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Hybrid shrinking iterative solutions to convex feasibility problems for countable families of relatively nonexpansive mappings and a system of generalized mixed equilibrium problems

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Abstract
We propose a new hybrid shrinking iterative scheme for approximating common elements of the set of solutions to convex feasibility problems for countable families of relatively nonexpansive mappings of a set of solutions to a system of generalized mixed equilibrium problems. A strong convergence theorem is established in the framework of Banach spaces. The results extend those of other authors, in which the involved mappings consist of just finitely many ones.

MSC: 47H09; 47H10; 47J25

Keywords: relatively nonexpansive mappings; hybrid iteration scheme; convex feasibility problems; generalized mixed equilibrium problems

1 Introduction

Throughout this paper we assume that E is a real Banach space with its dual E^*, C is a nonempty, closed, convex subset of E, and $J : E \to 2^{E^*}$ is the *normalized duality mapping* defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \forall x \in E. \tag{1.1}$$

In the sequel, we use $F(T)$ to denote the set of fixed points of a mapping T. A point p in C is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}$ which converges weakly to p such that the $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of T will be denoted by $\hat{F}(T)$. A mapping $T : C \to C$ is said to be nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \tag{1.2}$$

A mapping $T : C \to C$ is said to be relatively nonexpansive if $F(T) = \hat{F}(T) \ne \emptyset$ and

$$\phi(p, Tx) \le \phi(p, x), \quad \forall x \in C, p \in F(T), \tag{1.3}$$

where $\phi : E \times E \to \mathbb{R}^1$ denotes the *Lyapunov functional* defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x,y \in E. \tag{1.4}$$

It is obvious from the definition of ϕ that

$$\bigl(\|x\| - \|y\|\bigr)^2 \leq \phi(x,y) \leq \bigl(\|x\| + \|y\|\bigr)^2, \tag{1.5}$$

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle, \tag{1.6}$$

and

$$\phi(x,y) = \langle x, Jx - Jy\rangle + \langle y - x, Jy\rangle \leq \|x\|\|Jx - Jy\| + \|y - x\|\|y\|. \tag{1.7}$$

The asymptotic behavior of a relatively nonexpansive mapping was studied in [1–4]. In 1953, Mann [5] introduced the iteration as follows: a sequence $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \tag{1.8}$$

where the initial element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a sequence of real numbers in $[0,1]$. The Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results was proved by Reich [6]. In an infinite-dimensional Hilbert space, a Mann iteration can yield only weak convergence (see [7, 8]). Attempts to modify the Mann iteration method (1.8) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [9] proposed the following modification of Mann iteration method (1.8) for a nonexpansive mapping T from C into itself in a Hilbert space: from an arbitrary $x_0 \in C$,

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)Tx_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{1.9}$$

where P_K denotes the metric projection from a Hilbert space H onto a closed convex subset K of H and proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$. A projection onto the intersection of two half-spaces is computed by solving a linear system of two equations with two unknowns (see [10, Section 3]).

Let $\theta : C \times C \to \mathbb{R}^1$ be a bifunction, $\psi : C \to \mathbb{R}^1$ a real-valued function, and $B : C \to E^*$ a nonlinear mapping. The so-called *generalized mixed equilibrium problem* (*GMEP*) is to find an $u \in C$ such that

$$\theta(u,y) + \langle y - u, Bu\rangle + \psi(y) - \psi(u) \geq 0, \quad \forall y \in C, \tag{1.10}$$

whose set of solutions is denoted by $\Omega(\theta, B, \psi)$.

The equilibrium problem is a unifying model for several problems arising in physics, engineering, science optimization, economics, transportation, network and structural analysis, Nash equilibrium problems in noncooperative games, and others. It has been shown that variational inequalities and mathematical programming problems can be viewed as a special realization of the abstract equilibrium problems. Many authors have proposed some useful methods to solve the EP (equilibrium problem), GEP (generalized equilibrium problem), MEP (mixed equilibrium problem), and GMEP.

In 2007, Plubtieng and Ungchittrakool [11] established strong convergence theorems for a common fixed point of two relatively nonexpansive mappings in a Banach space by using the following hybrid method in mathematical programming:

$$\begin{cases} x_0 = x \in C, \\ y_n = J^{-1}[\alpha_n Jx_n + (1-\alpha_n)Jz_n], \\ z_n = J^{-1}[\beta_n^{(1)} Jx_n + \beta_n^{(2)} JTx_n + \beta_n^{(3)} JSx_n], \\ H_n = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ W_n = \{z \in C : \langle x_n - z, Jx - Jy\rangle \ge 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x, \quad \forall n \in \mathbb{N} \cup \{0\}. \end{cases} \tag{1.11}$$

Their results extended and improved the corresponding ones announced by Nakajo and Takahashi [9], Martinez-Yanes and Xu [12], and Matsushita and Takahashi [4].

Recently, Su and Qin [13] modified iteration (1.9), the so-called monotone CQ method for nonexpansive mapping, as follows: from an arbitrary $x_0 \in C$,

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_0 = \{z \in C : \|y_0 - z\| \le \|x_0 - z\|\}, \qquad Q_0 = C, \\ C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_0, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{1.12}$$

and proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

Inspired and motivated by the studies mentioned above, in this paper, we use a modified hybrid iteration scheme for approximating common elements of the set of solutions to convex feasibility problem for a countable families of relatively nonexpansive mappings, of set of solutions to a system of generalized mixed equilibrium problems. A strong convergence theorem is established in the framework of Banach spaces. The results extend those of the authors, in which the involved mappings consist of just finitely many ones.

2 Preliminaries

We say that E is *strictly convex* if the following implication holds for $x, y \in E$:

$$\|x\| = \|y\| = 1, \qquad x \neq y \quad \Rightarrow \quad \left\| \frac{x+y}{2} \right\| < 1. \tag{2.1}$$

It is also said to be *uniformly convex* if for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \qquad \|x - y\| \ge \epsilon \quad \Rightarrow \quad \left\| \frac{x+y}{2} \right\| \le 1 - \delta. \tag{2.2}$$

It is well known that if E is a uniformly convex Banach space, then E is reflexive and strictly convex. A Banach space E is said to be *smooth* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3}$$

exists for each $x, y \in S(E) := \{x \in E : \|x\| = 1\}$. E is said to be *uniformly smooth* if the limit (2.3) is attained uniformly for $x, y \in S(E)$.

Following Alber [14], the *generalized projection* $P_C : E \to C$ is defined by

$$P_C = \arg\inf_{y \in C} \phi(y, x), \quad \forall x \in E. \tag{2.4}$$

Lemma 2.1 [14] *Let E be a smooth, strictly convex and reflexive Banach space and C be a nonempty, closed, convex subset of E. Then the following conclusions hold*:

(1) $\phi(x, P_C y) + \phi(P_C y, y) \le \phi(x, y)$ *for all* $x \in C$ *and* $y \in E$.
(2) *If* $x \in E$ *and* $z \in C$, *then* $z = P_C x \Leftrightarrow \langle z - y, Jx - Jz\rangle \ge 0$, $\forall y \in C$.
(3) *For* $x, y \in E$, $\phi(x, y) = 0$ *if and only if* $x = y$.

Lemma 2.2 [15] *Let E be a uniformly convex and smooth Banach space and let $r > 0$. Then there exists a continuous, strictly increasing, and convex function $h : [0, 2r] \to [0, \infty)$ such that $h(0) = 0$ and*

$$h(\|x\| - \|y\|) \le \phi(x, y) \tag{2.5}$$

for all $x, y \in B_r := \{z \in E : \|z\| \le r\}$.

Lemma 2.3 [16] *Let E be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences of E. If $\phi(x_n, y_n) \to 0$, where ϕ is the function defined by* (1.4), *and either $\{x_n\}$ or $\{y_n\}$ is bounded, then $\|x_n - y_n\| \to 0$.*

Remark 2.4 The following basic properties for a Banach space E can be found in Cioranescu [17].

(i) If E is uniformly smooth, then J is uniformly continuous on each bounded subset of E.
(ii) If E is reflexive and strictly convex, then J^{-1} is norm-weak-continuous.
(iii) If E is a smooth, strictly convex and reflexive Banach space, then the normalized duality mapping $J : E \to 2^{E^*}$ is single valued, one-to-one, and onto.
(iv) A Banach space E is uniformly smooth if and only if E^* is uniformly convex.
(v) Each uniformly convex Banach space E has the *Kadec-Klee property*, *i.e.*, for any sequence $\{x_n\} \subset E$, if $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$ as $n \to \infty$.

Lemma 2.5 [18] *Let E be a real uniformly convex Banach space and let $B_r(0)$ be the closed ball of E with center at the origin and radius $r > 0$. Then there exists a continuous strictly increasing convex function $g : [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \gamma z\|^2 \le \lambda\|x\|^2 + \mu\|y\|^2 + \gamma\|z\|^2 - \lambda\mu g(\|x - y\|) \tag{2.6}$$

for all $x, y, z \in B_r(0)$ *and* $\lambda, \mu, \gamma \in [0, 1]$ *with* $\lambda + \mu + \gamma = 1$.

Lemma 2.6 [19] *The unique solutions to the positive integer equation*

$$n = i_n + \frac{(m_n - 1)m_n}{2}, \quad m_n \ge i_n, n = 1, 2, 3, \ldots \tag{2.7}$$

are

$$i_n = n - \frac{(m_n - 1)m_n}{2}, \quad m_n = -\left[\frac{1}{2} - \sqrt{2n + \frac{1}{4}}\right], n = 1, 2, 3, \ldots, \tag{2.8}$$

where $[x]$ *denotes the maximal integer that is not larger than* x.

3 Main results

Theorem 3.1 *Let* E *be a real uniformly smooth and strictly convex Banach space, and* C *be a nonempty, closed, convex subset of* E. *Let* $\{T_i\} : C \to C$ *and* $\{S_i\} : C \to C$ *be two sequences of relatively nonexpansive mappings with* $F := \bigcap_{i=1}^{\infty}(F(T_i) \cap F(S_i)) \neq \emptyset$. *Let* $\{x_n\}$ *be the sequence generated by*

$$\begin{cases} x_0 = x \in C, \qquad H_{-1} = W_{-1} = C, \\ y_n = J^{-1}[\lambda_n Jx_n + (1-\lambda_n)Jz_n], \\ z_n = J^{-1}[\alpha_n Jx_n + \beta_n JT_{i_n}x_n + \gamma_n JS_{i_n}x_n], \\ H_n = \{z \in H_{n-1} \cap W_{n-1} : \phi(z, y_n) \le \phi(z, x_n)\}, \\ W_n = \{z \in H_{n-1} \cap W_{n-1} : \langle x_n - z, Jx - Jy\rangle \ge 0\}, \\ x_{n+1} = P_{H_n \cap W_n}x, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{3.1}$$

where $\{\lambda_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, *and* $\{\gamma_n\}$ *are sequences in* $[0,1]$ *satisfying*

(1) $0 \le \lambda_n < 1$, $\forall n \in \mathbb{N} \cup \{0\}$; $\limsup_{n\to\infty} \lambda_n < 1$;

(2) $\alpha_n + \beta_n + \gamma_n = 1$; $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} \beta_n\gamma_n > 0$;

and i_n *is the solution to the positive integer equation* $n = i_n + \frac{(m_n-1)m_n}{2}$ $(m_n \ge i_n, n = 1, 2, \ldots)$, *that is, for each* $n \ge 1$, *there exists a unique* i_n *such that*

$$i_1 = 1, \quad i_2 = 1, \quad i_3 = 2, \quad i_4 = 1, \quad i_5 = 2, \quad i_6 = 3,$$
$$i_7 = 1, \quad i_8 = 2, \quad i_9 = 3, \quad i_{10} = 4, \quad i_{11} = 1, \quad \ldots.$$

Then $\{x_n\}$ *converges strongly to* $P_F x$, *where* $P_F x$ *is the generalized projection from* C *onto* F.

Proof We divide the proof into several steps.

(I) H_n and W_n ($\forall n \in \mathbb{N} \cup \{0\}$) both are closed and convex subsets in C.

This follows from the fact that $\phi(z, y_n) \le \phi(z, x_n)$ is equivalent to

$$2\langle z, Jx_n - Jy_n\rangle \le \|x_n\|^2 - \|y_n\|^2. \tag{3.2}$$

(II) F is a subset of $\bigcap_{n=0}^{\infty}(H_n \cap W_n)$.

In fact, we note by [4, Proposition 2.4] that for each $i \ge 1$, $F(S_i)$ and $F(T_i)$ are closed convex sets and so is F. It is clear that $F \subset C = H_{-1} \cap W_{-1}$. Suppose that $F \subset C_{n-1} \cap Q_{n-1}$ for some $n \in \mathbb{N}$. For any $u \in F$, by the convexity of $\|\cdot\|^2$, we have

$$\begin{aligned} \phi(u, z_n) &= \phi\bigl(u, J^{-1}[\alpha_n Jx_n + \beta_n JT_{i_n}x_n + \gamma_n JS_{i_n}x_n]\bigr) \\ &= \|u\|^2 - 2\langle u, \alpha_n Jx_n + \beta_n JT_{i_n}x_n + \gamma_n JS_{i_n}x_n\rangle \\ &\quad + \|\alpha_n Jx_n + \beta_n JT_{i_n}x_n + \gamma_n JS_{i_n}x_n\|^2 \\ &\le \|u\|^2 - 2\alpha_n\langle u, Jx_n\rangle - 2\beta_n\langle u, JT_{i_n}x_n\rangle - 2\gamma_n\langle u, JS_{i_n}x_n\rangle \end{aligned}$$

$$\begin{aligned}&+\alpha_n\|x_n\|^2+\beta_n\|T_{i_n}x_n\|^2+\gamma_n\|S_{i_n}x_n\|^2\\&=\alpha_n\phi(u,x_n)+\beta_n\phi(u,T_{i_n}x_n)+\gamma_n\phi(u,S_{i_n}x_n)\\&\le\alpha_n\phi(u,x_n)+\beta_n\phi(u,x_n)+\gamma_n\phi(u,x_n)\\&=\phi(u,x_n),\end{aligned}\tag{3.3}$$

and then

$$\begin{aligned}\phi(u,y_n)&=\phi\bigl(u,J^{-1}\bigl[\lambda_nJx_n+(1-\lambda_n)Jz_n\bigr]\bigr)\\&=\|u\|^2-2\bigl\langle u,\lambda_nJx_n+(1-\lambda_n)Jz_n\bigr\rangle+\bigl\|\lambda_nJx_n+(1-\lambda_n)Jz_n\bigr\|^2\\&\le\|u\|^2-2\lambda_n\langle u,Jx_n\rangle-2(1-\lambda_n)\langle u,Jz_n\rangle+\lambda_n\|x_n\|^2+(1-\lambda_n)\|z_n\|^2\\&=\lambda_n\bigl(\|u\|^2-2\langle u,Jx_n\rangle+\|x_n\|^2\bigr)+(1-\lambda_n)\bigl(\|u\|^2-2\langle u,Jz_n\rangle+\|z_n\|^2\bigr)\\&=\lambda_n\phi(u,x_n)+(1-\lambda_n)\phi(u,z_n)\\&\le\lambda_n\phi(u,x_n)+(1-\lambda_n)\phi(u,x_n)\\&=\phi(u,x_n).\end{aligned}\tag{3.4}$$

This implies that $F\subset H_n$. It follows from $x_n=P_{H_{n-1}\cap W_{n-1}}x$ and Lemma 2.1(2) that

$$\langle x_n-z,Jx-Jx_n\rangle\ge 0,\quad\forall z\in H_{n-1}\cap W_{n-1}.\tag{3.5}$$

Particularly,

$$\langle x_n-z,Jx-Jx_n\rangle\ge 0,\quad\forall u\in F,\tag{3.6}$$

and hence $F\subset W_n$, which yields $F\subset H_n\cap W_n$. By induction, $F\subset\bigcap_{n=0}^{\infty}(H_n\cap W_n)$.

(III) $\lim_{n\to\infty}\|x_n-T_{i_n}x_n\|=\lim_{n\to\infty}\|x_n-S_{i_n}x_n\|=0$.

In view of $x_{n+1}=P_{H_n\cap W_n}x\in H_n$ and the definition of H_n, we also have

$$\phi(x_{n+1},y_n)\le\phi(x_{n+1},x_n),\quad\forall n\in\mathbb{N}.\tag{3.7}$$

This implies that

$$\lim_{n\to\infty}\phi(x_{n+1},y_n)=\lim_{n\to\infty}\phi(x_{n+1},x_n)=0.\tag{3.8}$$

It follows from Lemma 2.2 that

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=\lim_{n\to\infty}\|x_{n+1}-x_n\|=0.\tag{3.9}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|Jx_{n+1}-Jy_n\|=\lim_{n\to\infty}\|Jx_{n+1}-Jx_n\|=0\tag{3.10}$$

and

$$\|Jx_{n+1}-Jy_n\|\ge(1-\lambda_n)\|Jx_{n+1}-Jz_n\|-\lambda_n\|Jx_{n+1}-Jx_n\|,\quad\forall n\in\mathbb{N}\cup\{0\}.\tag{3.11}$$

This implies that

$$\begin{aligned}\|Jx_{n+1} - Jz_n\| &\le \frac{1}{1-\lambda_n}\left(\|Jx_{n+1} - Jy_n\| + \lambda_n\|Jx_{n+1} - Jx_n\|\right)\\ &\le \frac{1}{1-\lambda_n}\left(\|Jx_{n+1} - Jy_n\| + \|Jx_{n+1} - Jx_n\|\right).\end{aligned} \tag{3.12}$$

From (3.10) and $\limsup_{n\to\infty} \lambda_n < 1$, we have $\lim_{n\to\infty} \|Jx_{n+1} - Jz_n\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - z_n\| = \lim_{n\to\infty} \left\|J^{-1}(Jx_{n+1}) - J^{-1}(Jz_n)\right\| = 0. \tag{3.13}$$

From $\|x_n - z_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1} - z_n\|$ we have $\lim_{n\to\infty} \|x_n - z_n\| = 0$. Since $\{x_n\}$ is bounded, $\phi(p, T_{i_n}x_n) \le \phi(p, x_n)$ and $\phi(p, S_{i_n}x_n) \le \phi(p, x_n)$ for any $p \in F$. We also find that $\{Jx_n\}$, $\{JT_{i_n}x_n\}$ and $\{JS_{i_n}x_n\}$ are bounded, and then there exists an $r > 0$ such that $\{Jx_n\}, \{JT_{i_n}x_n\}, \{JS_{i_n}x_n\} \subset B_r(0)$. Therefore Lemma 2.5 is applicable and we observe that

$$\begin{aligned}\phi(p, z_n) &= \|p\|^2 - 2\langle p, \alpha_n Jx_n + \beta_n JT_{i_n}x_n + \gamma_n JS_{i_n}x_n\rangle\\ &\quad + \|\alpha_n Jx_n + \beta_n JT_{i_n}x_n + \gamma_n JS_{i_n}x_n\|^2\\ &\le \|p\|^2 - 2\alpha_n\langle p, Jx_n\rangle - 2\beta_n\langle p, JT_{i_n}x_n\rangle - 2\gamma_n\langle p, JS_{i_n}x_n\rangle\\ &\quad + \alpha_n\|x_n\|^2 + \beta_n\|T_{i_n}x_n\|^2 + \gamma_n\|S_{i_n}x_n\|^2 - \beta_n\gamma_n g\left(\|JT_{i_n}x_n - JS_{i_n}x_n\|\right)\\ &= \alpha_n\phi(p, x_n) + \beta_n\phi(p, T_{i_n}x_n) + \gamma_n\phi(p, S_{i_n}x_n) - \beta_n\gamma_n g\left(\|JT_{i_n}x_n - JS_{i_n}x_n\|\right)\\ &\le \phi(p, x_n) - \beta_n\gamma_n g\left(\|JT_{i_n}x_n - JS_{i_n}x_n\|\right).\end{aligned} \tag{3.14}$$

That is,

$$\beta_n\gamma_n g\left(\|JT_{i_n}x_n - JS_{i_n}x_n\|\right) \le \phi(p, x_n) - \phi(p, z_n), \tag{3.15}$$

where $g : [0,\infty) \to [0,\infty)$ is a continuous strictly convex function with $g(0) = 0$.

Let $\{\|T_{i_{n_k}}x_{n_k} - S_{i_{n_k}}x_{n_k}\|\}$ be any subsequence of $\{\|T_{i_n}x_n - S_{i_n}x_n\|\}$. Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_{n_k}\}$ such that for any $p \in F$,

$$\lim_{j\to\infty} \phi(p, x_{n_j}) = \limsup_{k\to\infty} \phi(p, x_{n_k}) := a. \tag{3.16}$$

From (1.6) we have

$$\begin{aligned}\phi(p, x_{n_j}) &= \phi(p, z_{n_j}) + \phi(z_{n_j}, x_{n_j}) + 2\langle p - z_{n_j}, Jz_{n_j} - Jx_{n_j}\rangle\\ &\le \phi(p, z_{n_j}) + \phi(z_{n_j}, x_{n_j}) + M\|Jz_{n_j} - Jx_{n_j}\|\end{aligned} \tag{3.17}$$

for some appropriate constant $M > 0$. Since

$$\lim_{j\to\infty} \phi(z_{n_j}, x_{n_j}) = 0 = \lim_{j\to\infty} \|Jz_{n_j} - Jx_{n_j}\|, \tag{3.18}$$

it follows that

$$a = \liminf_{j\to\infty} \phi(p, x_{n_j}) \le \liminf_{j\to\infty} \phi(p, z_{n_j}). \tag{3.19}$$

From (3.3), we have

$$\limsup_{j\to\infty} \phi(p, z_{n_j}) \le \limsup_{j\to\infty} \phi(p, x_{n_j}) = a \tag{3.20}$$

and hence $\lim_{j\to\infty} \phi(p, x_{n_j}) = a = \lim_{j\to\infty} \phi(p, z_{n_j})$. By (3.15), we observe that, as $j \to \infty$,

$$\beta_{n_j}\gamma_{n_j} g\big(\|JT_{i_{n_j}}x_{n_j} - JS_{i_{n_j}}x_{n_j}\|\big) \le \phi(p, x_{n_j}) - \phi(p, z_{n_j}) \to 0. \tag{3.21}$$

Since $\liminf_{n\to\infty} \beta_n\gamma_n > 0$, it follows that $\lim_{j\to\infty} g(\|JT_{i_{n_j}}x_{n_j} - JS_{i_{n_j}}x_{n_j}\|) = 0$. By the properties of the mapping g, we have $\lim_{j\to\infty} \|JT_{i_{n_j}}x_{n_j} - JS_{i_{n_j}}x_{n_j}\| = 0$. Since J^{-1} is also uniformly norm-to-norm continuous on bounded sets, we obtain

$$\lim_{j\to\infty} \|T_{i_{n_j}}x_{n_j} - S_{i_{n_j}}x_{n_j}\| = \lim_{j\to\infty} \big\|J^{-1}(JT_{i_{n_j}}x_{n_j}) - J^{-1}(JS_{i_{n_j}}x_{n_j})\big\| = 0, \tag{3.22}$$

and then $\lim_{n\to\infty} \|T_{i_n}x_n - S_{i_n}x_n\| = 0$. Next, we note by the convexity of $\|\cdot\|^2$ and (1.7) that, as $n \to \infty$,

$$\begin{aligned}
\phi(T_{i_n}x_n, z_n) &= \|T_{i_n}x_n\|^2 - 2\langle T_{i_n}x_n, \alpha_n Jx_n + \beta_n JT_{i_n}x_n + \gamma_n JS_{i_n}x_n\rangle \\
&\quad + \|\alpha_n Jx_n + \beta_n JT_{i_n}x_n + \gamma_n JS_{i_n}x_n\|^2 \\
&\le \|T_{i_n}x_n\|^2 - 2\alpha_n\langle T_{i_n}x_n, Jx_n\rangle - 2\beta_n\langle T_{i_n}x_n, JT_{i_n}x_n\rangle - 2\gamma_n\langle T_{i_n}x_n, JS_{i_n}x_n\rangle \\
&\quad + \alpha_n\|x_n\|^2 + \beta_n\|T_{i_n}x_n\|^2 + \gamma_n\|S_{i_n}x_n\|^2 \\
&= \alpha_n\phi(T_{i_n}x_n, x_n) + \beta_n\phi(T_{i_n}x_n, S_{i_n}x_n) \to 0,
\end{aligned} \tag{3.23}$$

since $\alpha_n \to 0$. By Lemma 2.3, we have $\lim_{n\to\infty} \|T_{i_n}x_n - z_n\| = 0$ and hence

$$\|T_{i_n}x_n - x_n\| \le \|T_{i_n}x_n - z_n\| + \|z_n - x_n\| \to 0 \tag{3.24}$$

as $n \to \infty$. Moreover, we observe that

$$\|S_{i_n}x_n - x_n\| \le \|S_{i_n}x_n - T_{i_n}x_n\| + \|T_{i_n}x_n - x_n\| \to 0 \tag{3.25}$$

as $n \to \infty$.

(IV) $x_n \to P_F x$ as $n \to \infty$.

It follows from the definition of W_n and Lemma 2.1(2) that $x_n = P_{W_n}x$. Since $x_{n+1} = P_{H_n\cap W_n}x \in W_n$, we have

$$\phi(x_n, x) \le \phi(x_{n+1}, x), \quad \forall n \ge 1. \tag{3.26}$$

Therefore, $\{\phi(x_n, x)\}$ is nondecreasing. Using $x_n = P_{W_n}x$ and Lemma 2.1(1), we have

$$\phi(x_n, x) = \phi(P_{W_n}x, x) \le \phi(p, x) - \phi(p, x_n) \le \phi(p, x) \tag{3.27}$$

for all $p \in F$ and for all $n \in \mathbb{N}$, that is, $\{\phi(x_n, x)\}$ is bounded. Then

$$\lim_{n\to\infty} \phi(x_n, x) \text{ exists.} \tag{3.28}$$

In particular, by (1.5), the sequence $\{(\|x_n\| - \|x\|)^2\}$ is bounded. This implies that $\{x_n\}$ is bounded. Note again that $x_n = P_{W_n}x$ and for any positive integer k, $x_{n+k} \in W_{n+k-1} \subset W_n$. By Lemma 2.1(1),

$$\begin{aligned}\phi(x_{n+k}, x_n) &= \phi(x_{n+k}, P_{W_n}x)\\ &\le \phi(x_{n+k}, x) - \phi(P_{W_n}x, x)\\ &= \phi(x_{n+k}, x) - \phi(x_n, x).\end{aligned} \tag{3.29}$$

By Lemma 2.2, we have, for $m, n \in \mathbb{N}$ with $m > n$,

$$h(\|x_m - x_n\|) \le \phi(x_m, x_n) \le \phi(x_m, x) - \phi(x_n, x), \tag{3.30}$$

where $h : [0,\infty) \to [0,\infty)$ is a continuous, strictly increasing, and convex function with $h(0) = 0$. Then the properties of the function g show that $\{x_n\}$ is a Cauchy sequence in C, so there exists $x^* \in C$ such that

$$x_n \to x^* \quad (n \to \infty). \tag{3.31}$$

Now, set $\mathbb{N}_i = \{k \in \mathbb{N} : k = i + \frac{(m-1)m}{2}, m \ge i, m \in \mathbb{N}\}$ for each $i \in \mathbb{N}$. Note that $T_{i_k} = T_i$ and $S_{i_k} = S_i$ whenever $k \in \mathbb{N}_i$. By Lemma 2.6 and the definition of $\mathbb{N}_i$, we have $\mathbb{N}_1 = \{1, 2, 4, 7, 11, 16, \dots\}$ and $i_1 = i_2 = i_4 = i_7 = i_{11} = i_{16} = \cdots = 1$. Then it follows from (3.15) and (3.24) that

$$\lim_{\mathbb{N}_i \ni k\to\infty} \|T_i x_k - x_k\| = \lim_{\mathbb{N}_i \ni k\to\infty} \|S_i x_k - x_k\| = 0, \quad \forall i \in \mathbb{N}. \tag{3.32}$$

It then immediately follows from (3.31) and (3.32) that $x^* \in F(T_i) \cap F(S_i)$ for each $i \in \mathbb{N}$ and hence $x^* \in F$.

Put $u = P_F x$. Since $u \in F \subset H_n \cap W_n$ and $x_{n+1} = P_{H_n \cap W_n} x$, we have $\phi(x_{n+1}, x) \le \phi(u, x)$, $\forall n \in \mathbb{N}$. Then

$$\phi(x^*, x) = \lim_{n\to\infty} \phi(x_{n+1}, x) \le \phi(u, x), \tag{3.33}$$

which implies that $x^* = u$ since $u = P_F x$, and hence $x_n \to x^* = P_F x$ as $n \to \infty$. This completes the proof. □

Remark 3.2 Note that the algorithm (3.1) is based on the projection onto an intersection of two closed and convex sets. We first give an example [20] of how to compute such a projection onto the intersection of two half-spaces.

Let H be a Hilbert space and suppose that $(x, y, z) \in H^3$ satisfies

$$\{w \in H : \langle w - y, x - y\rangle \le 0\} \cap \{w \in H : \langle w - z, y - z\rangle \le 0\} \ne \emptyset. \tag{3.34}$$

Set

$$\pi = \langle x - y, y - z\rangle, \qquad \mu = \|x - y\|^2, \qquad \nu = \|y - z\|^2, \qquad \rho = \mu\nu - \pi^2, \tag{3.35}$$

and

$$Q(x,y,z) = \begin{cases} z, & \text{if } \rho = 0 \text{ and } \pi \geq 0; \\ x + (1 + \pi/\nu)(z - y), & \text{if } \rho > 0 \text{ and } \pi\nu \geq \rho; \\ y + (\nu/\rho)(\pi(x - y) + \mu(z - y)), & \text{if } \rho > 0 \text{ and } \pi\nu < \rho. \end{cases} \tag{3.36}$$

In [21], Haugazeau introduced the operator Q as an explicit description of the projector onto the intersection of the two half-spaces defined in (3.34). He proved in [21] that the sequence $\{y_n\}$ defined by $y_0 = x$ and

$$(\forall n \in \mathbb{N}) \quad y_{n+1} = Q\bigl(x, Q(x, y_n, P_B y_n), P_A Q(x, y_n, P_B y_n)\bigr) \tag{3.37}$$

converges strongly to $P_C x$.

Since the algorithm (3.1) involves the projection onto the intersection of two convex sets not necessarily half-spaces, we next give an example [22] to explain and illustrate how the projection is calculated in the general convex case.

Dykstra's algorithm Let $\Omega_1, \Omega_2, \dots, \Omega_p$ be closed and convex subsets of $\mathbb{R}^n$. For any $i = 1, 2, \dots, p$ and $x^0 \in \mathbb{R}^n$, the sequences $\{x_i^k\}$ are defined by the following recursive formulas:

$$\begin{cases} x_0^k = x_p^{k-1}, \\ x_i^k = P_{\Omega_i}(x_{i-1}^k - y_i^{k-1}), \quad i = 1, 2, \dots, p, \\ y_i^k = x_i^k - (x_{i-1}^k - y_i^{k-1}), \quad i = 1, 2, \dots, p, \end{cases} \tag{3.38}$$

for $k = 1, 2, \dots$ with initial values $x_p^0 = x^0$ and $y_i^0 = 0$ for $i = 1, 2, \dots, p$. If $\Omega := \bigcap_{i=1}^p \Omega_i \neq \emptyset$, then $\{x_i^k\}$ converges to $x^* = P_\Omega(x^0)$, where $P_\Omega(x) := \arg\inf_{y \in \Omega} \|y - x\|^2$, $\forall x \in \mathbb{R}^n$.

Note Another iterative method termed HAAR (*Haugazeau-like Averaged Alternating Reflections*) for finding the projection onto intersection of finitely many closed convex sets in a Hilbert space can be found in [20, Remark 3.4(iii)].

4 Applications

The so-called convex feasibility problem for a family of mappings $\{T_i\}_{i=1}^\infty$ is to find a point in the nonempty intersection $\bigcap_{i=1}^\infty F(T_i)$.

Note Although the problem mentioned above is indeed a convex feasibility problem, it is mainly referred to the finite case.

Let E be a smooth, strictly convex, and reflexive Banach space, and C be a nonempty, closed, convex subset of E. Let $\{B_i\}_{i=1}^\infty : C \to E^*$ be a sequence of β_i-inverse strongly monotone mappings, $\{\psi\}_{i=1}^\infty : C \to \mathbb{R}^1$ a sequence of lower semi-continuous and convex functions, and $\{\theta_i\}_{i=1}^\infty : C \times C \to \mathbb{R}^1$ a sequence of bifunctions satisfying the conditions:

(A_1) $\theta(x, x) = 0$;

(A_2) θ is monotone, *i.e.*, $\theta(x,y) + \theta(y,x) \leq 0$;
(A_3) $\limsup_{t\downarrow 0} \theta(x + t(z - x), y) \leq \theta(x,y)$;
(A_4) the mapping $y \mapsto \theta(x,y)$ is convex and lower semicontinuous.

A system of generalized mixed equilibrium problems (*GMEP*) for $\{\theta_i\}_{i=1}^{\infty}$, $\{B_i\}_{i=1}^{\infty}$ and $\{\psi_i\}_{i=1}^{\infty}$ is to find an $x^* \in C$ such that

$$\theta_i(x^*, y) + \langle y - x^*, B_i x^* \rangle + \psi_i(y) - \psi_i(x^*) \geq 0, \quad \forall y \in C, i \in \mathbb{N}, \tag{4.1}$$

whose set of common solutions is denoted by $\Omega := \bigcap_{i=1}^{\infty} \Omega_i$, where Ω_i denotes the set of solutions to generalized mixed equilibrium problem for θ_i, B_i, and ψ_i.

Define a countable family of mappings $\{S_{r,i}\}_{i=1}^{\infty} : E \to C$ with $r > 0$ as follows:

$$S_{r,i}(x) = \left\{ z \in C : \tau_i(z,y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \forall y \in C \right\}, \quad \forall i \in \mathbb{N}, \tag{4.2}$$

where $\tau_i(x,y) = \theta_i(x,y) + \langle y - x, B_i x \rangle + \psi_i(y) - \psi_i(x)$, $\forall x, y \in C$, $i \in \mathbb{N}$. It has been shown by Zhang [23] that

(1) $\{S_{r,i}\}_{i=1}^{\infty}$ is a sequence of single-valued mappings;
(2) $\{S_{r,i}\}_{i=1}^{\infty}$ is a sequence of closed relatively nonexpansive mappings;
(3) $\bigcap_{i=1}^{\infty} F(S_{r,i}) = \Omega$.

Theorem 4.1 *Let E be a smooth, strictly convex, and reflexive Banach space, and C be a nonempty, closed, convex subset of E. Let $\{T_i\}_{i=1}^{\infty} : C \to C$ be a sequence of relatively nonexpansive mappings and $\{S_{r,i}\}_{i=1}^{\infty} : C \to C$ be a sequence of mappings defined by* (4.2) *with $F := \bigcap_{i=1}^{\infty}(F(T_i) \cap F(S_{r,i})) \neq \emptyset$. Let $\{x_n\}$ be the sequence generated by*

$$\begin{cases} x_0 = x \in C, \qquad H_{-1} = W_{-1} = C, \\ y_n = J^{-1}[\lambda_n Jx_n + (1 - \lambda_n)Jz_n], \\ z_n = J^{-1}[\alpha_n Jx_n + \beta_n JT_{i_n}x_n + \gamma_n JS_{r,i_n}x_n], \\ H_n = \{z \in H_{n-1} \cap W_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ W_n = \{z \in H_{n-1} \cap W_{n-1} : \langle x_n - z, Jx - Jy \rangle \geq 0\}, \\ x_{n+1} = P_{H_n \cap W_n} x, \quad \forall n \in \mathbb{N} \cup \{0\}, \end{cases} \tag{4.3}$$

where $\{\lambda_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $[0,1]$ *satisfying*

(1) $0 \leq \lambda_n < 1$, $\forall n \in \mathbb{N} \cup \{0\}$; $\limsup_{n\to\infty} \lambda_n < 1$;
(2) $\alpha_n + \beta_n + \gamma_n = 1$; $\lim_{n\to\infty} \alpha_n = 0$ *and* $\liminf_{n\to\infty} \beta_n \gamma_n > 0$;

and i_n satisfies the equation $n = i_n + \frac{(m_n - 1)m_n}{2}$ ($m_n \geq i_n$, $n = 1, 2, \ldots$). Then $\{x_n\}$ converges strongly to $P_F x$, which is some common solution to the convex feasibility problem for $\{T_i\}_{i=1}^{\infty}$ and a system of generalized mixed equilibrium problems for $\{S_{r,i}\}_{i=1}^{\infty}$.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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