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Research Article

Approximate Solution of Volterra-Stieltjes Linear Integral Equations of the Second Kind with the Generalized Trapezoid Rule

Avyt Asanov,[1] Elman Hazar,[2,3] Mustafa Eroz,[2] Kalyskan Matanova,[1] and Elmira Abdyldaeva[1]

[1]*Department of Mathematics, Kyrgyz-Turkish Manas University, Bishkek, Kyrgyzstan*
[2]*Department of Mathematics, Sakarya University, Sakarya, Turkey*
[3]*Department of Applied Mathematics and Informatics, Kyrgyz-Turkish Manas University, Bishkek, Kyrgyzstan*

Correspondence should be addressed to Elman Hazar; ealiyev@sakarya.edu.tr

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The numerical solution of linear Volterra-Stieltjes integral equations of the second kind by using the generalized trapezoid rule is established and investigated. Also, the conditions on estimation of the error are determined and proved. A selected example is solved employing the proposed method.

1. Introduction

Various issues concerning Volterra and Volterra-Stieltjes integral equations were studied in [1–13]. Some practical and theoretical investigations were made in paper [1] for nonclassical Volterra integral equations of the first kind. Also, the approximate solution for the integral equation considered is obtained. In paper [2], various inverse problems including Volterra operator equations were studied. Some properties for Volterra-Stieltjes integral operators were given in [3]. In the studies [6, 7], existence and uniqueness of the solutions were given for Volterra integral and Volterra operator equations of the first and the second kinds. In papers [4, 6], quadratic integral equations of Urysohn-Stieltjes type and their applications were investigated. Various numerical solution methods for integral equations were presented in the studies [8–13]. The notion of derivative of a function by means of a strictly increasing function was given by Asanov in [14]. In the study [15], the generalized trapezoid rule was proposed to evaluate the Stieltjes integral approximately by employing the notion of derivative of a function by means of a strictly increasing function.

In this study, we investigate the numerical solution of linear Volterra-Stieltjes integral equations of the second kind by using the generalized trapezoid rule. Therefore, we need the concept of the derivative defined in the works [14, 15] and theorems connected with it.

2. Approximating Volterra-Stieltjes Integral Equations

Consider the linear integral equation of the second kind

$$u(x) = \int_a^x K(x,s)\,u(s)\,d\varphi(s) + f(x), \quad x \in [a,b], \tag{1}$$

where $K(x,s)$ is a given continuous function on $G = \{(x,s) : a \le s \le x \le b\}$, $f(x)$ are given continuous functions on $[a,b]$, $\varphi(s)$ is a given strictly increasing continuous function on $[a,b]$, and $u(x)$ is the sought function on $[a,b]$.

Definition 1. The derivative of a function $f(x)$ with respect to $\varphi(x)$ is the function $f'_\varphi(x)$, whose value at $x \in (a,b)$ is the number

$$f'_\varphi(x) = \lim_{\Delta\to 0} \frac{f(x+\Delta) - f(x)}{\varphi(x+\Delta) - \varphi(x)}, \tag{2}$$

where $\varphi(x)$ is a given strictly increasing continuous function in (a,b).

If the limit in (2) exists, we say that $f(x)$ has a derivative (is differentiable) with respect to $\varphi(x)$. The first derivative $f'_{\varphi}(x)$ may also be a differentiable function with respect to $\varphi(x)$ at every point $x \in (a,b)$. Then, its derivative

$$f''_{\varphi}(x) = \left(f'_{\varphi}(x)\right)'_{\varphi} \tag{3}$$

is called the second derivative of $f(x)$ with respect to $\varphi(x)$. Consequently, the nth derivative of $f(x)$ with respect to $\varphi(x)$ is defined by

$$f^{(n)}_{\varphi}(x) = \left(f^{(n-1)}_{\varphi}(x)\right)'_{\varphi}. \tag{4}$$

We need the following theorem which is given in [15].

Theorem 2. *Let $\varphi(x)$ and $\psi(x)$ be two strictly increasing continuous functions on $[a,b]$ and $f''_{\varphi}(x), f''_{\psi}(x) \in C[a,b]$. Then,*

$$\begin{aligned}\left|I - A_n\right| = {} & \frac{M_0}{12}\left(\varphi(b) - \varphi(a)\right)\left(\omega_{\varphi}(h)\right)^2 \\ & + \frac{M'_0}{12}\left(\psi(b) - \psi(a)\right)\left(\omega_{\psi}(h)\right)^2,\end{aligned} \tag{5}$$

where

$$\begin{aligned}
I &= \int_a^b f(x)\,d\varphi(x) - \int_a^b f(x)\,d\psi(x), \\
M_0 &= \left\|f''_{\varphi}(x)\right\|_C = \sup_{x\in[a,b]}\left|f''_{\varphi}(x)\right|, \\
M'_0 &= \left\|f''_{\psi}(x)\right\|_C = \sup_{x\in[a,b]}\left|f''_{\psi}(x)\right|, \\
\omega_{\varphi}(h) &= \sup_{|x-y|\le h}\left|\varphi(x) - \varphi(y)\right|, \\
\omega_{\psi}(h) &= \sup_{|x-y|\le h}\left|\psi(x) - \psi(y)\right|, \\
A_n &= \frac{1}{2}\sum_{i=1}^{n}\left[f(x_i) + f(x_{i-1})\right]\left[\varphi(x_i) - \varphi(x_{i-1})\right] \\
&\quad - \frac{1}{2}\sum_{i=1}^{n}\left[f(x_i) + f(x_{i-1})\right]\left[\psi(x_i) - \psi(x_{i-1})\right],
\end{aligned} \tag{6}$$

and $x_i = a + ih$, $i = 0, 1, \ldots, n$, $h = (b-a)/n$, $n \in N$ (N denotes the set of natural numbers).

Corollary 3. *Let $\varphi(x)$ be a strictly increasing continuous function on $[a,b]$, $\psi(x) = 0$ for all $x \in [a,b]$ and $f''_{\varphi}(x) \in C[a,b]$. Then,*

$$\begin{aligned}\left|I - A_n\right| &\le \frac{M_0}{12}\left(\varphi(b) - \varphi(a)\right)\left(\omega_{\varphi}(h)\right)^2, \\ \left|I_i - M_i\right| &\le \frac{M_0}{12}\left(\varphi(x_i) - \varphi(x_{i-1})\right)^3, \quad i = 1, 2, \ldots, n,\end{aligned} \tag{7}$$

where

$$\begin{aligned}I_i &= \int_{x_{i-1}}^{x_i} f(x)\,d\varphi(x), \\ M_i &= \frac{1}{2}\left[f(x_i) + f(x_{i-1})\right]\left[\varphi(x_i) - \varphi(x_{i-1})\right].\end{aligned} \tag{8}$$

Theorem 4. *Let $\varphi(x)$ be a strictly increasing continuous function on $[a,b]$, $K(x,s) \in C(G)$, and $f(x) \in C[a,b]$. Then, the integral equation (1) has a unique solution $u(x) \in C[a,b]$ and*

$$\|u(x)\|_C \le c_1\|f(x)\|_C, \tag{9}$$

where $c_1 = \exp\{K_0(\varphi(b) - \varphi(a))\}$ and $K_0 = \|K(x,s)\|_C = \sup_{(x,s)\in G}|K(x,s)|$.

Then, we will need the following theorem which is given in [16].

Theorem 5. *Let $F(x,s), F'_{\varphi(x)}(x,s) \in C(G)$, $\varphi(x)$ be strictly increasing continuous functions on $[a,b]$, and $P(x) = \int_a^x F(x,s)\,d\varphi(s)$, $x \in [a,b]$. Then,*

$$P'_{\varphi(x)}(x) = F(x,x) + \int_a^x F'_{\varphi(x)}(x,s)\,d\varphi(s), \quad x \in [a,b], \tag{10}$$

where

$$\begin{aligned}F'_{\varphi(x)}(x,s) &= \lim_{\Delta x\to 0}\frac{F(x+\Delta x, s) - F(x,s)}{\varphi(x+\Delta x) - \varphi(x)}, \\ &\qquad (x,s) \in \{(x,s) : a < s < x < b\}, \\ P'_{\varphi(x)}(a) &= \lim_{\Delta x\to 0+}\frac{P(a+\Delta x) - P(a)}{\varphi(a+\Delta x) - \varphi(a)}, \\ P'_{\varphi(x)}(b) &= \lim_{\Delta x\to 0-}\frac{P(b+\Delta x) - P(b)}{\varphi(b+\Delta x) - \varphi(b)}.\end{aligned} \tag{11}$$

Corollary 6. *Let $u(x) \in C[a,b]$ be a solution of the integral equation (1), $K'_{\varphi(x)}(x,s) \in C(G)$, and $f'_{\varphi(x)}(x) \in C[a,b]$. Then, $u'_{\varphi(x)}(x) \in C[a,b]$ and*

$$\begin{aligned}u'_{\varphi(x)}(x) = {} & K(x,x)\,u(x) + \int_a^x K'_{\varphi(x)}(x,s)\,u(s)\,d\varphi(s) \\ & + f'_{\varphi(x)}(x),\end{aligned} \tag{12}$$

where $x \in [a,b]$.

Corollary 7. *Let $u(x) \in C[a,b]$ be a solution of the integral equation (1), $K''_{\varphi(x)}(x,s) \in C(G)$, $K'_{\varphi(x)}(x,x) \in C[a,b]$, and $f''_{\varphi(x)}(x) \in C[a,b]$. Then, $u''_{\varphi(x)}(x) \in C[a,b]$ and*

$$\begin{aligned}u''_{\varphi(x)}(x) = {} & K(x,x)\,u'_{\varphi(x)}(x) \\ & + \left[\left(K(x,x)\right)'_{\varphi(x)} + K'_{\varphi(x)}(x,s)\Big|_{s=x}\right]u(x) \\ & + \int_a^x K''_{\varphi(x)}(x,s)\,u(s)\,d\varphi(s) + f''_{\varphi(x)}(x),\end{aligned} \tag{13}$$

where $x \in [a,b]$.

In this paper, we assume that $(K(x,x))'_{\varphi(x)} \in C[a,b]$, $K''_{\varphi(x)}(x,s)$, $K''_{\varphi(s)}(x,s) \in C(G)$, and $f''_{\varphi(x)}(x) \in C[a,b]$. Then, using Theorems 4 and 5 (and Corollaries 6 and 7), we show that the number M defined as

$$M = \sup_{(x,s)\in G} \left| [K(x,s)u(s)]''_{\varphi(s)} \right| \tag{14}$$

can be determined in terms of quantities $\|f(x)\|_C$, $\|f'_\varphi(x)\|_C$, $\|f''_\varphi(x)\|_C$, $\|(K(x,x))'_{\varphi(x)}\|_C$, $\|K'_{\varphi(x)}(x,s)\|_C$, $\|K'_{\varphi(s)}(x,s)\|_C$, $\|K''_{\varphi(x)}(x,s)\|_C$, and $\|K''_{\varphi(s)}(x,s)\|_C$.

Under these circumstances, using Theorem 2, the integral

$$\int_a^x K(x,s)u(s)\,ds \tag{15}$$

can be evaluated numerically by employing the generalized trapezoid rule.

3. Numerical Solution

In order to obtain the approximate solution of (1), we employ the generalized trapezoid rule given in [15] to the integral in (1). Let $n \in N$,

$$h = \frac{b-a}{n}, \qquad x_k = a + kh, \tag{16}$$

where $k = 0, 1, \ldots, n$. Let us substitute $x = x_k$ in the integral equation (1) and examine the following system of equations:

$$u(x_0) = f(x_0), \quad x_0 = a,$$
$$u(x_k) = \int_a^{x_k} K(x_k,s)u(s)\,ds + f(x_k), \quad k = 1, 2, \ldots, n. \tag{17}$$

To evaluate the integral term in (17), we employ the generalized trapezoid rule given in [15] at the nodes $x_0, x_1, \ldots, x_k$. So, we get

$$\int_a^{x_k} K(x_k,s)u(s)\,ds = \sum_{j=1}^{k} \frac{1}{2}\left[K(x_k,x_{j-1})u(x_{j-1}) + K(x_k,x_j)u(x_j)\right] \cdot \left[\varphi(x_j) - \varphi(x_{j-1})\right] + \sum_{j=1}^{k} R_j^{(n)}(u), \tag{18}$$

where

$$\left|R_j^{(n)}(u)\right| \le \frac{M}{12}\left[\varphi(x_j) - \varphi(x_{j-1})\right]^3, \tag{19}$$

$$M = \sup_{(x,s)\in G}\left|[K(x,s)u(s)]''_{\varphi(s)}\right| = \sup_{(x,s)\in G}\left|K(x,s)u''_{\varphi(s)}(s) + 2K'_{\varphi(s)}(x,s)u'_{\varphi(s)}(s) + K''_{\varphi(s)}(x,s)u(s)\right|. \tag{20}$$

Substituting (18) in (17), we get

$$u(x_0) = f(x_0), \quad x_0 = a,$$
$$u(x_k) = \sum_{j=1}^{k}\frac{1}{2}\left[K(x_k,x_{j-1})u(x_{j-1}) + K(x_k,x_j)u(x_j)\right] \cdot \left[\varphi(x_j) - \varphi(x_{j-1})\right] + \sum_{j=1}^{k} R_j^{(n)}(u) + f(x_k), \tag{21}$$

where $k = 1, 2, \ldots, n$.

Omitting the terms $\sum_{j=1}^{k} R_j^{(n)}(u)$ appearing in each equation of system (21) and $u_k \approx u(x_k)$, we obtain

$$u_0 = f(x_0), \quad x_0 = a,$$
$$u_k = \sum_{j=1}^{k}\frac{1}{2}\left[K(x_k,x_{j-1})u(x_{j-1}) + K(x_k,x_j)u(x_j)\right] \cdot \left[\varphi(x_j) - \varphi(x_{j-1})\right] + f(x_k), \tag{22}$$

where $k = 1, 2, \ldots, n$.

Let us assume that

$$\alpha = \frac{1}{2}\sup_{x\in[a,b]}|K(x,x)|\,\omega_\varphi(h) < 1, \tag{23}$$

where $\omega_\varphi(h)$ denotes the modulus of continuity of the function φ; that is,

$$\omega_\varphi(h) = \sup_{|x-y|\le h}\left|\varphi(x) - \varphi(y)\right|. \tag{24}$$

Under condition (23), the system of (22) has a unique solution which is given by the formulas

$$u_0 = f(a),$$
$$u_1 = \left[1 - \frac{1}{2}K(x_1,x_1)(\varphi(x_1) - \varphi(x_0))\right]^{-1} \cdot \left[\frac{1}{2}K(x_1,x_0)(\varphi(x_1) - \varphi(x_0))u_0 + f(x_1)\right],$$
$$u_k = \left[1 - \frac{1}{2}K(x_k,x_k)(\varphi(x_k) - \varphi(x_{k-1}))\right]^{-1} \cdot \left[\frac{1}{2}\sum_{j=1}^{k-1}K(x_k,x_j)(\varphi(x_{j+1}) - \varphi(x_{j-1}))u_j + \frac{1}{2}K(x_k,x_0)(\varphi(x_1) - \varphi(x_0))u_0 + f(x_k)\right] \tag{25}$$

for $k = 2, 3, \ldots, n$.

We give a concrete example below.

Example 8. Let us take the integral equation (1) for $a = 0$ and $b = 2$ with

$$\varphi(x) = \begin{cases} \sqrt{x}, & \text{for } 0 \le x \le 1, \\ x, & \text{for } 1 < x \le 2 \end{cases}$$
$$K(x,s) = a_1(x) b_1(s), \quad a_1(x) = \begin{cases} 2\sqrt{x}+1, & \text{for } 0 \le x \le 1, \\ 2x+1, & \text{for } 1 < x \le 2, \end{cases} \quad b_1(s) = \begin{cases} s, & \text{for } 0 \le s \le 1, \\ s^2, & \text{for } 1 < s \le 2, \end{cases} \tag{26}$$

and using

$$f(x) = \begin{cases} 1 - \dfrac{2}{3}x^2 - \dfrac{1}{3}x\sqrt{x}, & \text{for } 0 \le x \le 1, \\ 1 - \dfrac{x^3}{3}(2x+1), & \text{for } 1 < x \le 2. \end{cases} \tag{27}$$

It is easily seen that $u(x) \equiv 1,\ x \in [0,2]$, is the unique solution of the integral equation (1) and the conditions $f''_{\varphi(x)}(x) \in C[0,2]$, $(K(x,x))'_{\varphi(x)} \in C[0,2]$, $K''_{\varphi(x)}(x,s)$, and $K''_{\varphi(s)}(x,s) \in C(G)$ hold, where $G = \{(x,s) : 0 \le s \le x \le 2\}$.

Using the proposed method of this study, we get the following results. Here, 20 nodes are selected; that is, $n = 20$. In Table 1, we give the values of the approximate solution obtained by the proposed method of this study and the error in absolute values at the given nodes.

4. Estimation of the Error

In this section, we investigate the problem of convergence of the approximate solution u_k to the solution of integral (1) at the nodes as $n \to \infty$.

Theorem 9. *Let $\varphi(x)$ be a strictly increasing continuous function on $[a,b]$ and for all $x, y \in [a,b]$ the following inequality holds:*

$$|\varphi(x) - \varphi(y)| \le L|x-y|, \tag{28}$$

where $L > 0$ and L is independent of the variables x and y. Then, the inequality

$$|u(x_k) - u_k| \le \frac{ML^2[\varphi(b)-\varphi(a)]}{12(1-\alpha)} \exp\left\{\frac{K_0 L(b-a)}{1-\alpha}\right\} h^2, \quad k = 1,2,\dots,n, \tag{29}$$

holds in which $K_0 = \|K(x,s)\|_C$, $\alpha = (1/2)\|K(x,x)\|_C Lh < 1$, and the number M is determined by (20).

Proof. Let the error be denoted by $v_k = u(x_k) - u_k$ for $k = 0,1,\dots,n$. Taking into account (21) and (22), we have the following system of equations:

$$\begin{aligned} v_0 &= 0, \\ v_k &= \sum_{j=1}^{k} \frac{1}{2}\left[K(x_k, x_{j-1}) v_{j-1} + K(x_k, x_j) v_j\right] \\ &\quad \cdot \left[\varphi(x_j) - \varphi(x_{j-1})\right] + \sum_{j=1}^{k} R_j^{(n)}(u), \end{aligned} \tag{30}$$

where $k = 1,2,\dots,n$.

Rearranging the above system of equations, we get

$$\begin{aligned} &\left(1 - \frac{1}{2}K(x_1,x_1)[\varphi(x_1) - \varphi(x_0)]\right) v_1 = R_1^{(n)}(u), \\ &\left(1 - \frac{1}{2}K(x_k,x_k)[\varphi(x_k) - \varphi(x_{k-1})]\right) v_k \\ &\quad = \frac{1}{2}\sum_{j=1}^{k-1} K(x_k,x_j)\left[\varphi(x_{j+1}) - \varphi(x_{j-1})\right] v_j \\ &\quad + \sum_{j=1}^{k} R_j^{(n)}(u), \end{aligned} \tag{31}$$

where $k = 1,2,\dots,n$.

Along with the inequality $\omega_\varphi(h) \le Lh$, using conditions (19) and (23), we get the following inequality for v_k from (31):

$$\begin{aligned} |v_1| &\le \frac{1}{1-\alpha} R(h), \\ |v_k| &\le \frac{1}{1-\alpha}\left[R(h) + K_0 Lh \sum_{j=1}^{k-1} |v_j|\right], \end{aligned} \tag{32}$$

where $k = 1,2,\dots,n$, $R(h) = (M/12)L^2h^2[\varphi(b) - \varphi(a)]$.

Let the term ε_k for $k = 1,2,\dots,n$ be determined by

$$\varepsilon_k = \frac{1}{1-\alpha}\left[R(h) + K_0 Lh \sum_{j=1}^{k-1} \varepsilon_j\right], \quad k = 2,3,\dots,n, \tag{33}$$

and $\varepsilon_1 = R(h)/(1-\alpha)$ as an initial condition.

It is easily seen that $|v_k| \le \varepsilon_k$ for $k = 1,2,\dots,n$. This can be verified by mathematical induction as follows: for $k = 1$, it is

Table 1: The values of approximate solution, analytical solution, and the error at the nodes.

The nodes x_k	Real value at x_k $u(x_k)$	Approx. value at x_k u_k	The error at x_k $\lvert u(x_k) - u_k \rvert$
0	1	1	0
0.1	1	1.00271872	0.00271872
0.2	1	1.00331826	0.00331826
0.3	1	1.00376994	0.00376994
0.4	1	1.00417187	0.00417187
0.5	1	1.00455757	0.00455757
0.6	1	1.00494416	0.00494416
0.7	1	1.00534264	0.00534264
0.8	1	1.00576140	0.00576140
0.9	1	1.00620761	0.00620761
1.0	1	1.00668805	0.00668805
1.1	1	1.00720950	0.00720950
1.2	1	1.00777907	0.00777907
1.3	1	1.00840442	0.00840442
1.4	1	1.00909399	0.00909399
1.5	1	1.00985722	0.00985722
1.6	1	1.01070480	0.01070480
1.7	1	1.01164880	0.01164880
1.8	1	1.01270300	0.01270300
1.9	1	1.01388350	0.01388350
2.0	1	1.01520860	0.01520860

trivial. Let $|v_j| \le \varepsilon_j$ for $j = 1, \ldots, k-1$. Then, using inequality (32), we get

$$|v_k| \le \frac{1}{1-\alpha}\left[R(h) + K_0 L h \sum_{j=1}^{k-1} \varepsilon_j\right] = \varepsilon_k. \tag{34}$$

Let us show that

$$\varepsilon_j = \frac{R(h)}{1-\alpha}\left(1 + \frac{K_0 L h}{1-\alpha}\right)^{j-1}, \quad j = 1, 2, \ldots, n, \tag{35}$$

are the solution of the system of (33). Taking (35) into account, we get

$$\begin{aligned} &\frac{1}{1-\alpha}\left[R(h) + K_0 L h \sum_{j=1}^{k-1} \varepsilon_j\right] \\ &= \frac{R(h)}{1-\alpha}\left\{1 + \frac{K_0 L h}{1-\alpha}\sum_{j=1}^{k-1}\left(1 + \frac{K_0 L h}{1-\alpha}\right)^{j-1}\right\} \\ &= \frac{R(h)}{1-\alpha}\left\{1 + \left[\left(1 + \frac{K_0 L h}{1-\alpha}\right)^{k-1} - 1\right]\right\} = \varepsilon_k, \\ &\qquad k \ge 2. \end{aligned} \tag{36}$$

Here, we use the equality

$$(1+\gamma)^{k-1} - 1 = \gamma \sum_{j=1}^{k-1} (1+\gamma)^{j-1}, \quad k \ge 2, \tag{37}$$

where $\gamma = K_0 L h/(1-\alpha)$. Consequently, we get the following estimate for the error v_k for all values $k = 1, \ldots, n$:

$$|v_k| \le \frac{R(h)}{1-\alpha}\left(1 + \frac{K_0 L h}{1-\alpha}\right)^{k-1}. \tag{38}$$

Using the fact that $(1+t)^{1/t}$ is increasing and approaches the number e as $t \to 0+$, we get the following chain of inequalities:

$$\begin{aligned} \left(1 + \frac{K_0 L h}{1-\alpha}\right)^{k-1} &\le \left(1 + \frac{K_0 L h}{1-\alpha}\right)^{(b-a)/h} \\ &= \left[\left(1 + \frac{K_0 L}{1-\alpha} h\right)^{(1-\alpha)/K_0 L h}\right]^{K_0 L (b-a)/(1-\alpha)} \\ &\le e^{K_0 L (b-a)/(1-\alpha)} \end{aligned} \tag{39}$$

for $k \le n = (b-a)/h$. Hence, the proof is obtained. □

Remark 10. The function

$$\varphi(x) = \begin{cases} x, & \text{for } 0 \le x \le 1, \\ 2x - 1, & \text{for } 1 < x \le 2, \\ 3x - 3, & \text{for } 2 \le x \le 3 \end{cases} \tag{40}$$

is a strictly increasing continuous function on $[0,3]$, $\varphi'(x) \notin C[0,3]$. But, for all $x, y \in [0,3]$, the following inequality holds:

$$|\varphi(x) - \varphi(y)| \le 4|x - y|. \tag{41}$$

Theorem 11. *Let $\varphi(x)$ be a strictly increasing continuous function on $[a,b]$ and*

$$\beta = K_0[\varphi(b) - \varphi(a)] < 1. \tag{42}$$

Then, the inequality

$$|u(x_k) - u_k| \le \frac{M}{12(1-\beta)}(\omega_\varphi(h))^2[\varphi(b) - \varphi(a)], \quad k = 1, 2, \ldots, n, \tag{43}$$

holds in which $K_0 = \|K(x,s)\|_C$.

Proof. Let the error be denoted by $v_k = u(x_k) - u_k$ and set up the system of equations

$$\begin{aligned} v_1 &= \frac{1}{2} K(x_1, x_1)[\varphi(x_1) - \varphi(x_0)] v_1 + R_1^{(n)}(u), \\ v_k &= \sum_{j=1}^{k-1} \frac{1}{2} K(x_k, x_j)[\varphi(x_{j+1}) - \varphi(x_{j-1})] v_j \\ &\quad + \frac{1}{2} K(x_k, x_k)[\varphi(x_k) - \varphi(x_{k-1})] v_k \\ &\quad + \sum_{j=1}^{k} R_j^{(n)}(u) \end{aligned} \tag{44}$$

for $k = 2, 3, \ldots, n$. From this system of equations, we get

$$\begin{aligned} |v_k| &\le \frac{1}{2} K_0 \sup_j |v_j| \\ &\quad \cdot \left\{ \sum_{j=1}^{k-1} [\varphi(x_{j+1}) - \varphi(x_j) + \varphi(x_j) - \varphi(x_{j-1})] \right. \\ &\quad \left. + [\varphi(x_k) - \varphi(x_{k-1})] \right\} + \frac{M}{12} (\omega_\varphi(h))^2 [\varphi(b) \\ &\quad - \varphi(a)] = \frac{1}{2} K_0 \sup_j |v_j| [\varphi(x_k) - \varphi(x_1) \\ &\quad + \varphi(x_{k-1}) - \varphi(x_0) + \varphi(x_k) - \varphi(x_{k-1})] \\ &\quad + \frac{M}{12} (\omega_\varphi(h))^2 [\varphi(b) - \varphi(a)] \le K_0 \sup_j |v_j| [\varphi(b) \\ &\quad - \varphi(a)] + \frac{M}{12} (\omega_\varphi(h))^2 [\varphi(b) - \varphi(a)] \end{aligned} \tag{45}$$

for $k = 2, 3, \ldots, n$. Using condition (42), we get inequality (43). Therefore, Theorem 11 is proved. □

Competing Interests

The authors declare that they have no competing interests.

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