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Research Article

Composition Formulas of Bessel-Struve Kernel Function

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The object of this paper is to study and develop the generalized fractional calculus operators involving Appell's function $F_3(\cdot)$ due to Marichev-Saigo-Maeda. Here, we establish the generalized fractional calculus formulas involving Bessel-Struve kernel function $S_\alpha(\lambda z)$, $\lambda, z \in \mathbb{C}$ to obtain the results in terms of generalized Wright functions. The representations of Bessel-Struve kernel function in terms of exponential function and its relation with Bessel and Struve function are also discussed. The pathway integral representations of Bessel-Struve kernel function are also given in this study.

1. Introduction

Fractional calculus has found applications in various and extensive fields of engineering and science such as electromagnetics, fluid mechanics, signals processing, and control theory. It has been used to model physical and engineering processes that are found to be best described by fractional differential equations. Recent researches observed that the solutions of fractional-order differential equations could model real-life situations better, particularly in reaction-diffusion-type problems. Due to the potential applicability to a wide variety of problems, fractional calculus is developed to a large area of mathematics and other engineering applications [1–4]. The fractional integral operator involving several special functions has found great importance and applications in many subfields such as statistical distribution theory, control theory, fluid dynamics, stochastic dynamical system, nonlinear biological systems, astrophysics, and quantum mechanics (see [5–7]).

The influence of fractional integral operators involving various special functions in fractional calculus is very important due to its significance and applications in various subfields of applied mathematical analysis. Many studies related to the fractional calculus are found in the papers of Love [8], McBride [9], Agarwal and Nieto [10], Kalla [11], Kalla and Saxena [12, 13], Saigo [14–16], Saigo and Maeda [17], and Kiryakova [18]. A comprehensive explanation of such operators is given by Miller and Ross [19] and Kiryakova [18].

Recently, researchers investigated and studied about the fractional integration formulas for the Bessel function and generalized Bessel functions (see [20, 21]). The generalization of Bessel function and its applications in fractional transforms are found in [22, 23]. A useful generalization of the hypergeometric fractional integrals, including the Saigo operators ([14–16]), has been introduced by Marichev [24] (see details in Samko et al. [25, page 194]) and later extended and studied by Saigo and Maeda ([17, page 393]) in terms of any complex order with Appell function $F_3(\cdot)$ in the kernel, as follows.

Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ and $x > 0$; then the generalized fractional calculus operators involving the Appell function are defined as follows:

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} \cdot t^{-\alpha'} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1 - \frac{t}{x}, 1 - \frac{x}{t}\right) f(t)\, dt, \quad (1)$$

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} f\right)(x) = \frac{x^{-\alpha'}}{\Gamma(\gamma)} \int_x^{\infty} (t-x)^{\gamma-1} \cdot t^{-\alpha} F_3\left(\alpha, \alpha', \beta, \beta'; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)\, dt, \quad (2)$$

with $\mathrm{Re}(\gamma) > 0$. For more details about the above operators, see [17, 24]. The generalized fractional integral operators of the type (1) and (2) have been introduced by Marichev [24] and later extended and studied by Saigo and Maeda [17] (this operator known as the Marichev-Saigo-Maeda operator). For the definition of the Appell function $F_3(\cdot)$ the interested readers may refer to the monograph by Srivastava and Karlson [26] (see also Erdélyi et al. [27] and Prudnikov et al. [28]). The applications of fractional integral operators are found in many papers ([19, 22, 29–31]). The following results given in [17, 32] are needed in sequel.

Lemma 1. *Let $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$ such that* $\mathrm{Re}(\gamma) > 0$ *and*

$$\mathrm{Re}(\rho) > \max\left\{0, \mathrm{Re}\left(\alpha - \alpha' - \beta - \gamma\right), \mathrm{Re}\left(\alpha' - \beta'\right)\right\}. \quad (3)$$

Then there exists the relation

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right)(x) = \Gamma\left[\begin{matrix} \rho, \rho+\gamma-\alpha-\alpha'-\beta, \rho+\beta'-\alpha' \\ \rho+\beta', \rho+\gamma-\alpha-\alpha', \rho+\gamma-\alpha'-\beta \end{matrix}\right] x^{\rho-\alpha-\alpha'+\gamma-1}, \quad (4)$$

where

$$\Gamma\left[\begin{matrix} a, b, c \\ d, e, f \end{matrix}\right] = \frac{\Gamma(a)\Gamma(b)\Gamma(c)}{\Gamma(d)\Gamma(e)\Gamma(f)}. \quad (5)$$

Lemma 2. *Let $\alpha, \alpha', \beta, \beta', \gamma, \rho \in \mathbb{C}$ such that* $\mathrm{Re}(\gamma) > 0$ *and*

$$\mathrm{Re}(\rho) < 1 + \min\left\{\mathrm{Re}(-\beta), \mathrm{Re}\left(\alpha+\alpha'-\gamma\right), \mathrm{Re}\left(\alpha+\beta'-\gamma\right)\right\}. \quad (6)$$

Then there exists the relation

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\right)(x) = \Gamma\left[\begin{matrix} 1-\rho-\gamma+\alpha+\alpha', 1-\rho+\alpha+\beta', 1-\rho-\beta \\ 1-\rho, 1-\rho+\alpha+\alpha'+\beta+\beta'-\gamma, 1-\rho+\alpha-\beta \end{matrix}\right] x^{\rho-\alpha-\alpha'+\gamma-1}. \quad (7)$$

The generalized Wright hypergeometric function ${}_p\psi_q(z)$ is defined by the series

$${}_p\psi_q(z) = {}_p\psi_q\left[\begin{matrix} (a_i, \alpha_i)_{1,p} \\ (b_j, \beta_j)_{1,q} \end{matrix} \middle| z\right] = \sum_{k=0}^{\infty} \frac{\prod_{i=1}^{p} \Gamma(a_i + \alpha_i k)}{\prod_{j=1}^{q} \Gamma\left(b_j + \beta_j k\right)} \frac{z^k}{k!}. \quad (8)$$

Here $a_i, b_j \in \mathbb{C}$, and $\alpha_i, \beta_j \in \mathbb{R}$ $(i = 1, 2, \ldots, p;\ j = 1, 2, \ldots, q)$. Asymptotic behavior of this function for large values of argument of $z \in \mathbb{C}$ was studied in [33] and under the condition

$$\sum_{j=1}^{q} \beta_j - \sum_{i=1}^{p} \alpha_i > -1 \quad (9)$$

in [34–38]. The Bessel and modified Bessel functions of first kind, the Struve function $H_v(z)$, and modified Struve function $L_v(z)$ possess power series representation of the form [39]

$$J_v(z) = \sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{2k+v}}{\Gamma(k+v+1)\, k!},$$

$$I_v(z) = \sum_{k=0}^{\infty} \frac{(z/2)^{2k+v}}{\Gamma(k+v+1)\, k!},$$

$$H_v(z) = \left(\frac{z}{2}\right)^{v+1} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+3/2)\Gamma(k+v+1/2)} \left(\frac{z}{2}\right)^{2k},$$

$$L_v(z) = \left(\frac{z}{2}\right)^{v+1} \sum_{k=0}^{\infty} \frac{1}{\Gamma(k+3/2)\Gamma(k+v+1/2)} \left(\frac{z}{2}\right)^{2k}. \quad (10)$$

The Bessel-Struve kernel $S_\alpha(\lambda z)$, $\lambda \in \mathbb{C}$, [40] which is unique solution of the initial value problem $l_\alpha u(z) = \lambda^2 u(z)$ with the initial conditions $u(0) = 1$ and $u'(0) = \lambda\Gamma(\alpha+1)/(\sqrt{\pi}\Gamma(\alpha+3/2))$, is given by $S_\alpha(\lambda z) = j_\alpha(i\lambda z) - i h_\alpha(i\lambda z)$, $\forall z \in C$, where j_α and h_α are the normalized Bessel and Struve functions.

Moreover, the Bessel-Struve kernel is a holomorphic function on $\mathbb{C} \times \mathbb{C}$ and it can be expanded in a power series in the form

$$S_\alpha(\lambda z) = \sum_{n=0}^{\infty} \frac{(\lambda z)^n \Gamma(\alpha+1)\Gamma((n+1)/2)}{\sqrt{\pi} n! \Gamma(n/2+\alpha+1)}. \quad (11)$$

The present paper is organized as follows. The composition of integral transforms (1) and (2) with Bessel-Struve kernel function and the relation between Bessel-Struve function and other functions are given in Section 2. Section 3 investigates the pathway fractional integration of the Bessel-Struve kernel function and finally the concluding remark is drawn in Section 4.

2. Fractional Integral Formulas

In this section we will investigate the composition of integral transforms (1) and (2) with the Bessel-Struve kernel function defined in (11).

Theorem 3. *Let $\alpha, \alpha', \beta, \beta', \gamma, \rho, p, b, c, \lambda, \nu \in \mathbb{C}$. Suppose that $\mathrm{Re}(\gamma) > 0$ and $\mathrm{Re}(\rho + n) > \max\{0, \mathrm{Re}(\alpha + \alpha' + \beta - \gamma), \mathrm{Re}(\alpha' - \beta')\}$. Then*

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S_\nu(\lambda t)\right)(x) = x^{\rho+\gamma-\alpha-\alpha'-1} \frac{\Gamma(\nu+1)}{\sqrt{\pi}} \times {}_4\Psi_4 \left[\begin{array}{c} \left(\frac{1}{2}, \frac{1}{2}\right), (\rho, 1), \left(\rho+\gamma-\alpha-\alpha'-\beta, 1\right), \left(\rho+\beta'-\alpha', 1\right) \\ \left(\nu+1, \frac{1}{2}\right), \left(\rho+\beta', 1\right), \left(\rho+\gamma-\alpha-\alpha', 1\right), \left(\rho+\gamma-\alpha'-\beta, n\right) \end{array} \middle| \lambda x \right]. \quad (12)$$

Proof. Applying (11) and using the definition (1) to (12), we get

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S_\nu(\lambda t)\right)(x) = \left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} \sum_{n=0}^{\infty} \frac{(\lambda)^n \Gamma(\nu+1)\Gamma((n+1)/2)}{\sqrt{\pi} n! \Gamma(n/2+\nu+1)} t^{\rho+n-1}\right) \cdot (x). \quad (13)$$

By changing the order of integration and summation,

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S_\nu(\lambda t)\right)(x) = \sum_{n=0}^{\infty} \frac{(\lambda)^n \Gamma(\nu+1)\Gamma((n+1)/2)}{\sqrt{\pi} n! \Gamma(n/2+\nu+1)} \left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho+n-1}\right) \cdot (x). \quad (14)$$

Hence replacing ρ by $\rho + n$ in Lemma 1, after some simplification, we obtain the following expression:

$$\begin{aligned} \left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S_\nu(\lambda t)\right)(x) &= \sum_{n=0}^{\infty} \frac{\Gamma(\nu+1)\Gamma((n+1)/2)\lambda^n}{\sqrt{\pi} n! \Gamma(n/2+\nu+1)} \\ &\cdot \frac{\Gamma(\rho+n)\Gamma\left(\rho+n+\gamma-\alpha-\alpha'-\beta\right)\Gamma\left(\rho+n+\beta'-\alpha'\right)}{\Gamma(\rho+n+\beta')\Gamma(\rho+n+\gamma-\alpha-\alpha')\Gamma(\rho+n+\gamma-\alpha'-\beta)} \\ &\cdot x^{\rho+n+\gamma-\alpha-\alpha'-1} = \frac{x^{\rho+\gamma-\alpha-\alpha'-1}}{\sqrt{\pi}} \Gamma(\nu+1) \sum_{n=0}^{\infty} \frac{\Gamma(n/2+1/2)}{\Gamma(n/2+\nu+1)} \\ &\cdot \frac{\Gamma(\rho+n)\Gamma\left(\rho+n+\gamma-\alpha-\alpha'-\beta\right)\Gamma\left(\rho+n+\beta'-\alpha'\right)}{\Gamma(\rho+n+\beta')\Gamma(\rho+n+\gamma-\alpha-\alpha')\Gamma(\rho+n+\gamma-\alpha'-\beta)} \\ &\cdot \frac{(x\lambda)^n}{n!} \end{aligned} \quad (15)$$

whose last summation, in view of (8), is easily seen to arrive at the expression (12). □

Theorem 4. *Let $\alpha, \alpha', \beta, \beta', \gamma, \rho, p, b, c, \lambda, \nu \in \mathbb{C}$. Suppose that $\mathrm{Re}(\gamma) > 0$ and $\mathrm{Re}(\rho - n) < 1 + \min\{\mathrm{Re}(-\beta), \mathrm{Re}(\alpha + \alpha' - \gamma), \mathrm{Re}(\alpha + \beta' - \gamma)\}$. Then*

$$\begin{aligned} &\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1} S_\nu\left(\frac{\lambda}{t}\right)\right)(x) \\ &= \frac{x^{\rho-\alpha-\alpha'+\gamma-1}}{\sqrt{\pi}} \Gamma(\nu+1) \\ &\times {}_4\Psi_4 \left[\begin{array}{c} \left(\frac{1}{2}, \frac{1}{2}\right), \left(1-\rho-\gamma+\alpha+\alpha', 1\right), \left(1-\rho+\alpha+\beta', 1\right), \left(1-\rho-\beta', 1\right) \\ \left(\nu+1, \frac{1}{2}\right), (1-\rho, 1), \left(1-\rho+\alpha+\alpha'+\beta+\beta'-\gamma, 1\right), \left(1-\rho+n+\alpha+\beta, 1\right) \end{array} \middle| \lambda x \right]. \end{aligned} \quad (16)$$

Proof. Using (2) and (11) and then changing the order of integration and summation,

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}S_\nu\left(\frac{\lambda}{t}\right)\right)(x) = \sum_{n=0}^{\infty}\frac{(\lambda)^n\Gamma(\nu+1)\Gamma((n+1)/2)}{\sqrt{\pi}n!\Gamma(n/2+\nu+1)}\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-n-1}\right)\cdot(x). \quad (17)$$

Using Lemma 2, after a little simplification, we obtain the following expression:

$$\left(I_{0,-}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}S_\nu\left(\frac{\lambda}{t}\right)\right)(x) = \sum_{n=0}^{\infty}\frac{(\lambda)^n\Gamma(\nu+1)\Gamma((n+1)/2)}{\sqrt{\pi}n!\Gamma(n/2+\nu+1)}\frac{\Gamma(1-\rho-n-\gamma+\alpha+\alpha')\Gamma(1-\rho-n+\alpha+\beta')\Gamma(1-\rho-n-\beta)}{\Gamma(1-\rho-n)\Gamma(1-\rho-n+\alpha+\alpha'+\beta+\beta'-\gamma)\Gamma(1-\rho-n+\alpha+\beta)}x^{\rho-n-\alpha-\alpha'+\gamma-1}$$
$$= \frac{x^{\rho-\alpha-\alpha'+\gamma-1}}{\sqrt{\pi}}\frac{\Gamma(\nu+1)\Gamma((n+1)/2)}{\Gamma(n/2+\nu+1)}\frac{\Gamma(1-\rho+n-\gamma+\alpha+\alpha')\Gamma(1-\rho+n+\alpha+\beta')\Gamma(1-\rho+n-\beta)}{\Gamma(1-\rho+n)\Gamma(1-\rho+n+\alpha+\alpha'+\beta+\beta'-\gamma)\Gamma(1-\rho+n+\alpha+\beta)}\frac{(x\lambda)^n}{n!}. \quad (18)$$

In view of (8), we obtained the desired result (16). □

2.1. Representation of Bessel-Struve Kernel Function in terms of Exponential Function. In this subsection we represent the Bessel-Struve function in terms of exponential function. Also, we derive the Marichev-Saigo-Maeda operator representation of these special cases. The representation of Bessel-Struve kernel function in terms of exponential function is

$$S_{-1/2}(x) = e^x, \quad (19)$$

$$S_{1/2}(x) = \frac{-1+e^x}{x}. \quad (20)$$

Now, we have the following theorems.

Theorem 5. *Let* $\alpha,\alpha',\beta,\beta',\gamma,\rho,p,b,c,\lambda\in\mathbb{C}$. *Suppose that* $\mathrm{Re}(\gamma)>0$ *and* $\mathrm{Re}(\rho+n)>\max\{0,\mathrm{Re}(\alpha+\alpha'+\beta-\gamma),\mathrm{Re}(\alpha'-\beta')\}$. *Then*

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}e^t\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1}\times{}_3\Psi_3\left[\begin{matrix}(\rho,1),(\rho+\gamma-\alpha-\alpha'-\beta,1),(\rho+\beta'-\alpha',1),(1-\rho-\beta',1)\\(\rho+\beta',1),(\rho+\gamma-\alpha-\alpha',1),(\rho+\gamma-\alpha'-\beta,1)\end{matrix}\middle| x\right]. \quad (21)$$

Proof. From (1), (19), and the definition of Bessel-Struve kernel function (11), we have

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}e^t\right)(x) = \left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}\sum_{n=0}^{\infty}\frac{\Gamma(-1/2+1)\Gamma((n+1)/2)}{\sqrt{\pi}n!\Gamma(n/2-1/2+1)}t^n\right)\cdot(x)$$
$$= \sum_{n=0}^{\infty}\frac{\Gamma(1/2)\Gamma((n+1)/2)}{\sqrt{\pi}n!\Gamma((n+1)/2)}\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho+n-1}\right)(x). \quad (22)$$

This together with Lemma 1 yields

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma}t^{\rho-1}e^t\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1}\sum_{n=0}^{\infty}\frac{x^n}{n!}\Gamma\left[\begin{matrix}\rho+n,\rho+n+\gamma-\alpha-\alpha'-\beta,\rho+n+\beta'-\alpha'\\\rho+n+\beta',\rho+n+\gamma-\alpha-\alpha',\rho+n+\gamma-\alpha'-\beta\end{matrix}\right] \quad (23)$$

which is the desired result. □

Theorem 6. *Let* $\alpha,\alpha',\beta,\beta',\gamma,\rho,p,b,c,\lambda\in\mathbb{C}$. *Suppose that* $\mathrm{Re}(\gamma)>0$ *and* $\mathrm{Re}(\rho+n)>\max\{0,\mathrm{Re}(\alpha+\alpha'+\beta-\gamma),\mathrm{Re}(\alpha'-\beta')\}$. *Then*

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\left(\frac{-1+e^t}{t}\right)\right)(x) = x^{\rho-\alpha-\alpha'+\gamma-1} \times {}_4\Psi_4\left[\begin{matrix}\left(\frac{1}{2},\frac{1}{2}\right),(\rho,1),\left(\rho+\gamma-\alpha-\alpha'-\beta,1\right),\left(\rho+\beta'-\alpha',1\right)\\ \left(\frac{1}{2},\frac{3}{2}\right),\left(\rho+\beta',1\right),\left(\rho+\gamma-\alpha-\alpha',1\right),\left(\rho+\gamma-\alpha'-\beta,1\right)\end{matrix}\middle| x\right]. \quad (24)$$

Proof. The proof of the fractional integration formula (24) would run parallel to the proof of (21) by considering (20). Therefore, we omit the details. □

2.2. Relation between Bessel-Struve Kernel Function and Bessel and Struve Function of First Kind. In this subsection we show the relation between $S_\alpha(x)$ and modified Bessel function $I_v(x)$ and modified Struve function $L_v(x)$ by choosing particular values of α:

$$S_0(x) = I_0(x) + L_0(x), \quad (25)$$

$$S_1(x) = \frac{2I_1(x) + L_1(x)}{x}. \quad (26)$$

In the coming two theorems, we give the fractional integral representations of (25) and (26).

Theorem 7. *Let $\alpha,\alpha',\beta,\beta',\gamma,\rho,p,b,c,\lambda \in \mathbb{C}$. Suppose that* $\mathrm{Re}(\gamma) > 0$ *and* $\mathrm{Re}(\rho+n) > \max\{0, \mathrm{Re}(\alpha+\alpha'+\beta-\gamma), \mathrm{Re}(\alpha'-\beta')\}$. *Then*

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\left(I_0(t)+L_0(t)\right)\right)(x) = \frac{x^{\rho-\alpha-\alpha'+\gamma-1}}{\sqrt{\pi}} \times {}_4\Psi_4\left[\begin{matrix}\left(\frac{1}{2},\frac{1}{2}\right),(\rho,1),\left(\rho+\gamma-\alpha-\alpha'-\beta,1\right),\left(\rho+\beta'-\alpha',1\right)\\ \left(\frac{1}{2},1\right),\left(\rho+\beta',1\right),\left(\rho+\gamma-\alpha-\alpha',1\right),\left(\rho+\gamma-\alpha'-\beta,1\right)\end{matrix}\middle| x\right]. \quad (27)$$

Proof. Applying (26) and using the definition (1) to (27), we get

$$\begin{aligned}\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t_0^{\rho-1}\left(I_0(t)+L_0(t)\right)\right)(x) &= \left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\sum_{n=0}^{\infty}\frac{\Gamma(1)\Gamma((n+1)/2)}{\sqrt{\pi}n!\Gamma(n/2+1)}t^n\right)(x)\\ &= \sum_{n=0}^{\infty}\frac{\Gamma((n+1)/2)}{\sqrt{\pi}n!\Gamma(n/2+1)}\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho+n-1}\right)(x).\end{aligned} \quad (28)$$

Using Lemma 1, we obtain

$$\begin{aligned}\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t_0^{\rho-1}\left(I_0(t)+L_0(t)\right)\right)(x) &= \frac{x^{\rho-\alpha-\alpha'+\gamma-1}}{\sqrt{\pi}}\\ &\cdot \sum_{n=0}^{\infty}\frac{\Gamma((n+1)/2)}{\Gamma(n/2+1)}\frac{x^n}{n!}\\ &\cdot \Gamma\left[\begin{matrix}\rho+n, \rho+n+\gamma-\alpha-\alpha'-\beta, \rho+n+\beta'-\alpha'\\ \rho+n+\beta', \rho+n+\gamma-\alpha-\alpha', \rho+n+\gamma-\alpha'-\beta\end{matrix}\right].\end{aligned} \quad (29)$$

The use of (8) will give the desired result (27). □

Theorem 8. *Let $\alpha,\alpha',\beta,\beta',\gamma,\rho,p,b,c,\lambda \in \mathbb{C}$. Suppose that* $\mathrm{Re}(\gamma) > 0$ *and* $\mathrm{Re}(\rho+n) > \max\{0, \mathrm{Re}(\alpha+\alpha'+\beta-\gamma), \mathrm{Re}(\alpha'-\beta')\}$. *Then*

$$\left(I_{0,+}^{\alpha,\alpha',\beta,\beta',\gamma} t^{\rho-1}\frac{\left(2I_1(t)+L_1(t)\right)}{t}\right)(x) = \frac{x^{\rho-\alpha-\alpha'+\gamma-1}}{\sqrt{\pi}} \times {}_4\Psi_4\left[\begin{matrix}\left(\frac{1}{2},\frac{1}{2}\right),(\rho,1),\left(\rho+\gamma-\alpha-\alpha'-\beta,1\right),\left(\rho+\beta'-\alpha',1\right)\\ \left(\frac{1}{2},1\right),\left(\rho+\beta',1\right),\left(\rho+\gamma-\alpha-\alpha',1\right),\left(\rho+\gamma-\alpha'-\beta,1\right)\end{matrix}\middle| x\right]. \quad (30)$$

Proof. The details of the proof are omitted because it runs as parallel as the proof of (27). □

3. Pathway Fractional Integration of Bessel-Struve Kernel Function

The pathway fractional integral operator was introduced and studied by Mathai [41] and Nair [42] and developed further by Mathai and Haubold ([43, 44]) as follows.

Let $f(x) \in L(a,b)$, $\eta \in C$, $R(\eta) > 0$, $a > 0$ and the pathway parameter $\alpha < 1$; then

$$\left(P_{0+}^{(\eta,\alpha)} f\right)(x) = x^{\eta} \int_0^{[x/a(1-\alpha)]} 1 - \left[\frac{a(1-\alpha)t}{x}\right]^{\eta/(1-\alpha)} f(t)\,dt. \tag{31}$$

For a real scalar α, the pathway model for scalar random variables is represented by the following probability density function (p.d.f.):

$$f(x) = c|x|^{\gamma-1}\left[1 - a(1-\alpha)|x|^{\delta}\right]^{\beta/(1-\alpha)} \tag{32}$$

provided that $-\infty < x < \infty$, $\delta > 0$, $\beta \geq 0$, $[1 - a(1-\alpha)|x|^{\delta}] > 0$, and $\gamma > 0$, where c is the normalizing constant and α is called the pathway parameter (for details, see [42]).

Here, we investigate the composition formula of integral transform operator due to Nair, which is expressed in terms of the generalized Wright hypergeometric function, by inserting the generalized Bessel-Struve kernel $S_\alpha(\lambda z)$ which is defined in (11). The results given in this section are based on the preliminary assertions given by composition formula of pathway fractional integral (31) with a power function.

Lemma 9 (see [42]). *Let* $\eta \in C$, $\mathrm{Re}(\eta) > 0$, $\beta \in C$, *and* $\alpha < 1$. *If* $\mathrm{Re}(\beta) > 0$ *and* $\mathrm{Re}(\eta/(1-\alpha)) > -1$, *then*

$$\left\{P_{0+}^{(\eta,\alpha)}\left[t^{\beta-1}\right]\right\}(x) = \frac{x^{\eta+\beta}}{[a(1-\alpha)]^{\beta}} \frac{\Gamma(\beta)\Gamma(1+\eta/(1-\alpha))}{\Gamma(1+\eta/(1-\alpha)+\beta)}. \tag{33}$$

Now, we have the following theorems.

Theorem 10. *Let* $\eta, \sigma, p, b, c, \lambda \in \mathbb{C}$ *and* $\alpha < 1$ *such that* $\mathrm{Re}(\eta) > 0$, $\mathrm{Re}(\sigma) > 0$, $\mathrm{Re}(\sigma+n) > 0$, *and* $\mathrm{Re}(\eta/(1-\alpha)) > -1$; *then the following formula holds:*

$$\begin{aligned}\left(P_{0+}^{(\eta,\alpha)}\left[t^{\sigma-1}S_\nu(\lambda t)\right]\right)(x) &= x^{\eta+\sigma}\frac{\Gamma(\nu+1)\mu(1+\eta/(1-\alpha))}{\sqrt{\pi}[a(1-\alpha)]^{\sigma+p+1}} \\ &\times {}_2\Psi_2\left[\begin{matrix}\left(\frac{1}{2},\frac{1}{2}\right),(\rho,1) \\ \left(\nu+1,\frac{1}{2}\right),\left(1+\frac{\eta}{1-\alpha}+\sigma,1\right)\end{matrix};\lambda x\right].\end{aligned} \tag{34}$$

Proof. Applying (11) and (31) and changing the order of integration and summation, we get

$$\begin{aligned}\left(P_{0+}^{(\eta,\alpha)}\left[t^{\sigma-1}S_\nu(\lambda t)\right]\right)(x) &= \left(P_{0+}^{(\eta,\alpha)}\left[t^{\sigma-1}\sum_{n=0}^{\infty}\frac{\lambda^n\Gamma(\alpha+1)\Gamma((n+1)/2)}{\sqrt{\pi}n!\Gamma(n/2+\alpha+1)}t^n\right]\right)(x) \\ &= \sum_{k=0}^{\infty}\frac{\lambda^n\Gamma(\nu+1)\Gamma((n+1)/2)}{\sqrt{\pi}n!\Gamma(n/2+\nu+1)}\left(P_{0+}^{(\eta,\alpha)}\left\{t^{(n+\sigma)-1}\right\}\right)(x).\end{aligned} \tag{35}$$

Using the conditions mentioned in the statement of the theorem and $k \in K_0$, $R(p+n) > 0$, $\mathrm{Re}(\eta/(1-\alpha)) > -1$. Applying Lemma 9 and using (33) with β replaced by $\sigma + n$, we get

$$\begin{aligned}\left(P_{0+}^{(\eta,\alpha)}\left[t^{\sigma-1}S_\nu(\lambda t)\right]\right)(x) &= \sum_{k=0}^{\infty}\frac{\lambda^n\Gamma(\nu+1)\Gamma((n+1)/2)}{\sqrt{\pi}n!\Gamma(n/2+\nu+1)}\frac{x^{\eta+\alpha}}{[a(1-\alpha)^{\sigma+n}]} \\ &\cdot\frac{\Gamma(\sigma+n)\Gamma(1+\eta/(1-\alpha))}{\Gamma(1+\eta/(1-\alpha)+\sigma+n)} \\ &= \frac{x^{\eta+\alpha}\Gamma(\nu+1)\Gamma(1+\eta/(1-\alpha))}{\sqrt{\pi}[a(1-\alpha)^{\sigma}]} \\ &\cdot\sum_{k=0}^{\infty}\frac{\Gamma(n/2+1/2)\Gamma(\sigma+n)}{\Gamma(n/2+\nu+1)\Gamma(1+\eta/(1-\alpha)+\sigma+n)}\frac{(x\lambda)^n}{n!}\end{aligned} \tag{36}$$

which gives the desired result. □

By considering the relations given in (19) and (20), we obtain various new integral formulas for Bessel-Struve functions involved in the pathway fractional integration operators.

Theorem 11. *Let* $\eta, \sigma, p, b, c \in C$ *and* $\alpha < 1$ *such that* $\mathrm{Re}(\eta) > 0$, $\mathrm{Re}(\sigma+n) > 0$, *and* $\mathrm{Re}(\eta/(1-\alpha)) > -1$; *then the following formula holds:*

$$\begin{aligned}\left(P_{0+}^{(\eta,\alpha)}\left[t^{\sigma-1}e^t\right]\right)(x) &= x^{\eta+\sigma}\frac{\Gamma(1+\eta/(1-\alpha))}{[a(1-\alpha)]^{\sigma}} \\ &\times {}_1\Psi_1\left[\begin{matrix}(\sigma,1) \\ \left(1+\sigma+\frac{\eta}{1-\alpha},1\right)\end{matrix}\middle|\frac{x}{[a(1-\alpha)]}\right].\end{aligned} \tag{37}$$

Proof. Applying (19) and using (31) with the help of Lemma 9 we can easily prove (37), so the details are omitted. □

Theorem 12. *Let $\eta, \sigma, p, b, c \in C$ and $\alpha < 1$ such that* $\mathrm{Re}(\sigma + \eta) > 0$ *and* $\mathrm{Re}(\eta/(1-\alpha)) > -1$*; then the following formula holds:*

$$\left(P_{0+}^{(\eta,\alpha)}\left[t^{\sigma-1}\frac{-1+e^{t}}{t}\right]\right)(x) = x^{\eta+\sigma}\frac{\Gamma(1+\eta/(1-\alpha))}{2[a(1-\alpha)]^{\sigma}} \times {}_2\Psi_2\left[\begin{matrix}\left(\frac{1}{2},\frac{1}{2}\right),(\sigma,1)\\ \left(\frac{1}{2},\frac{1}{2}\right),\left(1+\sigma+\frac{\eta}{1-\alpha},1\right)\end{matrix}\middle|\frac{x\lambda}{[a(1-\alpha)]}\right]. \quad (38)$$

Proof. The details of proof are omitted because the result (38) can easily derive as similar as the procedure of the proof of Theorem 10 using (20), (11), and (31). □

Remark 13. As similar as the method discussed in Theorem 10, one can easily derive the pathway integral representation of (25) and (26).

4. Conclusion

Fractional integral formulas involving Bessel-Struve kernel function $S_\alpha(\lambda z)$, $\lambda, z \in \mathbb{C}$ have been developed and studied in this paper. The pathway integral representations Bessel-Struve kernel function and its relation between many other functions are also derived in this study. The results thus given in this paper are general in character and likely to find some applications in the theory of special functions.

Competing Interests

The authors declare that they have no competing interests.

Authors' Contributions

All authors have equal contributions. All authors read and approved the final paper.

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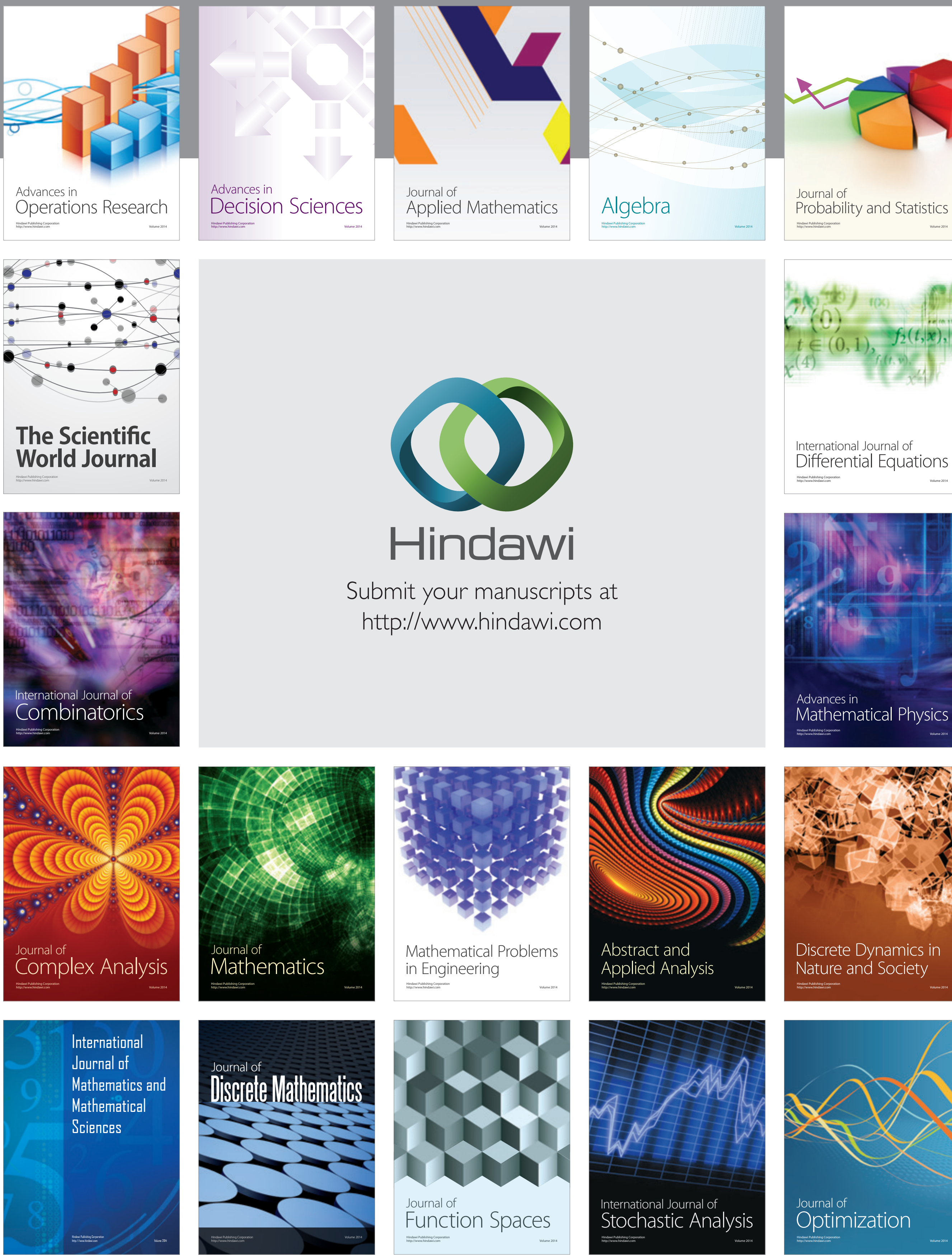
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