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Research Article

The Interactions of N-Soliton Solutions for the Generalized $2 + 1$-Dimensional Variable-Coefficient Fifth-Order KdV Equation

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A generalized (2 + 1)-dimensional variable-coefficient KdV equation is introduced, which can describe the interaction between a water wave and gravity-capillary waves better than the (1 + 1)-dimensional KdV equation. The N-soliton solutions of the (2 + 1)-dimensional variable-coefficient fifth-order KdV equation are obtained via the Bell-polynomial method. Then the soliton fusion, fission, and the pursuing collision are analyzed depending on the influence of the coefficient $e^{A_{ij}}$; when $e^{A_{ij}} = 0$, the soliton fusion and fission will happen; when $e^{A_{ij}} \neq 0$, the pursuing collision will occur. Moreover, the Bäcklund transformation of the equation is gotten according to the binary Bell-polynomial and the period wave solutions are given by applying the Riemann theta function method.

1. Introduction

In soliton theory, the nonlinear evolution equations (NLEEs) [1–3] have described natural phenomena in many aspects, such as in the nonlinear phonology [4, 5], water waves [6], hydromechanics [7], and super symmetry. Especially in the hydromechanics, the nonlinear evolution equations can explain the interaction of the waves by the different dispersion relations. As for the NLEEs, there are many methods to get the solutions, like the Hirota method [8–13], the Darboux transformation, Bell-polynomial approach [14–20], Bäcklund transformations [21, 22], and so on [23]. One can get the N-soliton solutions and analyze the interactions of the waves based on the Bell-polynomial approach.

Soliton interaction can be split into elastic and inelastic. As for the elastic, the amplitudes, velocities, and shapes of the soliton can be brought into correspondence with the initial soliton, but for the inelastic collision, after the interaction, one soliton can be divided into two or more solitons, a phenomenon called soliton fission, or contrarily, two or more solitons can be merged into one soliton which is called soliton fusion. What is more, the variable coefficient of the equation can lead to the soliton fission and fusion. In [14], the variable-coefficient KdV equation can be applied to describe the large-amplitudes internal waves of the atmosphere and the ocean. In recent years, the general KdV equation had been expanded to the fifth-order KdV equation and the generalized (2 + 1)-dimensional Korteweg-de Vries equation whose bilinear Bäcklund transformation and Darboux covariant Lax pair have been obtained.

In this paper, we introduced a generalized variable-coefficient fifth-order KdV equation as follows:

$$\begin{aligned} u_t + a(t)\,uu_x + b(t)\,u_{xxx} + c(t)\,u^2u_x + d(t)\,u_xu_{2x} \\ + e(t)\,uu_{xxx} + f(t)\,u_{xxxxx} + g(t)\,u_x\partial_x^{-1}u_y \\ + h(t)\,u_{xxy} + k(t)\,uu_y = 0, \end{aligned} \tag{1}$$

where u is a real function of space x, y, and time t and $a(t), b(t), c(t), d(t), e(t), f(t), g(t), h(t)$, and $k(t)$ are the functions of t. It can be used to describe the gravity-capillary wave on a fluid interface, which is influenced by both the effects of surface tension and gravity as well as by fluid inertia. Equation (1) has special issues.

(1) When $g(t) = h(t) = k(t) = 0$, $a(t) = 6$, $b(t) = 1$, and $c(t) = d(t) = e(t) = f(t) =$ constant, it can be changed into the fifth-order KdV equation [24],

$$u_t + 6uu_x + u_{xxx} + cu^2u_x + du_xu_{2x} + euu_{xxx} + fu_{xxxxx} = 0, \quad (2)$$

whose Darboux transformation, bilinear representation, N-soliton solutions, and bilinear Bäcklund transformation have been obtained.

(2) When $a(t) = 6b(t)$, $k(t) = 4h(t) = 2g(t)$, $c(t) = d(t) = e(t) = f(t) = 0$, and $b(t) = h(t) =$ constants, (1) can be changed into generalized $(2 + 1)$-dimensional Korteweg-de Vries equation [25],

$$u_t + 6buu_x + bu_{xxx} + 2hu_x\partial_x^{-1}u_y + hu_{xxy} + 4huu_y = 0, \quad (3)$$

whose Bell-polynomial, Bäcklund transformation, and Darboux covariant Lax pair have been got.

(3) When $a(t) = 3$, $b(t) = \gamma$, $d(t) = 2$, $e(t) = 1$, $f(t) = 2/15$, and $c(t) = g(t) = h(t) = k(t) = 0$, (1) takes the form of

$$u_t + 3uu_x - \gamma u_{xxx} + 2u_xu_{2x} + uu_{xxx} + \frac{2}{15}u_{xxxxx} = 0 \quad (4)$$

which occurs to a weakly nonlinear long-wave approximation to the general gravity-capillary water-wave problem and γ is a real scaled parameter.

The focus of the paper is to get the N-soliton solutions of the generalized variable-coefficient fifth-order $(2 + 1)$-dimensional equation and analyze the interaction of the water wave and the gravity-capillary wave [26–30]. The details of the paper are as follows: Section 2 introduces a variable-coefficient fifth-order $(2 + 1)$-dimensional KdV equation. The approach and the properties of the Bell-polynomial are presented in Section 3 and then give rise to the N-soliton solutions of the equation based on the Hirota approach. In the final part, we explain the soliton fission and fusion and the soliton pursuing collision of the variable-coefficient fifth-order $(2 + 1)$-dimensional KdV equation according to the different coefficients $e^{A_{ij}}$. Furthermore, the Bäcklund transformation is given.

2. The Introduction of the Bell-Polynomial

To start with, we briefly introduce the basic concepts and the properties of the Bell-polynomial.

(1) Let θ be a function of the variable x; then the formula [25]

$$Y_{nx} = Y_n(\theta_x, \theta_{xx}, \ldots, \theta_{nx}) = e^{-\theta}\partial_x^n e^{\theta}, \quad (n = 1, 2, \ldots) \quad (5)$$

is a polynomial concerning θ with respect to x, which is the definition of the one-dimensional Bell-polynomial. Moreover, we can calculate the initial explicit expressions by the definition as follows:

$$\begin{aligned} Y_x &= \theta_x, \\ Y_{2x} &= \theta_{2x} + \theta_x^2, \\ Y_{3x} &= \theta_{xxx} + 3\theta_x\theta_{xx} + \theta_x^2, \\ Y_{4x} &= \theta_{4x} + 4\theta_x\theta_{3x} + 3\theta_{xx}^2 + 6\theta_x^2\theta_{xx} + \theta_x^4. \end{aligned} \quad (6)$$

(2) Take θ as a C^∞ multivariables function; then the definition of the multivariables Bell-polynomial is as follows:

$$\begin{aligned} &Y_{n_1x_1n_2x_2\cdots n_rx_r}(\theta) \\ &\equiv Y_{n_1,n_2,\ldots,n_r}\left(\theta_{l_1x_1,\ldots,l_rx_r}\ (1 \le l_i \le n_i,\ 0 \le i \le r)\right) \\ &= e^{-\theta}\partial_{x_1}^{n_1}\partial_{x_2}^{n_2}\cdots\partial_{x_r}^{n_r}e^{\theta}. \end{aligned} \quad (7)$$

As for a special function θ with the variables x, y, we give rise to the following several initial values under the definition of the multivariables Bell-polynomial:

$$\begin{aligned} Y_{x,y}(\theta) &= \theta_{x,y} + \theta_x\theta_y, \\ Y_{x,2y}(\theta) &= \theta_{x,2y} + \theta_x\theta_{2y} + 2\theta_{x,y}\theta_y + \theta_x\theta_y^2 + \cdots. \end{aligned} \quad (8)$$

In view of the multivariables Bell-polynomial, the multivariables binary Bell-polynomial can be defined as follows:

$$\begin{aligned} &\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}(\varphi, \psi) \\ &\equiv Y_{n_1,\ldots,n_r}(\theta)\Big|_{\theta_{l_1x_1,\ldots,l_rx_r} = \begin{cases} \varphi_{l_1x_1,\ldots,l_rx_r} & \Sigma_{i=1}^r l_i \text{ is odd}, \\ \psi_{l_1x_1,\ldots,l_rx_r} & \Sigma_{i=1}^r l_i \text{ is even}, \end{cases}} \end{aligned} \quad (9)$$

where φ and ψ both are the C^∞ function with the variables $x_1, x_2, \ldots, x_r$; likewise, we can take some of the expressions depending on (9); for example,

$$\begin{aligned} \mathscr{Y}_x(\varphi) &= \varphi_x, \\ \mathscr{Y}_{2x}(\varphi, \psi) &= \varphi_x^2 + \psi_{xx}, \\ \mathscr{Y}_{x,y} &= \psi_{x,y} + \varphi_x\varphi_y, \\ \mathscr{Y}_{3x}(\varphi, \psi) &= \varphi_{3x} + 3\varphi_x\psi_{2x} + \varphi_x^3, \\ \mathscr{Y}_{2x,y}(\varphi, \psi) &= \varphi_{2x,y} + 2\varphi_x\psi_{x,y} + \varphi_x^2\varphi_y + \varphi_y\psi_{2x} \\ &\quad + \cdots. \end{aligned} \quad (10)$$

Next, we study the proposition of the Bell-polynomial.

Proposition 1. *Bell-polynomial (9) can be written as the Hirota D-operator through a transformational identity:*

$$\begin{aligned} &\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}\left(\varphi = \ln\frac{F}{G}, \psi = \ln FG\right) \\ &= (F \cdot G)^{-1} D_{x_1}^{n_1}\cdots D_{x_r}^{n_r} F \cdot G, \end{aligned} \quad (11)$$

where the Hirota operator is defined by

$$D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F\cdot G=\left(\partial_{x_1}-\partial_{x_1'}\right)^{n_1}\cdots\left(\partial_{x_r}-x_r'\right)\cdot F\left(x_1,\ldots,x_r\right)G\left(x_1',\ldots,x_r'\right)\Big|_{x_1'=x_1,\ldots,x_r'=x_r}. \quad (12)$$

Especially when $F=G$, (11) can be read as

$$F^{-2}D_{x_1}^{n_1}\cdots D_{x_r}^{n_r}F^2=\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}\left(\varphi=0,\psi=2\ln F\right)=\begin{cases}0, & \Sigma_{i=1}^{r}n_i \text{ is odd},\\ \mathscr{P}_{n_1x_1,\ldots,n_rx_r}(q), & \Sigma_{i=1}^{r}n_i \text{ is even}.\end{cases} \quad (13)$$

In (13), the Bell-polynomial is significant if and only if when $\Sigma_{i=1}^{r}n_i$ is even, one redefines a P-polynomial:

$$\mathscr{P}_{n_1x_1,\ldots,n_rx_r}(q)=\mathscr{Y}_{n_1x_1,\ldots,n_rx_r}\left(\varphi=0,\psi=2\ln F\right), \quad (14)$$

when $n_1+n_2+\ldots+n_r$ is even. The initial few P-polynomials are

$$\begin{aligned}\mathscr{P}_{2x}(q)&=q_{2x},\\ \mathscr{P}_{4x}(q)&=q_{4x}+3q_{2x}^2,\\ \mathscr{P}_{2x,2y}(q)&=q_{2x,2y}+q_{2x}q_{2y}+2q_{x,y}^2,\\ \mathscr{P}_{x,y}(q)&=q_{x,y},\\ \mathscr{P}_{3x,y}(q)&=q_{3x,y}+3q_{x,y}q_{2x},\\ \mathscr{P}_{6x}(q)&=q_{6x}+15q_{2x}q_{4x}+15q_{2x}^3+\cdots.\end{aligned} \quad (15)$$

As for the NLEEs, (9) and (15) are important to get the N-soliton solutions; one can get the bilinear equation provided that the NLEEs can express the linear combination of the P-polynomials.

3. The N-Soliton Solutions of the Variable-Coefficient Fifth-Order KdV Equation

As for (1), if we take

$$\begin{aligned}a(t)&=6b(t),\\ d(t)&=\frac{2}{3}c(t),\\ e(t)&=\frac{1}{3}c(t),\\ f(t)&=\frac{1}{30}c(t),\\ g(t)&=2h(t),\\ k(t)&=4h(t),\end{aligned} \quad (16)$$

then (1) can be rewritten as

$$u_t+b(t)\left(6uu_x+u_{xxx}\right)+\frac{1}{30}c(t)\left(30u^2u_x+20u_xu_{2x}+10uu_{xxx}+u_{xxxxx}\right)+h(t)\left(2u_x\partial_x^{-1}u_y+u_{xxy}+4uu_y\right)=0. \quad (17)$$

By virtue of the transformation $u=Q_{xx}$, (17) can be changed into

$$Q_{xxt}+b(t)\left(6Q_{xx}Q_{xxx}+Q_{5x}\right)+\frac{1}{30}c(t)\cdot\left(30Q_{xx}^2Q_{xxx}+20Q_{xxx}Q_{4x}+10Q_{xx}Q_{5x}+Q_{7x}\right)+h(t)\left(2Q_{3x}Q_{x,y}+Q_{4x,y}+4Q_{xx}Q_{xxy}\right)=0. \quad (18)$$

What is more, we can obtain the following formula by making use of the integral to (18) with respect to the variable x:

$$Q_{xt}+b(t)\left(3Q_{xx}^2+Q_{4x}\right)+\frac{1}{30}c(t)\left(10Q_{xx}^3+10Q_{xx}Q_{4x}+5Q_{3x}^2+Q_{6x}\right)+h(t)\left(2Q_{2x}Q_{x,y}+Q_{3x,y}+2\partial_x^{-1}Q_{xx}Q_{xxy}\right)=0. \quad (19)$$

Expression (19) is changed as follows with the aid of P-polynomial (15):

$$\mathscr{P}_{x,t}(Q)+b(t)\mathscr{P}_{4x}(Q)+\frac{c(t)}{30}\mathscr{P}_{6x}(Q)-\frac{c(t)}{18}\left(\partial_x^2\mathscr{P}_{4x}(Q)+3Q_{2x}\mathscr{P}_{4x}(Q)-\partial_x^{-1}\left(\partial_x^3\mathscr{P}_{4x}(Q)+6Q_{2x}\partial_x\mathscr{P}_{4x}(Q)\right)\right)+\frac{2}{3}h(t)\cdot\left(\mathscr{P}_{3x,y}(Q)\right)+\frac{1}{3}h(t)\partial_x^{-1}\partial_y\left(\mathscr{P}_{4x}(Q)\right)=0, \quad (20)$$

which cannot be written as the linear combination of P-polynomials, so we construct an auxiliary variable z which satisfies

$$Q_{4x}+3Q_{2x}^2+Q_{x,z}=0. \quad (21)$$

From (21), we can get a pair of P-polynomials as follows:

$$\mathscr{P}_{4x}(Q)+\mathscr{P}_{x,z}(Q)=0,$$

$$\mathscr{P}_{x,t}(Q)+b(t)\mathscr{P}_{4x}(Q)+\frac{c(t)}{30}\mathscr{P}_{6x}(Q)$$

$$
\begin{aligned}
&-\frac{c(t)}{18}\left(\mathscr{P}_{3x,z}(Q)+\mathscr{P}_{2z}(Q)\right)+\frac{2}{3}h(t)\mathscr{P}_{3x,y}(Q)\\
&-\frac{1}{3}h(t)\mathscr{P}_{y,z}(Q)=0.
\end{aligned}
\tag{22}
$$

Finally, generalized variable-coefficient (2 + 1)-dimensional fifth-order KdV equation (22) can be cast into a bilinear representation based on the transportation $Q = 2\ln f$,

$$
\begin{aligned}
&\left(D_x^4+D_xD_z\right)f\cdot f=0,\\
&\left(D_xD_t+b(t)D_x^4+\frac{c(t)}{30}D_x^6-\frac{c(t)}{18}\left(D_x^3D_z+D_z^2\right)\right.\\
&\left.+\frac{2}{3}h(t)D_x^3D_y-\frac{1}{3}h(t)D_yD_z\right)f\cdot f=0.
\end{aligned}
\tag{23}
$$

Once the bilinear representation of (19) is given, we can present the N-soliton solutions of (23) with the help of Hirota's bilinear approach and the symbolic computation.

After that, we begin to solve (17) on account of the Hirota method; set

$$
f=1+\varepsilon f_1+\varepsilon^2 f_2+\varepsilon^3 f_3+\cdots. \tag{24}
$$

Substitute (24) into (23) and compare the powers of ε; then the N-soliton solutions of (19) are gotten by making $\varepsilon = 1$ as follows:

$$
f=\sum_{\mu_i,\mu_j=0,1}\exp\left\{\sum_{i>j}^{N}A_{ij}\mu_i\mu_j+\sum_{j=1}^{N}\mu_j\gamma_j\right\}, \tag{25}
$$

with

$$
\begin{aligned}
\gamma_j&=k_j\left(x+p_jy\right)-\int\left(\frac{c(t)}{30}k_j^5+b(t)k_j^3+h(t)k_j^3p_j\right)dt,\\
e^{A_{ij}}&=\frac{(c(t)/6)\left(k_i-k_j\right)^2\left(k_i^2+k_j^2-k_ik_j\right)+3b(t)\left(k_i-k_j\right)^2+h(t)\left(k_i-k_j\right)\left(p_i\left(2k_i-k_j\right)-p_j\left(2k_j-k_i\right)\right)}{(c(t)/6)\left(k_i+k_j\right)^2\left(k_i^2+k_j^2+k_ik_j\right)+3b(t)\left(k_i+k_j\right)^2+h(t)\left(k_i+k_j\right)\left(p_i\left(2k_i-k_j\right)+p_j\left(2k_j-k_i\right)\right)},
\end{aligned}
\tag{26}
$$

where k_j and p_j are both the constants and $\sum_{\mu_i,\mu_j=0,1}$ indicate summation over all the different possible cases $\mu_i,\mu_j = 0,1$ $(i,j=1,2,3,\ldots)$.

For $N = 1$, we can read the one-soliton solution as

$$
u=\frac{1}{2}k_1^2\operatorname{sech}^2\left(\frac{\gamma_1}{2}\right). \tag{27}
$$

For $N = 2$, the two-soliton solutions can be written as

$$
\begin{aligned}
u&=2\frac{k_1^2e^{\gamma_1}+k_2^2e^{\gamma_2}+e^{A_{21}}\left(k_1+k_2\right)^2e^{\gamma_1+\gamma_2}}{1+e^{\gamma_1}+e^{\gamma_2}+e^{A_{21}}e^{\gamma_1+\gamma_2}}\\
&-2\frac{\left(k_1e^{\gamma_1}+k_2e^{\gamma_2}+e^{A_{21}}\left(k_1+k_2\right)e^{\gamma_1+\gamma_2}\right)^2}{\left(1+e^{\gamma_1}+e^{\gamma_2}+e^{A_{21}}e^{\gamma_1+\gamma_2}\right)^2}.
\end{aligned}
\tag{28}
$$

For $N = 3$, we can obtain the three-soliton solutions as

$$
\begin{aligned}
u=2&\left[\ln\left(1+e^{\gamma_1}+e^{\gamma_2}+e^{\gamma_3}+e^{\gamma_1+\gamma_2+A_{21}}+e^{\gamma_1+\gamma_3+A_{31}}\right.\right.\\
&\left.\left.+e^{\gamma_2+\gamma_3+A_{32}}+e^{\gamma_1+\gamma_2+\gamma_3+A_{21}+A_{31}+A_{32}}\right)\right]_{xx}.
\end{aligned}
\tag{29}
$$

4. The Bäcklund Transformation of the Variable-Coefficient Fifth-Order KdV Equation

For the NLEEs, the BT method provides a new idea to construct the solutions by the Bell-polynomial. In this section, we will obtain the BT of the known (2 + 1)-dimensional variable-coefficient fifth-order KdV equation. Suppose $Q,\overline{Q}$ are the two different solutions of (19), and consider the following form:

$$
\begin{aligned}
L=\;&\frac{1}{30}c(t)\left(10Q_{xx}^3 10Q_{xx}Q_{4x}+5Q_{3x}^2+Q_{6x}-10\overline{Q}_{xx}^3\right.\\
&\left.-10\overline{Q}_{xx}\overline{Q}_{4x}-5\overline{Q}_{3x}^2+\overline{Q}_{6x}\right)+h(t)\left(2Q_{2x}Q_{x,y}\right.\\
&+Q_{3x,y}+2\partial_x^{-1}Q_{xx}Q_{xxy}-2\overline{Q}_{2x}\overline{Q}_{x,y}-\overline{Q}_{3x,y}\\
&\left.-2\partial_x^{-1}\overline{Q}_{xx}\overline{Q}_{xxy}\right)+Q_{xt}-\overline{Q}_{xt}+b(t)\left(3Q_{xx}^2+Q_{4x}\right.\\
&\left.-3\overline{Q}_{xx}^2-\overline{Q}_{4x}\right)=0.
\end{aligned}
\tag{30}
$$

In order to obtain the BT of (30), we introduce the mixing variables as

$$
\begin{aligned}
\varphi&=\ln(F\cdot G),\\
\psi&=\ln\left(\frac{F}{G}\right),\\
Q&=\varphi+\psi=2\ln F,\\
\overline{Q}&=\psi-\varphi=2\ln G;
\end{aligned}
\tag{31}
$$

then

$$L = 2\Big(\varphi_{xt} + b(t)\left(6\varphi_{xx}\psi_{xx} + \varphi_{4x}\right) + \frac{c(t)}{30}\big(10\varphi_{2x}^3 + 30\varphi_{2x}\psi_{2x}^2 + 10\varphi_{2x}\psi_{4x} + 10\varphi_{4x}\psi_{2x} + 10\varphi_{3x}\psi_{3x} + \varphi_{6x}\big) + \frac{2}{3}h(t)\left(\varphi_{3x,y} + 3\varphi_{2x}\psi_{x,y} + 3\varphi_{x,y}\psi_{2x}\right) + \frac{1}{3}\cdot h(t)\,\partial_x^{-1}\partial_y\left(6\varphi_{xx}\psi_{xx} + \varphi_{4x}\right)\Big) = 2\partial_x\Big(\mathscr{Y}_t(\varphi) + b(t)\,\mathscr{Y}_{3x}(\varphi,\psi) + \frac{c(t)}{30}\mathscr{Y}_{5x}(\varphi,\psi) + h(t)\,\mathscr{Y}_{2x,y}(\varphi,\psi)\Big) + R(\varphi,\psi), \tag{32}$$

where

$$R(\varphi,\psi) = 2\Big(b(t)\left(3\varphi_{2x}\psi_{3x} - 3\varphi_x^2\varphi_{xx} - 3\varphi_x\psi_{4x}\right) + h(t)\left(2\varphi_{x,y}\psi_{xx} - 2\varphi_x\psi_{xxy} - \partial_x\left(\varphi_x^2\varphi_y + \varphi_y\psi_{2x}\right)\right) + \frac{c(t)}{30}\big(5\varphi_{2x}\psi_4 + 15\varphi_{2x}\psi_{2x}^2 + 10\varphi_{2x}^3 - 5\varphi_x^4\varphi_{2x} - 30\varphi_x^2\varphi_{xx}\psi_{xx} - 10\varphi_x^3\psi_{xxx} - 30\varphi_x\psi_{2x}\psi_{3x} - 20\varphi_x\varphi_{xx}\varphi_{3x} - 5\varphi_x\psi_{5x} - 10\varphi_x^2\varphi_{4x}\big) + \frac{1}{3}h(t)\cdot\partial_x^{-1}\left(6\varphi_{xxy}\psi_{xx} + 6\varphi_{xx}\psi_{2x,y}\right)\Big). \tag{33}$$

Finally, we derive the BT by introducing a spectrum parameter equation as

$$\mathscr{Y}_{2x}(\varphi,\psi) = \lambda \tag{34}$$

and have

$$L = 2\partial_x\Big(\mathscr{Y}_t(\varphi) + b(t)\,\mathscr{Y}_{3x}(\varphi,\psi) + \frac{c(t)}{30}\mathscr{Y}_{5x}(\varphi,\psi) + h(t)\,\mathscr{Y}_{2x,y}(\varphi,\psi) + \left(3\lambda b(t) + \frac{c(t)}{2}\lambda^2\right)\mathscr{Y}_x(\varphi) + 3\lambda h(t)\,\mathscr{Y}_y(\varphi)\Big); \tag{35}$$

then the Bell-polynomial-typed BT of the (2+1)-dimensional variable-coefficient KdV equation is the following:

$$\begin{aligned} &\mathscr{Y}_{2x}(\varphi,\psi) = \lambda, \\ &2\Big(\mathscr{Y}_t(\varphi) + b(t)\,\mathscr{Y}_{3x}(\varphi,\psi) + \frac{c(t)}{30}\mathscr{Y}_{5x}(\varphi,\psi) + h(t)\,\mathscr{Y}_{2x,y}(\varphi,\psi) + \left(3\lambda b(t) + \frac{c(t)}{2}\lambda^2\right)\mathscr{Y}_x(\varphi) + 3\lambda h(t)\,\mathscr{Y}_y(\varphi)\Big) = \vartheta(t), \end{aligned} \tag{36}$$

where $\vartheta(t)$ is a function about the variable t.

The BT of (36) can be read with the help of the expression of (11),

$$\begin{aligned} &\left(D_x^2 - \lambda\right)F\cdot G = 0, \\ &2\Big(D_t + b(t)D_x^3 + \frac{c(t)}{30}D_x^5 + D_x^2D_y + \left(3\lambda b(t) + \frac{c(t)}{2}\lambda^2\right)D_x + 3\lambda D_y - \vartheta(t)\Big)F\cdot G = 0. \end{aligned} \tag{37}$$

5. The Period Wave Solutions of the Fifth-Order KdV Equation

In this section, we want to get the period wave solution of the variable-coefficient fifth-order KdV equation by using the Hirota method. Nakamura gave a lucid approach to the period wave solutions according to the Riemann theta function [31]. Hence, we let the Riemann function of (23) as

$$f = \sum_{n=-\infty}^{n=\infty} e^{2\pi i n\gamma + \pi i n^2\tau}, \tag{38}$$

where $n \in Z$, $\tau \in C$, $\operatorname{Im}\tau > 0$, and $\gamma = k(x + py) - \int((c(t)/30)k^5 + b(t)k^3 + h(t)k^3p)dt$; in addition, the parameters k, p are constant to be determined. In order to simplify γ, let $w = (c(t)/30)k^5 + b(t)k^3 + h(t)k^3p$. Inserting (38) into (23), we have

$$\begin{aligned} Hf\cdot f &= H\left(D_x, D_y, D_t\right)\sum_{n=-\infty}^{n=\infty} e^{2\pi i n\gamma + \pi i n^2\tau}\sum_{m=-\infty}^{m=\infty} e^{2\pi i m\gamma + \pi i m^2\tau} = \sum_{n=-\infty}^{n=\infty}\sum_{m=-\infty}^{m=\infty} H\left(D_x, D_y, D_t\right) e^{2\pi i n\gamma + \pi i n^2\tau}\cdot e^{2\pi i m\gamma + \pi i m^2\tau} \\ &= \sum_{n=-\infty}^{n=\infty}\sum_{m=-\infty}^{m=\infty} H\left(2\pi i(n-m)k, 2\pi i(n-m)kp, 2\pi i(n-m)w\right) e^{2\pi i(n+m)\gamma + \pi i(n^2+m^2)\tau} \\ &= \sum_{q=-\infty}^{q=\infty}\left\{\sum_{n=-\infty}^{n=\infty} H\left(2\pi i(2n-q)k, 2\pi i(2n-q)kp, 2\pi i(2n-q)w\right) e^{\pi i(n^2+(q-n)^2)\tau}\right\} e^{2\pi i q\gamma} = \sum_{q=-\infty}^{q=\infty}\overline{H}(q)\,e^{2\pi i q\gamma}. \end{aligned} \tag{39}$$

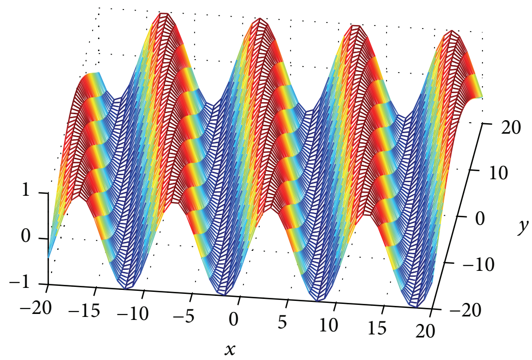

FIGURE 1: One period wave solution to (17) with the parameters $k = 0.1$, $p = 1$, $t = 0.5$, $\tau = 2i$, $c(0.5) = 30$, $b(0.5) = 1$, and $h(0.5) = 1$.

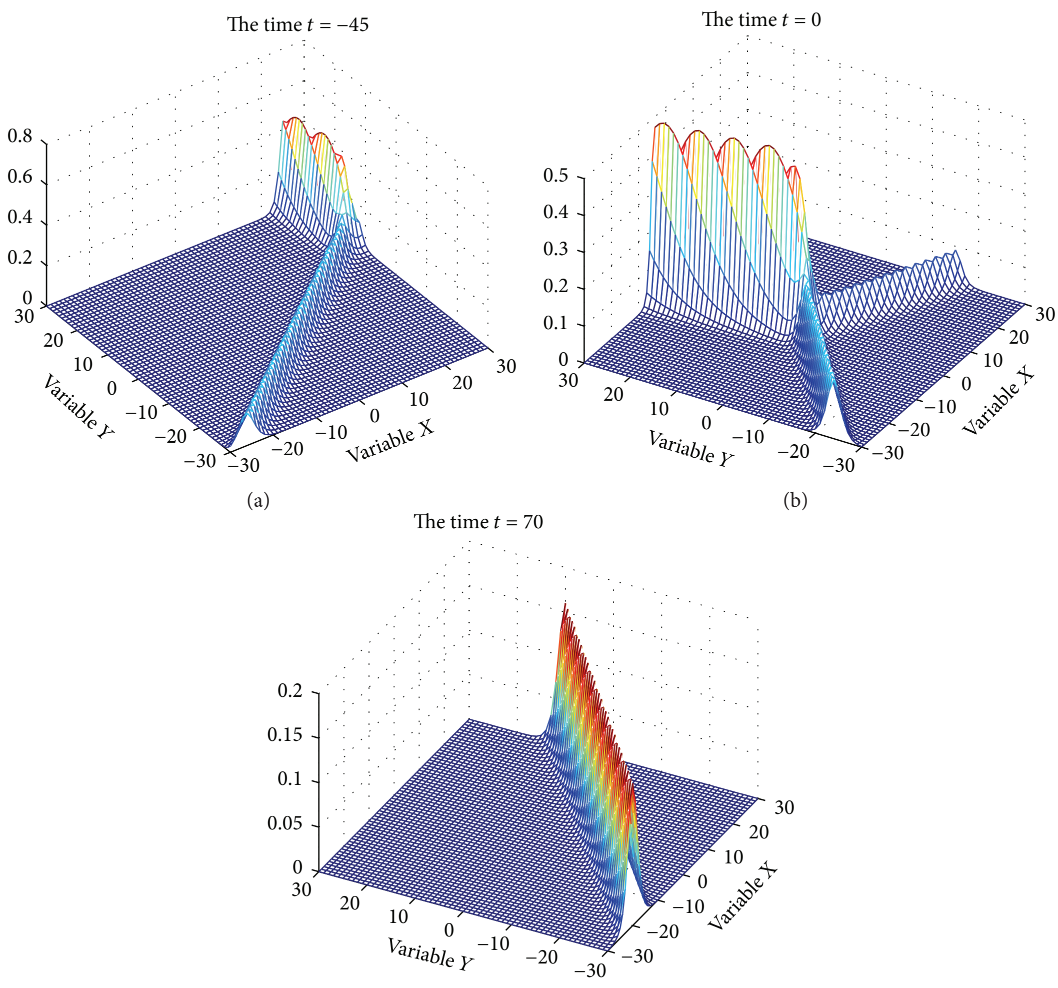

FIGURE 2: The process of the soliton fusion for the two solitary waves (46) with $k_1 = -0.8$, $k_2 = 0.6$, $p_1 = -1.25$, and $p_2 = 2$, and (a) implies $t = -45$, (b) implies $t = 0$, and (c) implies $t = 70$.

Based on the calculation of (39), we can obtain

$$\overline{H}(q) = \sum_{n=-\infty}^{n=\infty} H\left(2\pi i(2n-q)k, 2\pi i(2n-q)\right.$$
$$\left. \cdot kp, 2\pi i(2n-q)w\right) e^{\pi i(n^2+(q-n)^2)\tau}$$
$$= \sum_{h=-\infty}^{h=\infty} H\left(2\pi iBk, 2\pi iBkp, 2\pi iBw\right) e^{\pi i(n^2+(q-h-2)^2)\tau}$$
$$\cdot e^{2\pi i(q-1)\tau}, \quad (40)$$

where $B = 2h - q + 2$, $q = m + n$, $h = n - 1$.

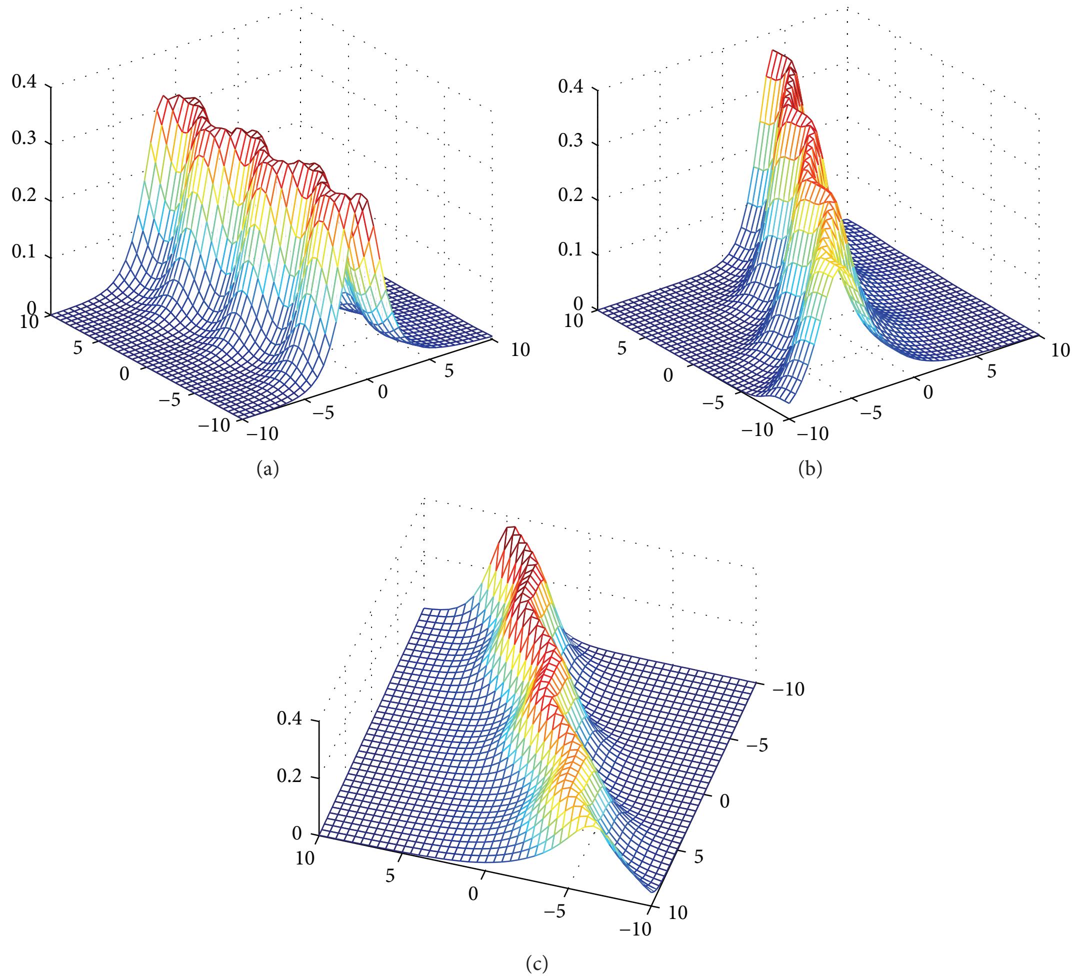

Figure 3: The process of (29) with $k_1 = -0.8$, $k_2 = 0.6$, $k_3 = 0.4$, $p_1 = p_2 = p_3 = 0$, and $c(t) = h(t) = 0$, and (a) implies $b(t) = \cos(t)$, (b) implies $b(t) = \cos(t) + 1$, and (c) implies $b(t) = \cos(t) - 1$.

From the characters of (40), we can get the following recursion formula:

$$\overline{H}(q) = \begin{cases} \overline{H}(0)\, e^{\pi i n q \tau}, & q = 2n, \\ \overline{H}(1)\, e^{\pi i (2n^2+2n)\tau}, & q = 2n+1. \end{cases} \tag{41}$$

If we set $\overline{H}(0) = \overline{H}(1) = 0$, it can satisfy with (23); that is,

$$\begin{aligned} \overline{H}(0) &= \sum_{n=-\infty}^{n=\infty} \Big(b(t)\, 256\pi^4 n^4 k^4 - 16\pi^2 n^2 k w \\ &\quad - \frac{c(t)}{30}\left(4096\pi^6 n^6 k^6\right) + h(t)\, 256\pi^4 n^4 k^4 p \Big) e^{2\pi i n^2 \tau} \\ &= 0, \\ \overline{H}(1) &= \sum_{n=-\infty}^{n=\infty} \Big(b(t)\, 16\pi^4 (2n-1)^4 k^4 \\ &\quad - 4\pi^2 (2n-1)^2 k w - \frac{c(t)}{30}\left(64\pi^6 (2n-1)^6 k^6\right) \\ &\quad + h(t)\, 16\pi^4 (2n-1)^4 k^4 p \Big), \\ e^{\pi i (2n^2-2n+1)\tau} &= 0. \end{aligned} \tag{42}$$

With the purpose of computational convenience, set

$$\begin{aligned} q_{11} &= -\sum_{n=-\infty}^{n=\infty} 16\pi^2 n^2 k e^{2\pi i n^2 \tau}, \\ q_{12} &= \sum_{n=-\infty}^{n=\infty} \Big(b(t)\, 16\pi^4 (2n-1)^4 k^4 \\ &\quad - \frac{c(t)}{30}\left(64\pi^6 (2n-1)^6 k^6\right) \Big) e^{\pi i (2n^2-2n+1)\tau}, \\ q_{21} &= \sum_{n=-\infty}^{n=\infty} e^{\pi i (2n^2-2n+1)\tau}, \\ q_{22} &= -\sum_{n=-\infty}^{n=\infty} 4\pi^2 (2n-1)^2 k e^{\pi i (2n^2-2n+1)\tau}, \end{aligned}$$

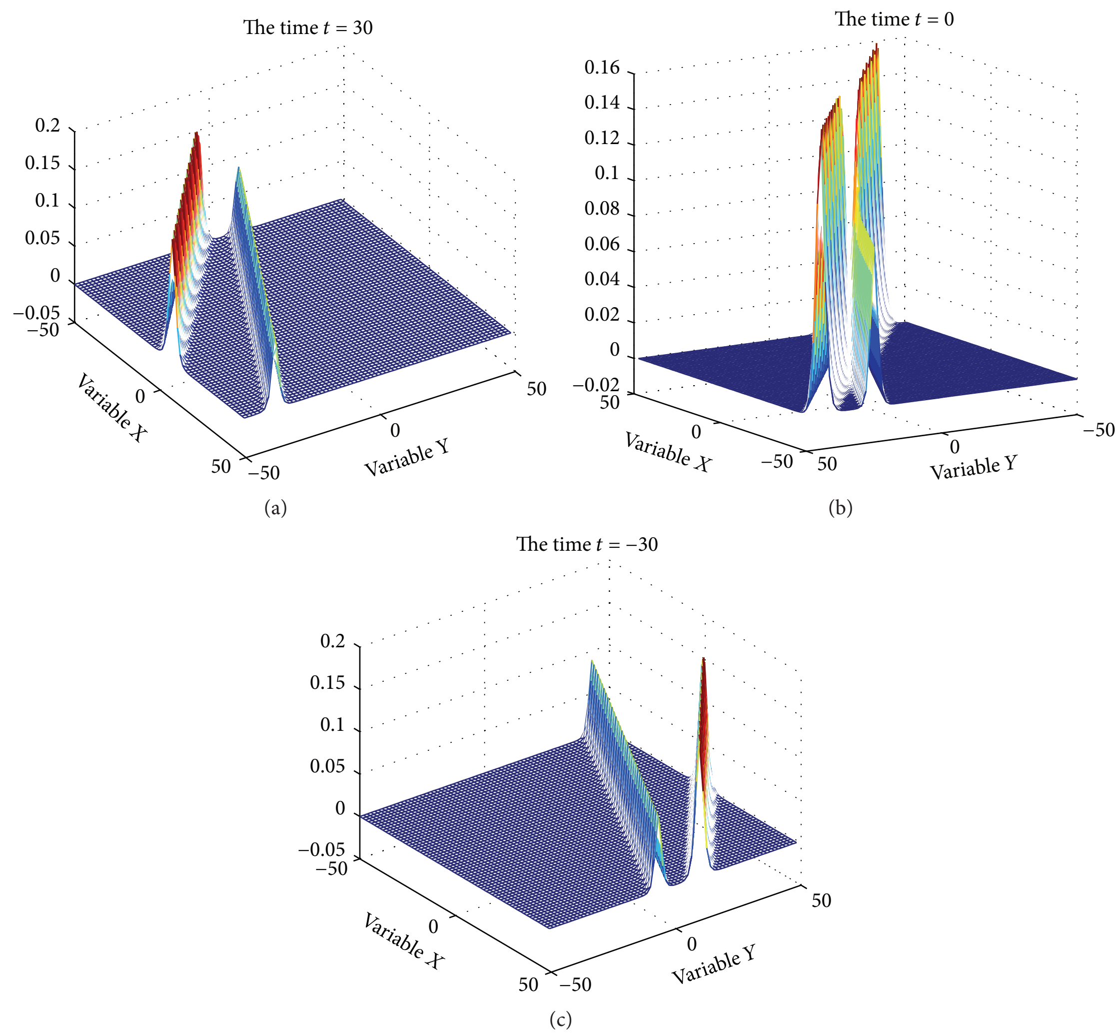

FIGURE 4: The process of (27) with $k_1 = 0.8$, $k_2 = 0.6$, $p_1 = 1.25$, $p_2 = 2$, $c(t) = 30$, $b(t) = 1$, and $h(t) = 1$, and (a) implies $t = 30$, (b) implies $t = 0$, and (c) implies $t = -30$.

$$q_{31} = \sum_{n=-\infty}^{n=\infty} e^{2\pi i n^2 \tau},$$
$$q_{13} = \sum_{n=-\infty}^{n=\infty} \left(b(t)\, 256\pi^4 n^4 k^4 - \frac{c(t)}{30} \left(4096\pi^6 n^6 k^6\right)\right) \cdot e^{2\pi i n^2 \tau}. \tag{43}$$

Then (42) can be changed into

$$q_{11} w + q_{13} + h(t)\, 256\pi^4 n^4 k^4 p q_{31} = 0,$$
$$q_{22} w + q_{12} + h(t)\, 16\pi^4 (2n-1)^4 k^4 p q_{21} = 0. \tag{44}$$

The parameters w, p can be got by (44) as

$$w = \frac{16n^4 q_{12} q_{31} - (2n-1)^4 q_{21} q_{13}}{(2n-1)^4 q_{21} q_{11} - 16n^4 q_{31} q_{22}},$$
$$p = \frac{q_{12} q_{11} - q_{22} q_{13}}{h(t)\, \pi^4 k^4 \left(256 n^4 q_{22} q_{31} - 16(2n-1)^4 q_{11} q_{21}\right)}. \tag{45}$$

Therefore, we can obtain the period wave solution of (17) in Figure 1.

6. The Interaction of the Soliton Waves

In this part, we will discuss the interaction of the soliton waves. From Section 3, we get the N-soliton solutions, which will show soliton fusion or fission when $e^{A_{ij}} = 0$; on the other hand, when $e^{A_{ij}} \neq 0$, there will occur the soliton pursuing collision; after the collision, the waves are still spread along the previous direction but cannot keep the previous amplitude. Then we can describe the interaction of the soliton waves by Figures 2–5.

If $e^{A_{ij}} = 0$, two-soliton solutions (27) become the following resonance solution:

$$u = 2\left[\ln\left(1 + e^{\gamma_1} + e^{\gamma_2}\right)\right]_{xx}. \tag{46}$$

We can analyze the soliton fusion and fission under the approximation form.

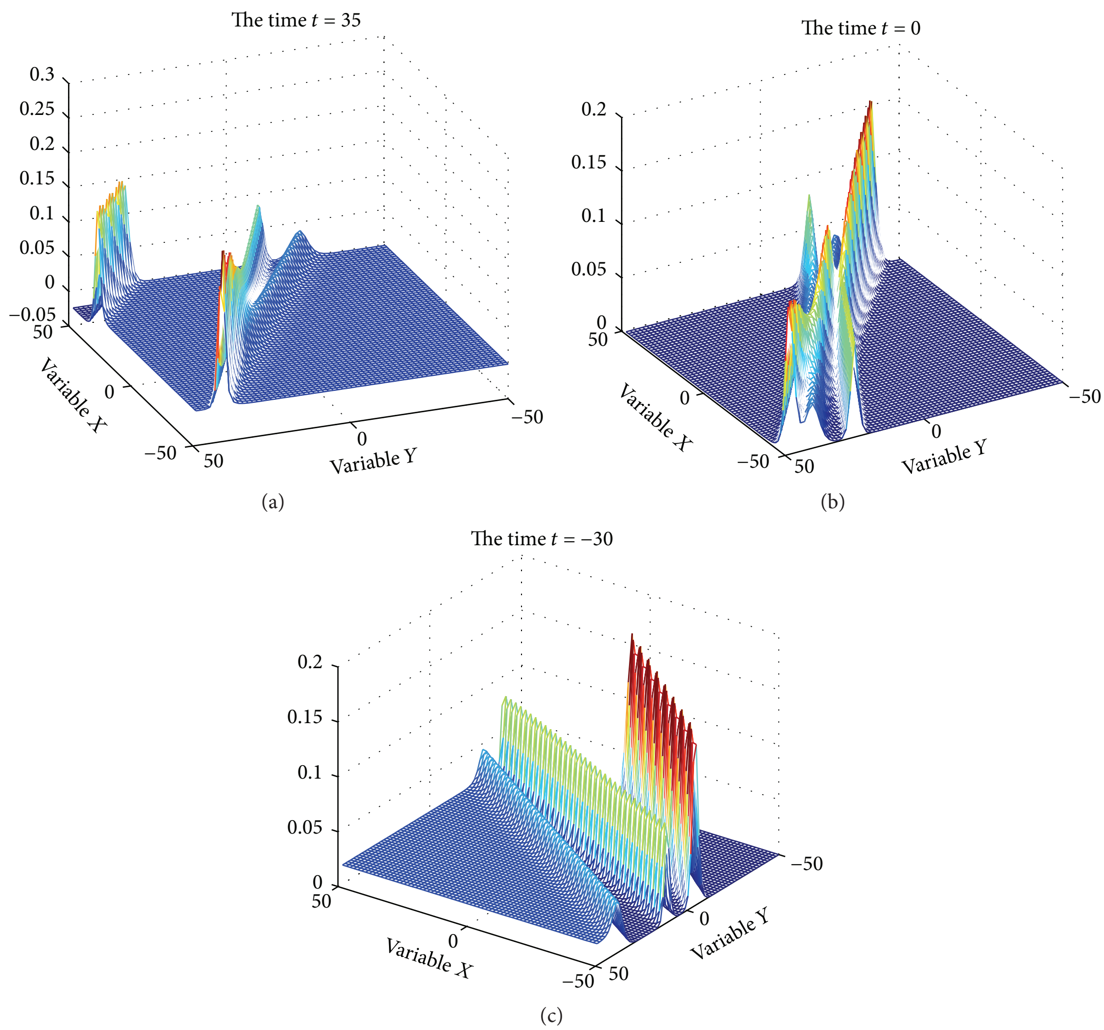

Figure 5: The process of (28) with $k_1 = 0.8$, $k_2 = 0.6$, $k_3 = 0.4$, $p_1 = 1.25$, $p_2 = 2$, $p_3 = 1.5$, $c(t) = 30$, $b(t) = 1$, and $h(t) = 1$, and (a) implies $t = 35$, (b) implies $t = 0$, and (c) implies $t = -30$.

Figure 2(a) indicates that there appear two waves on the time 45; in Figure 2(b), it causes the collision between the two waves; during the collision, it creates new waves because of the acting force; in Figure 2(c), the two waves merge into one wave after the collision and spread placidly.

Next, we discuss the influence of the variable coefficient; due to the different coefficient, the different waves shapes will occur; the specific progress is as shown in Figure 3.

In the end, we provide the interaction of the soliton solutions based on the coefficient $e^{A_{ij}} \neq 0$. As time goes on, there will happen the soliton pursuing collision because of the different soliton speed; the wave with the faster speed will catch up with the slower speed wave; then collision of the two-soliton solution can happen; after the collision, some of the solitons can spread along the previous direction, but the other soliton will be far from the previous direction and spread with a new direction; the faster speed wave will be in front of the slower waves; Figures 4 and 5 give a visual description of the collision.

As for two-soliton solutions (27), we get the soliton pursuing collision in Figure 4.

As for three-soliton solutions (28), we describe the three-soliton pursuing collision in Figure 5.

7. Conclusion

In this paper, we first introduce a generalized (2 + 1)-dimensional variable-coefficient KdV equation, which can describe the interaction between a water wave and gravity-capillary waves better than the (1 + 1)-dimensional KdV equation. Secondly, we get the N-soliton solutions of the (2+1)-dimensional variable-coefficient KdV equation via the Bell-polynomial approach and explain the interactions of the N-soliton solutions. The main conclusions of this paper can be summarized as follows:

(1) The Bell-polynomial of the (2 + 1)-dimensional variable-coefficient fifth-order KdV equation has

been got based on the Bell-polynomial method; then the Bell-polynomial-typed BT is obtained by virtue of the mixing variables.

(2) N-soliton solutions of the (2 + 1)-dimensional variable-coefficient fifth-order KdV equation are obtained with the Hirota approach; then the explicit one-soliton solutions, two-soliton solutions, and three-soliton solutions are showed; under the explicit solutions and the different coefficient of $e^{A_{ij}}$, we draw the pictures of the interaction of the soliton solutions; furthermore, through the graph, we can see the soliton fusion and the fission clearly in Figure 1 with $e^{A_{ij}} = 0$; in Figure 2, we analyze the influence of the variable coefficient; last but not least, the soliton pursuing collision is presented in Figures 3 and 4 and can get that, after the collision, the amplitude and the spread direction of the soliton will be changed under the condition of $e^{A_{ij}} \neq 0$.

(3) In the last part, we get the Bell-polynomial-typed BT of the variable-coefficient fifth-order KdV equation and via the expression of (11), we finally give rise to the bilinear-typed BT. Then, we give the period wave solutions with the help of the Riemann theta function.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

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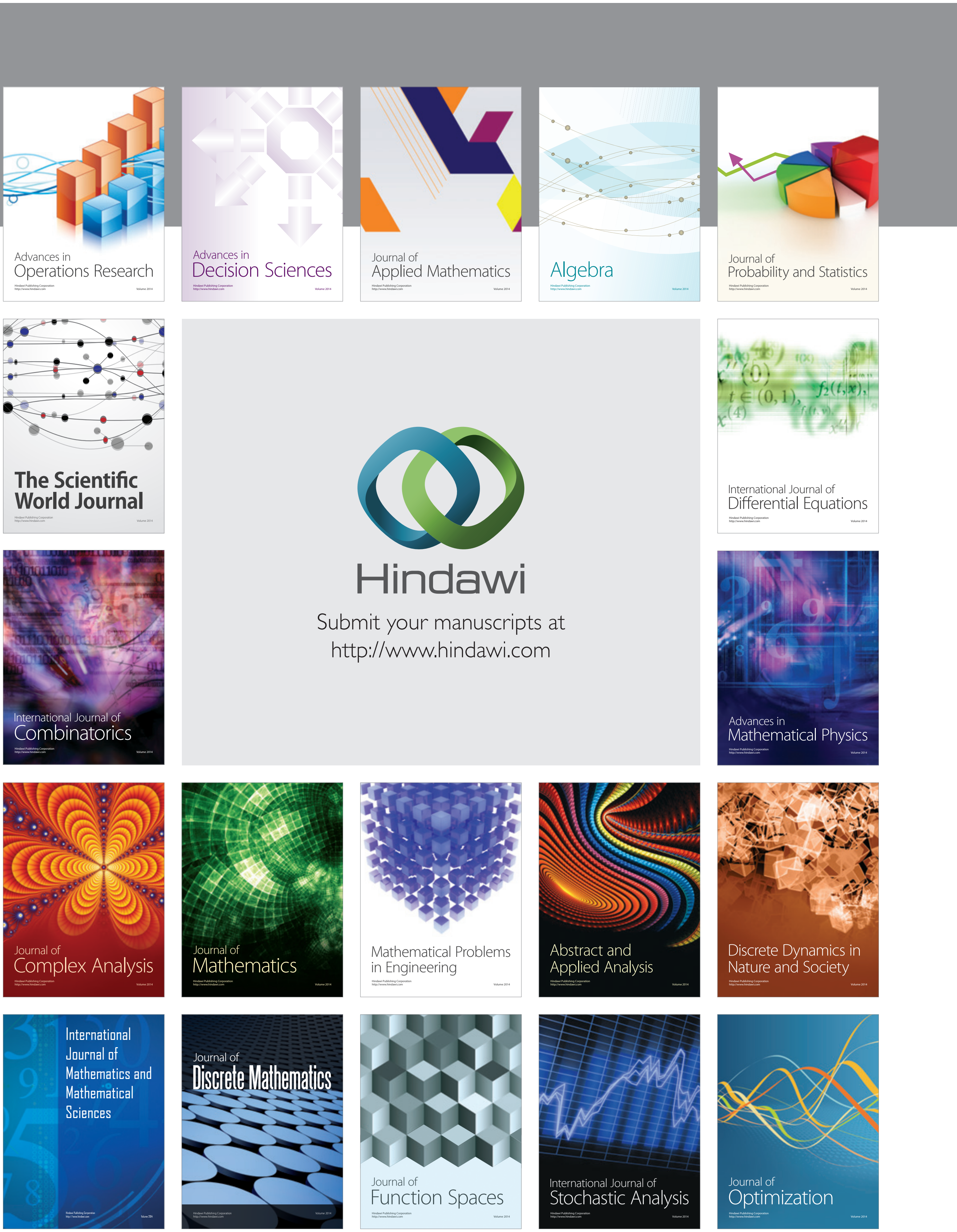

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