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FINITE AG-GROUPOID WITH LEFT IDENTITY AND LEFT ZERO

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ABSTRACT. A groupoid G whose elements satisfy the left invertive law: $(ab)c = (cb)a$ is known as Abel-Grassman's groupoid (AG-groupoid). It is a nonassociative algebraic structure midway between a groupoid and a commutative semigroup. In this note, we show that if G is a finite AG-groupoid with a left zero then, under certain conditions, G without the left zero element is a commutative group.

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1. Preliminaries. An Abel-Grassman's groupoid [6], abbreviated as AG-groupoid, is a groupoid G whose elements satisfy the left invertive law: $(ab)c = (cb)a$. It is also called a left almost semigroup [2, 3, 4, 5]. In [1], the same structure is called left invertive groupoid. In this note we call it AG-groupoid.

It is a nonassociative algebraic structure midway between a groupoid and a commutative semigroup. The structure is medial [5], that is, $(ab)(cd) = (ac)(bd)$ for all $a, b, c, d \in G$. It has been shown in [5] that if an AG-groupoid contains a left identity then it is unique. It has been proved also that an AG-groupoid with right identity is a commutative monoid, that is, a semigroup with identity element. An element a_0 of an AG-groupoid G is called a left (right) zero if $a_0a = a_0$ $(aa_0 = a_0)$ for all $a \in G$.

Let a, b, c, and d belong to an AG-groupoid with left identity and $ab = cd$. Then it has been shown in [5] that $ba = dc$.

An element a^{-1} of an AG-groupoid with left identity e is called a left inverse if $a^{-1}a = e$. It has been shown in [5] that if a^{-1} is a left inverse of a then it is unique and is also the right inverse of a.

If for all a, b, c in an AG-groupoid G, $ab = ac$ implies that $b = c$, then G is known as left cancellative. Similarly, if $ba = ca$, implies that $b = c$, then G is called right cancellative. It is known [5] that every left cancellative AG-groupoid is right cancellative but the converse is not true. However, every right cancellative AG-groupoid with left identity is left cancellative.

In this note, we show that if G is a finite AG-groupoid with left identity and a left zero a_0, under certain conditions $G\backslash\{a_0\}$ is a commutative group without a left zero.

2. Results. We need the following theorem from [4] for our main result.

THEOREM 2.1 [4]. *A cancellative AG-groupoid G is a commutative semigroup if $a(bc) = (cb)a$ for all $a, b, c \in G$.*

We now state and prove our main result.

Theorem 2.2. *Let $(G,\circ)$ be a finite AG-groupoid with at least two elements. Suppose that it contains a left identity and a left zero a_0. Then $G^0 = G\backslash\{a_0\}$ is a commutative group under the binary operation $(\circ)$ provided there is another binary operation $(*)$ such that*

(i) $(G,*)$ *is an AG-groupoid with left identity and left inverses,*
(ii) $a_0 * a = a$, *for all* $a \in G$,
(iii) $(a*b)\circ c = (a\circ c)*(b\circ c)$, *for all* $a,b,c \in G$,
(iv) $a\circ b = a_0$ *implies that either* $a = a_0$ *or* $b = a_0$ *for all* $a,b \in G$,
(v) $a\circ(b\circ c) = (c\circ b)\circ a$, *for all* a, b, $c \in G$.

Proof. Suppose that $G = \{a_0, a_1, \dots, a_m\}$, where m is a positive integer, is an AG-groupoid with left identity under the binary operation $(\circ)$. Let e be the identity element of G. It is certainly different from a_0 because of (ii) and because a_0 is the left zero under $(\circ)$. The left invertive law together with (iv) implies that $(a\circ a_0)\circ e = (e\circ a_0)\circ a = a_0\circ a = a_0$, where $e \neq a_0$. That is,

$$a_0\circ a = a\circ a_0 = a_0. \tag{2.1}$$

Now consider the subset G^0 of G which is obtained from it by deleting a_0, so that $G^0 = \{a_i : i = 1,2,\dots,m\}$. In view of the facts that a_0 is a zero under the binary operation $(\circ)$ and it is the left identity under $(*)$ and that $(G,\circ)$ is a finite AG-groupoid with left identity. $(G^0,\circ)$ is also a finite AG-groupoid with left identity having the same e as the left identity in which all elements are distinct.

We now examine whether an element a of G^0 has an inverse in G^0 under $(\circ)$ or not. We construct a set $H_k = \{a_k\circ a_1, a_k\circ a_2, \dots, a_k\circ a_m\}$, where $a_k \neq a_0$. If $a_k = a_0$, then because a_0 is a left zero in G under $(\circ)$ and the left identity under $(*)$, the ultimate form of the set H_k will be $\{a_0\}$. Therefore it validates our supposition that $a_k \neq a_0$.

We assert that H_k contains m elements. Suppose otherwise and let

$$a_k\circ a_r = a_k\circ a_s, \tag{2.2}$$

for some $r,s = 1,2,\dots,m$ and $r \neq s$. Since H_k is an AG-groupoid with left identity under $(\circ)$, therefore (2.2) implies that

$$a_r\circ a_k = a_s\circ a_k, \tag{2.3}$$

for some $r,s = 1,2,\dots,m$ and $r \neq s$. Consider now the element $(a_s * a_r^{-1})\circ a_k$, which is certainly an element of G, where a_r^{-1} is the left inverse of a_r in G with respect to $(*)$. Now,

$$\begin{aligned}(a_s * a_r^{-1})\circ a_k &= (a_s\circ a_k)*(a_r^{-1}\circ a_k) = (a_r\circ a_k)*(a_r^{-1}\circ a_k)\\ &= (a_r * a_r^{-1})\circ a_k = a_0\circ a_k = a_0.\end{aligned} \tag{2.4}$$

Because of (iii), equation (2.3) and the facts that a_r^{-1} is the inverse of a_r under $(*)$. Thus $(a_s * a_r^{-1})\circ a_k = a_0$. Since $a_k \neq a_0$, therefore because of (iv), $a_s * a_r^{-1} = a_0$. Next $(a_s * a_r^{-1})\circ a_r = a_0 * a_r$ implies that $(a_s * a_r^{-1})\circ a_r = a_r$ because a_0 is the left identity in G under $(*)$. Hence, $a_r = (a_s * a_r^{-1}) * a_r = (a_r * a_r^{-1}) * a_s = a_0 * a_s = a_s$, that is, $a_r = a_s$. Since $|H_k| = m$, therefore the result $a_r = a_s$ contradicts our assumption; thus proving that H_k contains distinct elements. Since H_k is contained in G^0 and $|G^0| = m$ we have $H_k = G^0$.

Also, since G^0 is an AG-groupoid under $(\circ)$ with the left identity e, so is H_k and hence H_k contains the left identity e. So, e will be of the form $a_i \circ a_j$, that is, $e = a_i \circ a_j$ implying that a_i is the left inverse of a_j under the binary operation $(\circ)$. But in an AG-groupoid with left identity, if it contains left inverses, every left inverse is a right inverse. Thus a_j is the right inverse of a_j under $(\circ)$.

Since $k = 1, 2, \ldots, m$ has been chosen arbitrarily, we have shown that G^0 is an AG-groupoid with left identity and inverses under the binary operation $(\circ)$.

If $a_i, a_j, a_k \in G^0$ such that $a_i \circ a_k = a_j \circ a_k$, then $(a_i \circ a_k) \circ a_k^{-1} = (a_j \circ a_k) \circ a_k^{-1}$ implies that $(a_k^{-1} \circ a_k) \circ a_i = (a_k^{-1} \circ a_k) \circ a_j$ and so $a_i = a_j$. Thus G^0 is right cancellative under $(\circ)$. But G^0 being right cancellative under $(\circ)$, is left cancellative also, therefore G^0 is cancellative. Since G^0 is cancellative whose elements satisfy condition (v), therefore by applying Theorem 2.1, we conclude that G^0 is a commutative group under $(\circ)$. □

Corollary 2.3. *If $(G, \circ)$ is a finite AG-groupoid with left identity and a left zero a_0, then $(G \backslash \{a_0\}, \circ)$ is a cancellative AG-groupoid with left identity and inverses provided there is another binary operation $(*)$ such that*

(i) *$(G, *)$ is an AG-groupoid with left identity and left inverses,*
(ii) *$a_0 * a = a$, for all $a \in G$,*
(iii) *$(a * b) \circ c = (a \circ c) * (b \circ c)$, for all $a, b, c \in G$,*
(iv) *$a \circ b = a_0$ implies that either $a = a_0$ or $b = a_0$ for all a, $b \in G$.*

Proof. The proof is analogous to the proof of Theorem 2.2. □

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