty}E|\omega_{ni}\zeta_i|^s}{W_n^2(p)}\\
&\quad=\frac{\sum_{i=-\infty}^{\infty}|\omega_{ni}|^s E|\zeta_0|^s}{W_n^s(p)}\\
&\quad=\left(\frac{W_n(s)}{W_n(p)}\right)^s E|\zeta_0|^s\\
&\quad=o(1/\log n),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2}\left\{-\frac{\varepsilon^2 W_n^2(p)}{2(t+2)^2 e^t\sum_{i=-\infty}^{\infty}\operatorname{Var}(\omega_{ni}\zeta_i I(|\omega_{ni}\zeta_i|\le W_n(p)))}\right\}\\
&\quad\le C\sum_{n=1}^{\infty} n^{r-2}\left\{-\frac{\varepsilon^2\log n}{2(t+2)^2 e^t o(1)}\right\}<\infty.
\end{aligned}
$$

□

The next theorem treats the case $r=1$.

Theorem 2.2 *Let $1\le p<2$. Let $\{a_i, i\in\mathbb{Z}\}$ be a sequence of real numbers with*

$$\sum_{i=-\infty}^{\infty}|a_i|^\theta<\infty\quad\text{for some } 0<\theta<p.$$

Suppose that $\{X_n, n\ge 1\}$ is the linear process of a sequence $\{\zeta_i, i\in\mathbb{Z}\}$ of i.i.d. random variables with mean zero and $E|\zeta_0|^p<\infty$. Furthermore, assume that

$$W_n(\theta)/W_n(p)=O\big(n^{1/\theta-1/p}\big)$$

and

$$W_n(q)/W_n(p) = O\big(n^{1/q-1/p}\big) \quad \textit{for some } q \in (p,2).$$

Then

$$\sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{k=1}^{n} X_k\right| > W_n(p)\varepsilon\right) < \infty \quad \textit{for all } \varepsilon > 0.$$

Proof The proof is similar to that of Theorem 2.1(1). We proceed with two cases $1 \le \theta < p$ and $0 < \theta < 1$.

For the case $1 \le \theta < p$, we have by Lemmas 2.1 and 2.3 that

$$\begin{aligned}
\sum_{n=1}^{\infty} n^{-1} P\big(|S'_n| > W_n(p)\varepsilon/2\big) &\le \sum_{n=1}^{\infty} n^{-1} \frac{2^\theta E|S'_n|^\theta}{\varepsilon^\theta W_n^\theta(p)} \\
&\le C\sum_{n=1}^{\infty} n^{-1} \frac{\sum_{i=-\infty}^{\infty} |\omega_{ni}|^\theta E|\zeta_0|^\theta I(|\zeta_0| > n^{1/p})}{W_n^\theta(p)} \\
&= C\sum_{n=1}^{\infty} n^{-1} \left(\frac{W_n(\theta)}{W_n(p)}\right)^\theta E|\zeta_0|^p I\big(|\zeta_0| > n^{1/p}\big) \\
&\le C\sum_{n=1}^{\infty} n^{-1} n^{(1/\theta-1/p)\theta} E|\zeta_0|^p I\big(|\zeta_0| > n^{1/p}\big) \\
&\le CE|\zeta_0|^p < \infty.
\end{aligned}$$

As in the proof of Theorem 2.1(1), we have that

$$\sum_{n=1}^{\infty} n^{-1} P\big(|S''_n| > W_n(p)\varepsilon/2\big) \le CE|\zeta_0|^p < \infty.$$

For the case $0 < \theta < 1$, we rewrite $\sum_{k=1}^{n} X_k$ as

$$\begin{aligned}
\sum_{k=1}^{n} X_k &= \sum_{i=-\infty}^{\infty} \omega_{ni}\zeta_i I\big(|\zeta_i| > n^{1/p}\big) + \sum_{i=-\infty}^{\infty} \omega_{ni}\big[\zeta_i I\big(|\zeta_i| \le n^{1/p}\big) - E\zeta_i I\big(|\zeta_i| \le n^{1/p}\big)\big] \\
&\quad - \sum_{i=-\infty}^{\infty} \omega_{ni} E\zeta_i I\big(|\zeta_i| > n^{1/p}\big) \\
&:= S'_n + S''_n - S'''_n.
\end{aligned}$$

If $0 < \theta < 1$, then $\sum_{n=1}^{\infty} |a_n| \le (\sum_{n=1}^{\infty} |a_n|^\theta)^{1/\theta} < \infty$. It follows by Lemma 2.4 that

$$\begin{aligned}
W_n^{-1}(p)\big|S'''_n\big| &\le W_n^{-1}(p) \sum_{i=-\infty}^{\infty} |\omega_{ni}| E|\zeta_0| I\big(|\zeta_0| > n^{1/p}\big) \\
&\le Cn^{1-1/p} E|\zeta_0| I\big(|\zeta_0| > n^{1/p}\big) \\
&\le CE|\zeta_0|^p I\big(|\zeta_0| > n^{1/p}\big) \to 0
\end{aligned}$$

as $n \to \infty$. Hence

$$\sum_{n=1}^{\infty} n^{-1} P\left(\left|\sum_{k=1}^{n} X_k\right| > W_n(p)\varepsilon\right) \le C\sum_{n=1}^{\infty} n^{-1} P\left(\left|S_n'\right| > W_n(p)\varepsilon/3\right) + C\sum_{n=1}^{\infty} n^{-1} P\left(\left|S_n''\right| > W_n(p)\varepsilon/3\right).$$

The rest of the proof is the same as that of the previous case and is omitted. □

The following corollary extends Theorem 1.1 to the short-range dependent linear processes.

Corollary 2.1 *Let $r \ge 1$, $1 \le p < 2$, and $rp > 1$. Let $\{a_i, i \in \mathbb{Z}\}$ be an absolutely summable sequence of real numbers. Suppose that $\{X_n, n \ge 1\}$ is the linear process of a sequence $\{\zeta_i, i \in \mathbb{Z}\}$ of i.i.d. random variables with mean zero and $E|\zeta_0|^{rp} < \infty$. Then*

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\left|\sum_{k=1}^{n} X_k\right| > n^{1/p}\varepsilon\right) < \infty \quad \text{for all } \varepsilon > 0.$$

Proof We first note that

$$\sum_{i=-\infty}^{\infty} |a_i|^p \le \left(\sum_{i=-\infty}^{\infty} |a_i|\right)^p < \infty.$$

If $1 < p < 2$, then we take θ such that $1 \le \theta < p$. Then

$$\sum_{i=-\infty}^{\infty} |a_i|^\theta \le \left(\sum_{i=-\infty}^{\infty} |a_i|\right)^\theta < \infty.$$

By Lemma 2.4, for any $t > 0$, there exist positive constants C_1 and C_2 independent of n such that

$$C_1 n^{1/t} \le W_n(t) \le C_2 n^{1/t} \quad \text{for all } n \ge 1.$$

Then all conditions on $W_n(\cdot)$ in Theorems 2.1 and 2.2 are easily satisfied. Hence the proof follows from Theorems 2.1 and 2.2. □

Remark 2.1 In Corollary 2.1, the case $rp = 1$ (i.e., $r = 1$ and $p = 1$) is not considered. In fact, Corollary 2.1 does not hold for this case (see Sung [18]).

3 An estimation of $W_n(t)$ for the long-range dependent case

As we have seen in Sections 1 and 2, it is easy to estimate $W_n(t)$ for the short-range dependent case. In this section, we propose a method to estimate $W_n(t)$ for the long-range dependent case. It is not easy to estimate $W_n(t)$ when the sequence $\{a_i, i \in \mathbb{Z}\}$ is not absolutely summable. For simplicity, we will consider non-increasing sequences of positive numbers. For the finiteness of $W_n(t)$, without loss of generality, it is necessary to assume that $a_i = 0$ if $i \le 0$ and $\sum_{i=1}^{\infty} a_i^t < \infty$.

Lemma 3.1 *Let $t > 0$. Let $\{a_i, i \in \mathbb{Z}\}$ be a non-increasing sequence of positive real numbers satisfying $a_i = 0$ if $i \le 0$ and $\sum_{i=1}^{\infty} a_i^t < \infty$. Then*

$$\frac{n}{2}(a_1 + \cdots + a_{[n/2]})^t + n^t \sum_{i=n}^{\infty} a_i^t \le W_n^t(t) \le 2n(a_1 + \cdots + a_n)^t + n^t \sum_{i=n}^{\infty} a_i^t.$$

Proof Since $a_i = 0$ if $i \le 0$ and $0 < a_i \downarrow$, we get that

$$\begin{aligned} W_n^t(t) &= \sum_{i=1}^{n}\left(\sum_{j=1}^{i} a_j\right)^t + \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{i+j}\right)^t + \sum_{i=n+1}^{\infty}\left(\sum_{j=1}^{n} a_{i+j}\right)^t \\ &\le 2n(a_1 + \cdots + a_n)^t + n^t \sum_{i=n+1}^{\infty} a_{i+1}^t \\ &\le 2n(a_1 + \cdots + a_n)^t + n^t \sum_{i=n}^{\infty} a_i^t. \end{aligned}$$

Similarly,

$$\begin{aligned} W_n^t(t) &= \sum_{i=1}^{n-1}\left(\sum_{j=1}^{i} a_j\right)^t + \sum_{i=1}^{\infty}\left(\sum_{j=0}^{n-1} a_{i+j}\right)^t \\ &\ge \sum_{i=[n/2]}^{n-1}\left(\sum_{j=1}^{i} a_j\right)^t + n^t \sum_{i=n}^{\infty} a_i^t \\ &\ge \frac{n}{2}(a_1 + \cdots + a_{[n/2]})^t + n^t \sum_{i=n}^{\infty} a_i^t. \end{aligned}$$

Thus the proof is completed. □

The following lemma can be found in Martikainen [19].

Lemma 3.2 (Martikainen [19]) *Let $\{b_n, n \ge 1\}$ be a non-decreasing sequence of positive real numbers. Then*

$$\sum_{i=n}^{\infty} \frac{1}{ib_i} = O\bigl(b_n^{-1}\bigr) \quad \Longleftrightarrow \quad \liminf_{n\to\infty} \frac{b_{rn}}{b_n} > 1 \quad \textit{for some integer } r \ge 2.$$

Similarly, we can obtain a counterpart of Lemma 3.2.

Lemma 3.3 *Let $\{b_n, n \ge 1\}$ be a non-decreasing sequence of positive real numbers. Then*

$$\sum_{i=1}^{n} \frac{b_i}{i} = O(b_n) \quad \Longleftrightarrow \quad \liminf_{n\to\infty} \frac{b_{rn}}{b_n} > 1 \quad \textit{for some integer } r \ge 2.$$

Proof The proof is similar to that of Lemma 3.2 and is omitted. □

Using Lemmas 3.2 and 3.3, we have the following lemma.

Lemma 3.4 *Let $t > 1$ and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers satisfying $na_n \uparrow$, $na_n^t \downarrow$, and*

$$\frac{1}{r} < \liminf_{n\to\infty} \frac{a_{rn}}{a_n} \leq \limsup_{n\to\infty} \frac{a_{rn}}{a_n} < \left(\frac{1}{r}\right)^{1/t} \quad \textit{for some integer } r \geq 2.$$

Then the following statements hold:

(i) $\sum_{i=n}^{\infty} a_i^t = O(na_n^t)$.

(ii) $\sum_{i=1}^{n} a_i = O(na_n)$.

Proof The proof of (i) follows from Lemma 3.2. The proof of (ii) follows from Lemma 3.3. □

Now we present a method to estimate $W_n(t)$ for the long-range dependent case.

Theorem 3.1 *Let $t > 1$, and let $\{a_n, n \geq 1\}$ be a sequence of positive real numbers satisfying the same conditions as in Lemma* 3.4. *Then there exist positive constants C_1 and C_2 independent of n such that*

$$C_1 n^{1+t} a_n^t \leq W_n^t(t) \leq C_2 n^{1+t} a_n^t \quad \textit{for all } n \geq 1,$$

where $a_i = 0$ if $i \leq 0$.

Proof By the condition $na_n^t \downarrow$, we have $(a_{n+1}/a_n)^t \leq n/(n+1)$, which implies $0 < a_n \downarrow$. The upper bound of $W_n^t(t)$ follows by Lemmas 3.1 and 3.4. For the lower bound, we have by $\liminf_{n\to\infty} a_{rn}/a_n > 1/r$ that

$$a_{rn}/a_n \geq 1/r \quad \text{for all large } n.$$

It follows that for all large n

$$n^t \sum_{i=n}^{\infty} a_i^t \geq n^t \sum_{i=n}^{rn} a_i^t \geq (r-1)n^{1+t} a_{rn}^t \geq (r-1)r^{-t} n^{1+t} a_n^t.$$

Since $0 < a_n \downarrow$,

$$n(a_1 + \cdots + a_{[n/2]})^t \geq n[n/2]^t a_{[n/2]}^t \geq n[n/2]^t a_n^t.$$

Hence the lower bound follows from Lemma 3.1. □

Finally, we give two long-range dependent linear processes.

Example 3.1 Let $a_i = 1/i$ if $i \geq 1$ and $a_i = 0$ if $i \leq 0$. Then the series $\sum_{i=-\infty}^{\infty} a_i$ diverges, but $\sum_{i=-\infty}^{\infty} a_i^t$ converges if $t > 1$. Observe that

$$\ln(n+1) \leq \sum_{i=1}^{n} a_i \leq 1 + \ln n.$$

If $t > 1$, then

$$\frac{1}{t-1}n^{-t+1} \le \sum_{i=n}^{\infty} a_i^t \le n^{-t} + \frac{1}{t-1}n^{-t+1}.$$

By Lemma 3.1, for any $t > 1$, there exist positive constants C_1 and C_2 independent of n such that

$$C_1 n(\ln n)^t \le W_n^t(t) \le C_2 n(\ln n)^t \quad \text{for all } n \ge 2.$$

Let $X_n = \sum_{i=-\infty}^{\infty} a_{i+n}\zeta_i$ be the long-range dependent linear process of a sequence $\{\zeta_i\}$ of i.i.d. random variables with mean zero and $E|\zeta_0|^{rp} < \infty$, where $r > 1$ and $1 < p < 2$. Then all conditions of Theorem 2.1 are easily satisfied. By Theorem 2.1,

$$\sum_{n=1}^{\infty} n^{r-2} P\left(\left| \sum_{k=1}^{n} X_k \right| > n^{1/p} \ln n\varepsilon \right) < \infty \quad \text{for all } \varepsilon > 0.$$

Example 3.2 Let $1 < p < 2$. Let $a_i = 1/i^d$ if $i \ge 1$ and $a_i = 0$ if $i \le 0$, where $1/p < d < 1$. Then the series $\sum_{i=-\infty}^{\infty} a_i$ diverges, but $\sum_{i=-\infty}^{\infty} a_i^t$ converges if $t > 1/d$. Since $a_{2n}/a_n = 2^{-d}$, we have by Theorem 3.1 that

$$C_1 n^{1+t-dt} \le W_n^t(t) \le C_2 n^{1+t-dt} \quad \text{for all } n \ge 1.$$

Let $X_n = \sum_{i=-\infty}^{\infty} a_{i+n}\zeta_i$ be the long-range dependent linear process of a sequence $\{\zeta_i\}$ of i.i.d. random variables with mean zero and $E|\zeta_0|^p < \infty$. Take θ such that $1/d < \theta < p$. Then all conditions of Theorem 2.2 are easily satisfied. By Theorem 2.2,

$$\sum_{n=1}^{\infty} n^{-1} P\left(\left| \sum_{k=1}^{n} X_k \right| > n^{1/p+1-d}\varepsilon \right) < \infty \quad \text{for all } \varepsilon > 0.$$

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