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Aspects of univalent holomorphic functions involving multiplier transformation and Ruscheweyh derivative

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Abstract

Making use multiplier transformation and Ruscheweyh derivative,we introduce a new class of analytic functions $\mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$ defined on the open unit disc, and investigate its various characteristics. Further we obtain distortion bounds, extreme points and radii of close-to-convexity, starlikeness and convexity and neighborhood property for functions belonging to the class $\mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions of the form $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, which are analytic and univalent in the open unit disc $U = \{z : z \in \mathbb{C} : |z| < 1\}$. $\mathcal{T}$ is a subclass of $\mathcal{A}$ consisting of the functions of the form $f(z) = z - \sum_{j=2}^{\infty} |a_j| z^j$. For functions $f, g \in \mathcal{A}$ given by $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$, we define the Hadamard product (or convolution) of f and g by $(f * g)(z) = z + \sum_{j=2}^{\infty} a_j b_j z^j$, $z \in U$.

Definition 1.1 (Ruscheweyh [1]) For $f \in \mathcal{A}$, $n \in \mathbb{N}$, the operator R^n is defined by $R^n : \mathcal{A} \to \mathcal{A}$,

$$R^0 f(z) = f(z),$$

$$R^1 f(z) = zf'(z), \qquad \ldots,$$

$$(n+1)R^{n+1} f(z) = z\bigl(R^n f(z)\bigr)' + nR^n f(z), \quad z \in U.$$

Remark 1.1 If $f \in \mathcal{A}$, $f(z) = z + \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z + \sum_{j=2}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.
If $f \in \mathcal{T}$, $f(z) = z - \sum_{j=2}^{\infty} a_j z^j$, then $R^n f(z) = z - \sum_{j=t+1}^{\infty} \frac{(n+j-1)!}{n!(j-1)!} a_j z^j$, $z \in U$.

Definition 1.2 [2, 3] For $f \in \mathcal{A}$, $n \in \mathbb{N} \cup \{0\}$, $\lambda, l \geq 0$, the operator $I(n,\lambda,l)f(z)$ is defined by the following infinite series:

$$I(n,\lambda,l)f(z) := z + \sum_{j=2}^{\infty} \left(\frac{\lambda(j-1)+l+1}{l+1}\right)^n a_j z^j.$$

Remark 1.2 [4] It follows from the above definition that

$$I(0,\lambda,l)f(z) = f(z),$$
$$(l+1)I(n+1,\lambda,l)f(z) = (l+1-\lambda)I(n,\lambda,l)f(z) + \lambda z\big(I(n,\lambda,l)f(z)\big)', \quad z\in U.$$

Remark 1.3 The operator $I(n,\lambda,0) = D_\lambda^n$ is the generalized Sălăgean operator introduced by Al-Oboudi [5], and $I(n,1,0) = S^n$ is the Sălăgean differential operator [6].

Definition 1.3 [7, 8] Let $\gamma,\lambda,l\geq 0$, $n\in\mathbb{N}$. Denote by $RI_{n,\lambda,l}^{\gamma}$ the operator given by $RI_{n,\lambda,l}^{\gamma}:\mathcal{A}\to\mathcal{A}$,

$$RI_{n,\lambda,l}^{\gamma}f(z) = (1-\gamma)R^nf(z) + \gamma I(n,\lambda,l)f(z), \quad z\in U.$$

Remark 1.4 If $f\in\mathcal{A}$, $f(z) = z+\sum_{j=2}^{\infty}a_jz^j$, then

$$RI_{n,\lambda,l}^{\gamma}f(z) = z+\sum_{j=2}^{\infty}\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^j, \quad z\in U.$$

If $f\in\mathcal{T}$, $f(z) = z-\sum_{j=2}^{\infty}a_jz^j$, then

$$RI_{n,\lambda,l}^{\gamma}f(z) = z-\sum_{j=2}^{\infty}\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^j, \quad z\in U.$$

Remark 1.5 The operator $RI_{n,\lambda,0}^{\gamma}f(z) = RD_{\lambda,\gamma}^{n}f(z)$ which was introduced in [9] and the operator $RI_{n,1,0}^{\gamma}f(z) = L_{\gamma}^{n}f(z)$ which was introduced in [10].

Following the work of Najafzadeh and Pezeshki [11] we can define the class $\mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$ as follows.

Definition 1.4 For $\gamma,\lambda,l\geq 0$, $0\leq\alpha<1$ and $0<\beta\leq 1$, let $\mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$ be the subclass of $\mathcal{T}$ consisting of functions that satisfying the inequality

$$\left|\frac{RI_{n,\lambda,l}^{\mu,\gamma}f(z)-1}{2\nu(RI_{n,\lambda,l}^{\mu,\gamma}f(z)-\alpha)-(RI_{n,\lambda,l}^{\mu,\gamma}f(z)-1)}\right| < \beta, \tag{1.1}$$

where

$$RI_{n,\lambda,l}^{\mu,\gamma}f(z) = (1-\mu)\frac{RI_{n,\lambda,l}^{\gamma}f(z)}{z} + \mu\big(RI_{n,\lambda,l}^{\gamma}f(z)\big)', \tag{1.2}$$

$0<\nu\leq 1$.

Remark 1.6 If $f\in\mathcal{T}$, $f(z) = z-\sum_{j=2}^{\infty}a_jz^j$, then

$$RI_{n,\lambda,l}^{\mu,\gamma}f(z) = 1-\sum_{j=t+1}^{\infty}\big[1+\mu(j-1)\big] \times\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^{j-1}, \quad z\in U.$$

Remark 1.7 The class $\mathcal{RI}(\gamma,\lambda,0,\alpha,\beta) = \mathcal{RD}(\gamma,\lambda,\alpha,\beta)$ defined and studied in [12] and $\mathcal{RI}(\gamma,1,0,\alpha,\beta) = \mathcal{L}(\gamma,\alpha,\beta)$ defined and studied in [13].

2 Coefficient bounds

In this section we obtain coefficient bounds and extreme points for functions in $\mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$.

Theorem 2.1 *Let the function $f \in \mathcal{T}$. Then $f \in \mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$ if and only if*

$$\sum_{j=t+1}^{\infty}\bigl(1+\mu(j-1)\bigr)\bigl[1+\beta(2\nu-1)\bigr]\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j < 2\beta\nu(1-\alpha). \tag{2.1}$$

The result is sharp for the function $F(z)$ defined by

$$F(z) = z - \frac{2\beta\nu(1-\alpha)}{(1+\mu(j-1))[1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}z^{j}, \quad j \geq t+1.$$

Proof Suppose f satisfies (2.1). Then for $|z|<1$, we have

$$\begin{aligned}
&\bigl|RI_{n,\lambda,l}^{\mu,\gamma}f(z)-1\bigr| - \beta\bigl|2\nu\bigl(RI_{n,\lambda,l}^{\mu,\gamma}f(z)-\alpha\bigr)-\bigl(RI_{n,\lambda,l}^{\mu,\gamma}f(z)-1\bigr)\bigr| \\
&\quad = \left|-\sum_{j=t+1}^{\infty}\bigl(1+\mu(j-1)\bigr)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_jz^{j-1}\right| \\
&\qquad - \beta\left|2\nu(1-\alpha)-(2\nu-1)\sum_{j=t+1}^{\infty}\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}\right. \\
&\qquad \left.\times\bigl[1+\mu(j-1)\bigr]a_jz^{j-1}\right| \\
&\quad \leq \sum_{j=t+1}^{\infty}\bigl[1+\mu(j-1)\bigr]\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_k - 2\beta\nu(1-\alpha) \\
&\qquad + \sum_{j=t+1}^{\infty}\beta(2\nu-1)\bigl(1+\mu(j-1)\bigr)\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j \\
&\quad = \sum_{j=t+1}^{\infty}\bigl[1+\mu(j-1)\bigr]\bigl[1+\beta(2\nu-1)\bigr]\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}a_j \\
&\qquad - 2\beta\nu(1-\alpha) < 0.
\end{aligned}$$

Hence, by using the maximum modulus Theorem and (1.1), $f \in \mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$. Conversely, assume that

$$\begin{aligned}
&\left|\frac{RI_{n,\lambda,l}^{\mu,\gamma}f(z)-1}{2\nu(RI_{n,\lambda,l}^{\mu,\gamma}f(z)-\alpha)-(RI_{n,\lambda,l}^{\mu,\gamma}f(z)-1)}\right| \\
&\quad = \left|\frac{-\sum_{j=t+1}^{\infty}[1+\mu(j-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}a_jz^{j-1}}{2\nu(1-\alpha)-\sum_{j=t+1}^{\infty}[1+\mu(j-1)](2\nu-1)\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}a_jz^{j-1}}\right| \\
&\quad < \beta, \quad z \in U.
\end{aligned}$$

Since $\mathrm{Re}(z) \le |z|$ for all $z \in U$, we have

$$\mathrm{Re}\left\{\frac{\sum_{j=t+1}^{\infty}[1+\mu(j-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}a_j z^{j-1}}{2\nu(1-\alpha)-\sum_{j=t+1}^{\infty}[1+\mu(j-1)](2\nu-1)\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}a_j z^{j-1}}\right\} < \beta. \tag{2.2}$$

By choosing choose values of z on the real axis so that $RI_{n,\lambda,l}^{\mu,\gamma}f(z)$ is real and letting $z \to 1$ through real values, we obtain the desired inequality (2.1). □

Corollary 2.2 *If $f \in \mathcal{T}$ is in $\mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$, then*

$$a_j \le \frac{2\beta\nu(1-\alpha)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}, \quad j \ge t+1, \tag{2.3}$$

with equality only for functions of the form $F(z)$.

Theorem 2.3 *Let $f_1(z) = z$ and*

$$f_j(z) = z - \frac{2\beta\nu(1-\alpha)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}z^j, \quad j \ge t+1, \tag{2.4}$$

for $0 \le \alpha < 1$, $0 < \beta \le 1$, $\gamma,\lambda,l \ge 0$ and $0 < \nu \le 1$. Then $f(z)$ is in the class $\mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$ if and only if it can be expressed in the form

$$f(z) = \sum_{j=t}^{\infty}\omega_j f_j(z), \tag{2.5}$$

where $\omega_j \ge 0$ and $\sum_{j=1}^{\infty}\omega_j = 1$.

Proof Suppose $f(z)$ can be written as in (2.5). Then

$$f(z) = z - \sum_{j=t+1}^{\infty}\omega_j\frac{2\beta\nu(1-\alpha)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}z^j.$$

Now,

$$\sum_{j=t+1}^{\infty}\frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)}\omega_j \times \frac{2\beta\nu(1-\alpha)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}} = \sum_{j=t+1}^{\infty}\omega_j = 1-\omega_1 \le 1.$$

Thus $f \in \mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$.

Conversely, let $f \in \mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$. Then by using (2.3), setting

$$\omega_j = \frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)}a_j, \quad j \ge t+1,$$

and $\omega_1 = 1 - \sum_{j=2}^{\infty} \omega_j$, we have $f(z) = \sum_{j=t}^{\infty} \omega_j f_j(z)$. This completes the proof of Theorem 2.3. □

3 Distortion bounds

In this section we obtain distortion bounds for the class $\mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$.

Theorem 3.1 *If* $f \in \mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$, *then*

$$r - \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} r^{t+1} \le |f(z)| \le r + \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} r^{t+1} \tag{3.1}$$

holds if the sequence $\{\sigma_j(\gamma, \lambda, l, \beta, \nu)\}_{j=t+1}^{\infty}$ *is non-decreasing, and*

$$1 - \frac{2\beta\nu(1-\alpha)(t+1)}{(1+\mu t)[1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} r^{t} \le |f'(z)| \le 1 + \frac{2\beta\nu(1-\alpha)(t+1)}{(1+\mu t)[1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} r^{t} \tag{3.2}$$

holds if the sequence $\{\frac{\sigma_j(\gamma, \lambda, l, \beta, \nu)}{j}\}_{j=t+1}^{\infty}$ *is non-decreasing, where*

$$\sigma_j(\gamma, \beta, \nu) = [1+\mu(j-1)][1+\beta(2\nu-1)]\left\{\gamma\left(\frac{1+\lambda(j-1)+l}{l+1}\right)^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\right\}.$$

The bounds in (3.1) *and* (3.2) *are sharp, for* $f(z)$ *given by*

$$f(z) = z - \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} z^{t+1}, \quad z = \pm r. \tag{3.3}$$

Proof In view of Theorem 2.1, we have

$$\sum_{j=t+1}^{\infty} a_j \le \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}}. \tag{3.4}$$

We obtain

$$|z| - |z|^{t+1}\sum_{j=t+1}^{\infty} a_j \le |f(z)| \le |z| + |z|^{t+1}\sum_{j=t+1}^{\infty} a_j.$$

Thus

$$r - \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} r^{t+1} \le |f(z)| \le r + \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} r^{t+1}. \tag{3.5}$$

Hence (3.1) follows from (3.5). Further,

$$\sum_{j=t+1}^{\infty} ja_j \le \frac{2\beta\nu(1-\alpha)}{(1+\mu t)[1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}}.$$

Hence (3.2) follows from

$$1 - r^t \sum_{j=t+1}^{\infty} ja_j \le \left|f'(z)\right| \le 1 + r^t \sum_{j=t+1}^{\infty} ja_j.$$

□

4 Radius of starlikeness and convexity

The radii of close-to-convexity, starlikeness, and convexity for the class $\mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$ are given in this section.

Theorem 4.1 *Let the function $f \in \mathcal{T}$ belong to the class $\mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$, Then $f(z)$ is close-to-convex of order δ, $0 \le \delta < 1$, in the disc $|z| < r$, where*

$$r := \inf_{j\ge t+1}\left[\frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu j(1-\alpha)}\right]^{\frac{1}{t}}. \tag{4.1}$$

The result is sharp, with extremal function $f(z)$ given by (2.3).

Proof For given $f \in \mathcal{T}$ we must show that

$$\left|f'(z) - 1\right| < 1 - \delta. \tag{4.2}$$

By a simple calculation we have

$$\left|f'(z) - 1\right| \le \sum_{j=t+1}^{\infty} ja_j|z|^t.$$

The last expression is less than $1-\delta$ if

$$\sum_{j=t+1}^{\infty} \frac{j}{1-\delta} a_j|z|^t < 1.$$

We use the fact that $f \in \mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$ if and only if

$$\sum_{j=t+1}^{\infty} \frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)} a_j \le 1.$$

Equation (4.2) holds true if

$$\frac{j}{1-\delta}|z|^t \le \sum_{j=t+1}^{\infty} \frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)}.$$

Or, equivalently,

$$|z|^t \leq \sum_{j=t+1}^{\infty} \frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu j(1-\alpha)},$$

which completes the proof. □

Theorem 4.2 *Let $f \in \mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$. Then*

1. *f is starlike of order δ, $0 \leq \delta < 1$, in the disc $|z| < r_1$, where*

$$r_1 = \inf_{j\geq t+1}\left\{\frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)(j-\delta)}\right\}^{\frac{1}{t}}.$$

2. *f is convex of order δ, $0 \leq \delta < 1$, in the disc $|z| < r_2$ where,*

$$r_2 = \inf_{j\geq t+1}\left\{\frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu j(j-1)(1-\alpha)}\right\}^{\frac{1}{t}}.$$

Each of these results is sharp for the extremal function $f(z)$ given by (2.5).

Proof 1. For $0 \leq \delta < 1$ we need to show that

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\delta. \tag{4.3}$$

We have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \left|\frac{\sum_{j=t+1}^{\infty}(j-1)a_j|z|^t}{1-\sum_{j=t+1}^{\infty}a_j|z|^t}\right|.$$

The last expression is less than $1-\delta$ if

$$\sum_{j=t+1}^{\infty}\frac{(j-\delta)}{1-\delta}a_j|z|^t < 1.$$

We use the fact that $f \in \mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$ if and only if

$$\sum_{j=t+1}^{\infty}\frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)}a_j < 1.$$

Equation (4.3) holds true if

$$\frac{j-\delta}{1-\delta}|z|^t < \frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)}.$$

Or, equivalently,

$$|z|^t < \frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)(j-\delta)},$$

which yields the starlikeness of the family.

2. Using the fact that f is convex if and only zf' is starlike, we can prove (2) with a similar way of the proof of (1). The function f is convex if and only if

$$\left|zf''(z)\right| < 1 - \delta. \tag{4.4}$$

We have

$$\left|zf''(z)\right| \leq \left|\sum_{j=t+1}^{\infty} j(j-1)a_j|z|^{t-1}\right| < 1 - \delta,$$

$$\sum_{j=t+1}^{\infty} \frac{j(j-1)}{1-\delta} a_j|z|^{t-1} < 1.$$

We use the fact that $f \in \mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$ if and only if

$$\sum_{j=t+1}^{\infty} \frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)} a_j < 1.$$

Equation (4.4) holds true if

$$\frac{j(j-1)}{1-\delta}|z|^{t-1} < \frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu(1-\alpha)},$$

or, equivalently,

$$|z|^{t-1} < \frac{(1-\delta)[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}{2\beta\nu j(j-1)(1-\alpha)},$$

which yields the convexity of the family. □

5 Neighborhood property

In this section we study neighborhood property for functions in the class $\mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$.

Definition 5.1 For functions f belong to $\mathcal{A}$ of the form and $\varepsilon \geq 0$, we define $(\eta - \varepsilon)$-neighborhood of f by

$$N_\varepsilon^\eta(f) = \left\{g(z) \in \mathcal{A} : g(z) = z + \sum_{j=2}^{\infty} b_j z^j, \sum_{j=2}^{\infty} j^{\eta+1}|a_j - b_j| \leq \varepsilon\right\},$$

where η is a fixed positive integer.

By using the following lemmas we will investigate the $(\eta - \varepsilon)$-neighborhood of function in $\mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$.

Lemma 5.1 *Let* $-1 \leq \beta < 1$, *if* $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ *satisfies*

$$\sum_{j=2}^{\infty} j^{\rho+1}|b_j| \leq \frac{2\beta\nu(1-\alpha)}{1+\beta(2\nu-1)}$$

then $g(z) \in \mathcal{RI}(\gamma, \lambda, l, \alpha, \beta)$.

Proof By using of Theorem 2.1, it is sufficient to show that

$$\frac{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^{\rho}+(1-\gamma)\frac{(\rho+j-1)!}{\rho!(j-1)!}\}}{2\beta\nu(1-\alpha)}=\frac{j^{\rho+1}}{2\beta\nu(1-\alpha)}\big[1+\beta(2\nu-1)\big].$$

But

$$\frac{[1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^{\rho}+(1-\gamma)\frac{(\rho+j-1)!}{\rho!(j-1)!}\}}{2\beta\nu(1-\alpha)}\leq\frac{j^{\rho+1}}{2\beta\nu(1-\alpha)}\big[1+\beta(2\nu-1)\big].$$

Therefore it is enough to prove that

$$Q(j,\rho)=\frac{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^{\rho}+(1-\gamma)\frac{(\rho+j-1)!}{\rho!(j-1)!}}{j^{\rho+1}}\leq 1,$$

the result follows because the last inequality holds for all $j\geq t+1$. □

Lemma 5.2 *Let* $f(z)=z-\sum_{k=2}^{\infty}a_kz^k\in\mathcal{T}$, $\gamma,\lambda,l\geq 0$, $0\leq\alpha<1$, $0<\beta\leq 1$ *and* $\varepsilon\geq 0$. *If* $\frac{f(z)+\epsilon z}{1+\epsilon}\in\mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$, *then*

$$\sum_{j=t+1}^{\infty}j^{\rho+1}a_j\leq\frac{2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\rho+1}}{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^{n}+(1-\gamma)\frac{(n+t)!}{n!t!}\}},$$

where either $\rho=0$ *or* $\rho=1$. *The result is sharp with the extremal function*

$$f(z)=z-\frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^{n}+(1-\gamma)\frac{(n+t)!}{n!t!}\}}z^{t+1},\quad z\in U.$$

Proof Letting $g(z)=\frac{f(z)+\epsilon z}{1+\epsilon}$ we have

$$g(z)=z-\sum_{j=t+1}^{\infty}\frac{a_j}{1+\epsilon}z^j,\quad z\in U.$$

In view of Theorem 2.3, $g(z)=\sum_{j=1}^{\infty}\eta_jg_j(z)$ where $\eta_j\geq 0$, $\sum_{j=1}^{\infty}\eta_j=1$,

$$g_1(z)=z$$

and

$$g_j(z)=z-\frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}z^j,\quad j\geq t+1.$$

So we obtain

$$\begin{aligned}g(z)&=\eta_1z+\sum_{j=t+1}^{\infty}\eta_j\Big[z-\frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}z^j\Big]\\&=z-\sum_{j=t+1}^{\infty}\eta_k\frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^{n}+(1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}z^j.\end{aligned}$$

Since $\eta_j \geq 0$ and $\sum_{j=2}^{\infty} \eta_j \leq 1$, it follows that

$$\sum_{j=t+1}^{\infty} a_k \leq \sup_{j\geq t+1} j^{\rho+1} \frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}.$$

Since whenever $\rho = 0$ or $\rho = 1$ we conclude that

$$W(j,\rho,\gamma,\alpha,\beta,\epsilon) = j^{\rho+1} \frac{2\beta\nu(1-\alpha)(1+\epsilon)}{[1+\mu(j-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda(j-1)+l}{l+1})^n + (1-\gamma)\frac{(n+j-1)!}{n!(j-1)!}\}}$$

is a decreasing function of j, the result will follow. The proof is complete. □

Theorem 5.1 *Let $\rho = 0$ or $\rho = 1$ and suppose $0 \leq \beta < 1$ and*

$$\begin{aligned} -1 &\leq \theta \\ &< \frac{[1+\mu(t-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\} - 2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\eta+1}}{[1+\mu(t-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}}, \end{aligned}$$

$f(z) \in \mathcal{T}$ and $\frac{f(z)+\epsilon z}{1+\epsilon} \in \mathcal{RI}(\gamma,\lambda,l,\alpha,\beta)$, then the $(\eta-\varepsilon)$-neighborhood of f is the subset of $\mathcal{RI}(\lambda,\lambda,l,\alpha,\beta)$, where

$$\begin{aligned} \varepsilon &\leq 2(1-\alpha)\biggl\{\theta\gamma\bigl[1+\mu(t-1)\bigr]\bigl[1+\beta(2\nu-1)\bigr]\biggl\{\gamma\biggl(\frac{1+\lambda t+l}{l+1}\biggr)^n + (1-\gamma)\frac{(n+t)!}{n!t!}\biggr\} \\ &\quad - \beta\gamma\bigl[1+\theta(2\nu-1)\bigr](1+\epsilon)(t+1)^{\eta+1}\biggr\}\Big/\biggl(\bigl[1+\theta(2\nu-1)\bigr]\bigl[1+\mu(t-1)\bigr] \\ &\quad \times \bigl[1+\beta(2\nu-1)\bigr]\biggl\{\gamma\biggl(\frac{1+\lambda t+l}{l+1}\biggr)^n + (1-\gamma)\frac{(n+t)!}{n!t!}\biggr\}\biggr). \end{aligned}$$

The result is sharp.

Proof For $f(z) = z - \sum_{j=2}^{\infty} |a_j| z^j$, let $g(z) = z + \sum_{j=2}^{\infty} b_j z^j$ be in $N_\varepsilon^\eta(f)$. So by Lemma 5.2, we have

$$\begin{aligned} \sum_{j=2}^{\infty} j^{\eta+1}|b_j| &= \sum_{j=2}^{\infty} j^{\eta+1}|a_j - b_j - a_j| \\ &\leq \varepsilon + \frac{2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\eta+1}}{[1+\mu(t-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}}. \end{aligned}$$

By using Lemma 5.1, $g(z) \in \mathcal{L}(\gamma,\alpha,\beta)$ if

$$\varepsilon + \frac{2\beta\nu(1-\alpha)(1+\epsilon)(t+1)^{\eta+1}}{[1+\mu(t-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n + (1-\gamma)\frac{(n+t)!}{n!t!}\}} \leq \frac{2\theta\nu(1-\alpha)}{1+\theta(2\nu-1)},$$

that is, $\varepsilon \leq \frac{2(1-\alpha)\{\theta\gamma[1+\mu(t-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n+(1-\gamma)\frac{(n+t)!}{n!t!}\}-\beta\gamma[1+\theta(2\nu-1)](1+\epsilon)(t+1)^{\eta+1}\}}{[1+\theta(2\nu-1)][1+\mu(t-1)][1+\beta(2\nu-1)]\{\gamma(\frac{1+\lambda t+l}{l+1})^n+(1-\gamma)\frac{(n+t)!}{n!t!}\}}$, and the proof is complete. □

Competing interests
The author declares that she has no competing interests.

Author's contributions
The author drafted the manuscript, read and approved the final manuscript.

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