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Research Article

A Posteriori Error Estimate for Finite Volume Element Method of the Second-Order Hyperbolic Equations

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We establish a posteriori error estimate for finite volume element method of a second-order hyperbolic equation. Residual-type a posteriori error estimator is derived. The computable upper and lower bounds on the error in the H^1-norm are established. Numerical experiments are provided to illustrate the performance of the proposed estimator.

1. Introduction

The finite volume element method is a class of important numerical tools for solving partial differential equations. Due to the local conservation property and some other attractive properties, it is wildly used in many engineering fields, such as heat and mass transfer, fluid mechanics, and petroleum engineering, especially for those arising from conservation laws including mass, momentum, and energy. For the second-order hyperbolic equations, Li et al. [1] have proved the optimal order of convergence in H^1-norm. In [2], Kumar et al. have proved optimal order of convergence in L^2 and H^1-norm for the semidiscrete scheme and quasi-optimal order of convergence in maximum norm.

Since the pioneering work of Babuvška and Rheinboldt [3], the adaptive finite element methods based on a posteriori error estimates have become a central theme in scientific and engineering computations. Adaptive algorithm is among the most important means to boost accuracy and efficiency of the finite element discretization. The main idea of adaptive algorithm is to use the error indicator as a guide which shows whether further refinement of meshes is necessary. A computable a posteriori error estimator plays a crucial role in an adaptive procedure. A posteriori error analysis for the finite volume element method has been studied in [4–6] for the second-order elliptic problem, in [7–9] for the convection-diffusion equations, in [10] for the parabolic problems, in [11] for a model distributed optimal problem governed by linear parabolic equations, in [12] for the Stokes problem in two dimensions, and in [13] for the second-order hyperbolic equations.

However, to the best of our knowledge, there are few works related to the a posteriori error estimates of the finite volume element method for the second-order hyperbolic problems. The aim of this paper is to establish residual-type a posteriori error estimator of the finite volume element method for the second-order hyperbolic equation. We first construct a computable a posteriori error estimator of the finite volume element method. Then we analyze the residual-type a posteriori error estimates and obtain the computable upper and lower bounds on the error in the H^1-norm.

The organization of this paper is stated as follows. In Section 2, we present the framework of the finite volume element method for the second-order hyperbolic equation. In Section 3, we establish the residual-type a posteriori error estimator of the finite volume element method and derive the upper and lower bounds on the error in the H^1-norm. We provide some numerical experiments to illustrate the performance of the error estimator in Section 4.

2. Finite Volume Element Formulation

We use standard notation for Sobolev spaces $W^{s,p}(\Omega)$ with the norm $\|u\|_{s,p,\Omega}$ [14]. In order to simplify the notation, we

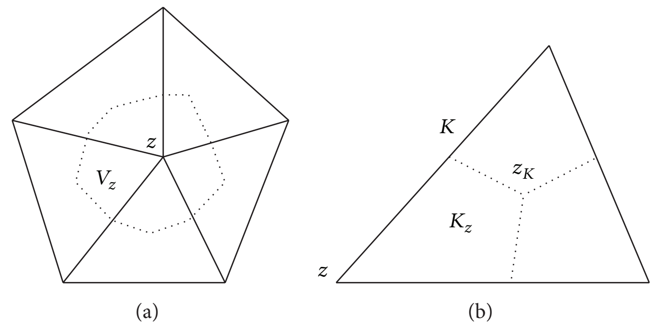

FIGURE 1: (a) The dotted line shows the boundary of the corresponding control volume V_z with z, a common vertex. (b) A triangle K is partitioned into three subregions K_z.

denote $W^{s,2}(\Omega)$ by $H^s(\Omega)$ and omit the index $p = 2$ and Ω whenever possible.

In this paper, we consider the following second-order hyperbolic problem:

$$\begin{aligned} u_{tt} - \nabla \cdot (a(x)\nabla u) &= f(x,t), \quad \text{in } \Omega \times (0,T], \\ u(x,t) &= 0, \quad \text{on } \partial\Omega \times (0,T], \\ u(x,0) &= u_0(x), \\ u_t(x,0) &= v_0(x), \\ &\text{in } \Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal bounded cross section, possessed with a Lipschitz boundary $\partial\Omega$. For simplicity, the right-hand side f is assumed to be measurable and square-integrable on $\Omega \times (0,T]$ and to be continuous with respect to time. The initial datum u_0 and v_0 are assumed to be measurable and square-integrable on Ω. $a(x,t) = (a_{ij}(x,t))_{i,j=1}^2$ is a real-valued smooth matrix function, uniformly symmetric, and positive definite in Ω.

The corresponding variational problem is to find $u \in H_0^1(\Omega)$, for $t > 0$, satisfying

$$(u_{tt}, v) + a(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega), \tag{2}$$

where the bilinear form $a(\cdot,\cdot)$ is defined by

$$a(u,v) = \int_\Omega a(x)\nabla u \cdot \nabla v\, dx, \quad \forall u,v \in H_0^1(\Omega). \tag{3}$$

Denote by T_h the primal quasi-uniform triangulation of Ω with $h = \max h_K$, where h_K is the diameter of the triangle $K \in T_h$. Let $\mathcal{U}_h$ be the standard conforming finite element space of piecewise linear functions, defined on the triangulation T_h:

$$\mathcal{U}_h = \left\{u \in C\left(\overline{\Omega}\right) : u|_K \text{ is linear and } u|_{\partial\Omega} = 0,\ \forall K \in T_h\right\}. \tag{4}$$

Denote by T_h^* the dual partition which is constructed in the same way as in [1, 15]. Let z_K be the barycenter of K. We connect z_K with the midpoints of the edges of K by straight line, thus partitioning K into three quadrilaterals K_z, $z \in Z_h(K)$, where $Z_h(K)$ are the vertices of K. Then with each vertex $z \in Z_h = \cup_{K\in T_h} Z_h(K)$, we associate a control volume V_z, which consists of the union of the subregions K_z, sharing the vertex z (see Figure 1). Finally, we obtain a group of control volumes covering the domain Ω, which is called the dual partition T_h^* of the triangulation T_h. Denote by Z_h^0 the set of interior vertices of Z_h and denote by $\mathscr{E}_h$ the set of all interior edges of T_h, respectively.

The partition T_h^* is regular or quasi-uniform, if there exists a positive constant $C > 0$ such that

$$C^{-1}h^2 \le \operatorname{meas}(V_z) \le Ch^2, \quad \forall V_z \in T_h^*. \tag{5}$$

The dual partition T_h^* will also be quasi-uniform [5] if the finite element triangulation T_h is quasi-uniform. The test function space $\mathscr{V}_h$ is defined by

$$\mathscr{V}_h = \left\{v \in L^2(\Omega) : v|_{V_z} \text{ is constant and } v|_{\partial\Omega} = 0\ \forall V_z \in T_h^*\right\}. \tag{6}$$

For any $u_h \in \mathcal{U}_h$, we define an interpolation operator $\Pi_h : \mathcal{U}_h \to \mathscr{V}_h$, such that

$$\Pi_h u_h = \sum_{z\in Z_h^0} u_h(z)\Psi_z, \tag{7}$$

where Ψ_z is the characteristic function of the control volume V_z.

According to [16], for each $u_h \in \mathcal{U}_h$, there exists a positive constant C independent of h, such that Π_h satisfies the following inequality:

$$\left\|u_h - \Pi_h u_h\right\|_{0,K} \le Ch_K \left|u_h\right|_{1,K}, \quad \forall K \in T_h. \tag{8}$$

Introduce the following adjoint elliptic problem:

$$-\nabla \cdot (a(x)\nabla u) = f \quad \text{in } \Omega, \text{ with } u = 0 \text{ on } \partial\Omega. \tag{9}$$

Denote by $\mathscr{T} : L^2(\Omega) \to H^2(\Omega)\bigcap H_0^1(\Omega)$ the solution operator of this problem, so that

$$a(\mathscr{T} f, \varphi) = (f,\varphi), \quad \forall \varphi \in H_0^1(\Omega). \tag{10}$$

Define negative norms by

$$\|v\|_{-s} = \sup\left\{\frac{(v,\varphi)}{\|\varphi\|_s};\ \varphi \in H^s(\Omega)\right\}, \quad \text{for } s \geq 0 \text{ integer}. \tag{11}$$

In fact, by Cauchy-Schwarz inequality, we obtain

$$\frac{(v,\varphi)}{\|\varphi\|_1} \leq \frac{\|v\|\,\|\varphi\|}{\|\varphi\|_1} \leq \frac{\|v\|\,\|\varphi\|_1}{\|\varphi\|_1} = \|v\|. \tag{12}$$

For our error analysis in the next section, it will be convenient to introduce such a norm defined by

$$|v|_{-s} = \left(\mathcal{T}^s v, v\right)^{1/2}, \quad \text{for } s \geq 0 \text{ integer}. \tag{13}$$

According to Thomée [17], we have the following lemma.

Lemma 1. *The norm $|v|_{-s}$ is equivalent to $\|v\|_{-s}$ and $(\mathcal{T} f, g) = (f, \mathcal{T} g)$, where s is a nonnegative integer. Particularly, $\|\mathcal{T} v\|_1$ is equivalent to $\|v\|_{-1}$ when $s = 1$.*

In order to get the fully discrete finite volume element method of (1), we give a partition of the time interval $[0,T]$: $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$. Let $\tau_n = t_n - t_{n-1}$, $\tau = \max_{1\leq n\leq N}\tau_n$, $U_h^n = U_h(t_n)$, and $U_h^{n,1/2} = (U_h^{n+1} + U_h^{n-1})/2$. With the help of Π_h, we obtain the fully discrete finite volume element method of (1): to find $U_h^n \in \mathcal{U}_h$, for $1 \leq n \leq N$, such that

$$\begin{gathered}\left(\partial_t\bar{\partial}U_h^n, \Pi_h\chi\right) + a\left(U_h^{n,1/2}, \Pi_h\chi\right) = \left(f^n, \Pi_h\chi\right), \quad \forall\chi \in \mathcal{U}_h, \\ U_h^0 = u_0, \\ \bar{\partial}U_h^1 = v_0,\end{gathered} \tag{14}$$

where

$$\partial_t\bar{\partial}U_h^n = \frac{\partial_t U_h^n - \partial_t U_h^{n-1}}{\tau_n} = \frac{\left(U_h^{n+1} - U_h^n\right)/\tau_{n+1} - \left(U_h^n - U_h^{n-1}\right)/\tau_n}{\tau_n}. \tag{15}$$

By setting $v = \partial u/\partial t = u_t$ and $\mathcal{Y} = \binom{u}{v}$, the notation $\nabla\cdot(a(x)\nabla)\phi = \nabla\cdot(a(x)\nabla\phi)$, (1) can equivalently be written as

$$\mathcal{Y}_t - \begin{pmatrix} 0 & 1 \\ \nabla\cdot(a(x)\nabla) & 0\end{pmatrix}\mathcal{Y} = F, \tag{16}$$

where $F = \binom{0}{f}$.

Let $V_h^n = \bar{\partial}U_h^{n+1}$; we define

$$\begin{aligned} U_\tau &= \frac{t - t_{n-1}}{\tau_n}U_h^n + \left(1 - \frac{t - t_{n-1}}{\tau_n}\right)U_h^{n-1}, \quad 1 \leq n \leq N, \\ V_\tau &= \frac{t - t_{n-1}}{\tau_n}V_h^n + \left(1 - \frac{t - t_{n-1}}{\tau_n}\right)V_h^{n-1}, \quad 1 \leq n \leq N.\end{aligned} \tag{17}$$

The residual system, with $Y_\tau = \binom{U_\tau}{V_\tau}$, is defined as follows:

$$\begin{gathered}\left(\mathcal{Y} - Y_\tau\right)_t - \begin{pmatrix} 0 & 1 \\ \nabla\cdot(a(x)\nabla) & 0\end{pmatrix}\left(\mathcal{Y} - Y_\tau\right) = \begin{pmatrix} P_u \\ P_v\end{pmatrix} \quad \text{in } \Omega\times[0,T], \\ u - U_\tau = 0 \quad \text{on } \partial\Omega\times[0,T], \\ \left(\mathcal{Y} - Y_\tau\right)(\cdot,0) = 0 \quad \text{in } \Omega,\end{gathered} \tag{18}$$

where the quantities P_u in $L^1(0,T;L^2(\Omega))$ and P_v in $L^1(0,T;H^{-1}(\Omega))$ are affine functions on each interval $[t_{n-1},t_n]$, $1 \leq n \leq N$, that

$$P_u(\cdot,t) = \begin{cases} V_\tau - V_h^{n-1}, & 2 \leq n \leq N, \\ 0, & n = 1.\end{cases} \tag{19}$$

And the quantities P_v are defined as follows.

From the fully discrete algorithm (14), for any $\varphi \in H_0^1(\Omega)$, $v \in \mathcal{U}_h$, we have

$$\begin{aligned}&\left(\partial_t\bar{\partial}U_h^n, \varphi\right) + a\left(U_h^{n,1/2}, \varphi\right) \\ &\quad = -\left(f^n - \partial_t\bar{\partial}U_h^n, \varphi - \Pi_h v\right) + \left(f^n, \varphi\right) \\ &\qquad + a\left(U_h^{n,1/2}, \varphi\right) - a\left(U_h^{n,1/2}, \Pi_h v\right).\end{aligned} \tag{20}$$

Since $(V_\tau)_t = \partial_t\bar{\partial}U_h^n$, by (2) and (20), for $t \in (t^{n-1}, t^n]$, we get

$$\begin{aligned}&\left((v - V_\tau)_t, \varphi\right) + a\left(u - U_h^{n,1/2}, \varphi\right) \\ &\quad = \left(f^n - \partial_t\bar{\partial}U_h^n, \varphi - \Pi_h v\right) + \left(f - f^n, \varphi\right) \\ &\qquad - a\left(U_h^{n,1/2}, \varphi\right) + a\left(U_h^{n,1/2}, \Pi_h v\right).\end{aligned} \tag{21}$$

Adding the term $a(U_h^{n,1/2} - U_\tau, \varphi)$ into the two hand sides of (21), we get

$$\begin{aligned}&\left((v - V_\tau)_t, \varphi\right) + a\left(u - U_\tau, \varphi\right) \\ &\quad = \left(f^n - \partial_t\bar{\partial}U_h^n, \varphi - \Pi_h v\right) + \left(f - f^n, \varphi\right) \\ &\qquad - \left[a\left(U_h^{n,1/2}, \varphi\right) - a\left(U_h^{n,1/2}, \Pi_h v\right)\right] \\ &\qquad + a\left(U_h^{n,1/2} - U_\tau, \varphi\right).\end{aligned} \tag{22}$$

So on each interval $[t_{n-1},t_n]$ $(2 \leq n \leq N)$, we have

$$\begin{aligned}\left(P_v, \varphi\right) &= \left(f^n - \partial_t\bar{\partial}U_h^n, \varphi - \Pi_h v\right) \\ &\quad - \left[a\left(U_h^{n,1/2}, \varphi\right) - a\left(U_h^{n,1/2}, \Pi_h v\right)\right] \\ &\quad + a\left(U_h^{n,1/2} - U_\tau, \varphi\right) + \left(f - f^n, \varphi\right), \\ &\qquad \forall\varphi \in H_0^1(\Omega),\ v \in \mathcal{U}_h.\end{aligned} \tag{23}$$

We define

$$\left(L^n,\varphi\right)=\left(f^n-\partial_t\bar{\partial}U_h^n,\varphi-\Pi_h v\right)-\left[a\left(U_h^{n,1/2},\varphi\right)-a\left(U_h^{n,1/2},\Pi_h v\right)\right]. \tag{24}$$

Then the term P_v on the interval $[t_{n-1},t_n]$ $(2\le n\le N)$ can be written as

$$\left(P_v,\varphi\right)=\left(L^n,\varphi\right)+a\left(U_h^{n,1/2}-U_\tau,\varphi\right)+\left(f-f^n,\varphi\right),\quad \forall\varphi\in H_0^1(\Omega),\ v\in\mathcal{U}_h. \tag{25}$$

When $t\in[0,t_1]$,

$$P_v(\cdot,t)=f(\cdot,t)+\nabla\cdot\left(a(x)\nabla\left(u_0+tv_0\right)\right). \tag{26}$$

3. Residual-Type A Posteriori Error Estimates

In this section, we will construct the residual-type a posteriori error estimates of the finite volume element method for (1). We introduce the jump of a vector-valued function across the edge $E\in\mathcal{E}_h$ which will be used in the residual-type a posteriori error estimates. Let E be an interior edge shared by elements K_+ and K_-. Define the unit normal vectors $\mathbf{n}_{K_+}$ and $\mathbf{n}_{K_-}$ on E pointing exterior to K_+ and K_-, respectively. Let $\mathbf{v}$ be a vector-valued function that is smooth inside each of the elements K_+ and K_-. $\mathbf{v}^+$ and $\mathbf{v}^-$ denote the traces of $\mathbf{v}$ on E taken from within the interior of K_+ and K_-, respectively. Then the jump of $\mathbf{v}$ on the edge E is defined by $[\mathbf{v}]_E=\mathbf{v}^+\cdot\mathbf{n}_{K_+}+\mathbf{v}^-\cdot\mathbf{n}_{K_-}$. We denote space refinement indicator by η_s^n defined by

$$\begin{aligned}\mathcal{R}_K^n&=f^n-\partial_t\bar{\partial}U_h^n+\nabla\cdot\left(a(x)\nabla U_h^{n,1/2}\right),\\ \mathcal{R}_E^n&=-\left[a(x)\nabla U_h^{n,1/2}\right]_E,\\ \eta_s^n&=\left(\sum_{K\in T_h}h_K^2\left\|\mathcal{R}_K^n\right\|_{0,K}^2+\sum_{E\in\mathcal{E}_h}h_E\left\|\mathcal{R}_E^n\right\|_{0,E}^2\right)^{1/2}.\end{aligned} \tag{27}$$

We define time refinement indicator η_t^n as

$$\eta_t^n=\tau_n\left\|U_h^n-U_h^{n-1}\right\|_1+\tau_n\left\|V_h^n-V_h^{n-1}\right\|. \tag{28}$$

3.1. Upper Bound. The Scott-Zhang interpolation function $\mathcal{I}_h:H_0^1(\Omega)\to\mathcal{U}_h$ is introduced in the following lemma [18].

Lemma 2. *For each $\varphi\in H_0^1(\Omega)$, a positive constant C is independent of h_K and h_E such that, for any $K\in T_h$, $E\in\mathcal{E}_h$*

$$\begin{aligned}\left\|\mathcal{I}_h\varphi\right\|_{1,\Omega}&\le C\|\varphi\|_{1,\Omega},\\ \left\|\varphi-\mathcal{I}_h\varphi\right\|_{0,K}&\le Ch_K\|\varphi\|_{1,\omega_K},\\ \left\|\varphi-\mathcal{I}_h\varphi\right\|_{0,E}&\le Ch_E^{1/2}\|\varphi\|_{1,\omega_E},\end{aligned} \tag{29}$$

where $\omega_K=\bigcup_{K'\cap K\neq\emptyset}K'$ and $\omega_E=\bigcup_{K\cap E\neq\emptyset}K$.

We also introduce the trace theorem [14].

Lemma 3 (trace theorem). *There exists a positive constant C independent of h_E such that*

$$\|\omega\|_{0,E}^2\le C\left(h_E^{-1}\|\omega\|_{0,K}^2+h_E\|\nabla\omega\|_{0,K}^2\right),\quad \forall\omega\in H^1(K),\ E\in\partial K,\ \forall K\in T_h. \tag{30}$$

Then we can get the following theorem for the upper bound of the error.

Theorem 4. *The following a posteriori error estimate holds between the solution u of (1) and the solution $(U_h^n)_{1\le n\le N}$ of (14), for $2\le m\le N$:*

$$\begin{aligned}&\left\|u^m-U_h^m\right\|+\left\|\sum_{k=1}^{k=m}\int_{t_{k-1}}^{t_k}\left(U_\tau-u\right)dt\right\|_1\\ &\le C\sum_{n=2}^{m}\left(\tau_n\left(\eta_t^n+\eta_s^n\right)\right)+C\sum_{n=2}^{m}\int_{t_{n-1}}^{t_n}\left\|f(\cdot,t)-f^n\right\|dt\\ &+C\int_0^{t_1}\left\|f(\cdot,t)+\nabla\cdot\left(a(x)\nabla\left(u_0+tv_0\right)\right)\right\|dt.\end{aligned} \tag{31}$$

Proof. Taking the inner product of (18) with $\binom{u-U_\tau}{\mathcal{T}(v-V_\tau)}$ and setting

$$Z(t)=\left(\left\|u-U_\tau\right\|^2+\left|v-V_\tau\right|_{-1}^2\right)^{1/2}, \tag{32}$$

we obtain, for $t\in[t_{n-1},t_n]$,

$$\begin{aligned}\frac{1}{2}\frac{dZ^2}{dt}&=\left(P_u,u-U_\tau\right)+\left(P_v,\mathcal{T}\left(v-V_\tau\right)\right)\le\left\|P_u\right\|\left\|u-U_\tau\right\|+\left\|\nabla\cdot\left(a(x)\nabla\left(U_h^{n,1/2}-U_\tau\right)\right)\right\|_{-1}\\ &\cdot\left\|\mathcal{T}\left(v-V_\tau\right)\right\|_1+\left\|L^n\right\|_{-1}\left\|\mathcal{T}\left(v-V_\tau\right)\right\|_1+\left\|f(\cdot,t)-f^n\right\|\left\|\mathcal{T}\left(v-V_\tau\right)\right\|\le\left\|P_u\right\|\left\|u-U_\tau\right\|+C\left\|\nabla\right.\\ &\left.\cdot\left(a(x)\nabla\left(U_h^{n,1/2}-U_\tau\right)\right)\right\|_{-1}\left\|v-V_\tau\right\|_{-1}+C\left\|L^n\right\|_{-1}\cdot\left\|v-V_\tau\right\|_{-1}+C\left\|f(\cdot,t)-f^n\right\|\left\|v-V_\tau\right\|_{-1}\\ &\le C\left(\left\|P_u\right\|^2+\left\|L^n\right\|_{-1}^2+\left\|f(\cdot,t)-f^n\right\|^2+\left\|\nabla\cdot\left(a(x)\nabla\left(U_h^{n,1/2}-U_\tau\right)\right)\right\|_{-1}^2\right)^{1/2}Z;\end{aligned} \tag{33}$$

hence,

$$\begin{aligned}\frac{dZ}{dt}&\le C\left(\left\|P_u\right\|^2+\left\|L^n\right\|_{-1}^2+\left\|f(\cdot,t)-f^n\right\|^2+\left\|\nabla\cdot\left(a(x)\nabla\left(U_h^{n,1/2}-U_\tau\right)\right)\right\|_{-1}^2\right)^{1/2}\le C\left(\left\|P_u\right\|\right.\\ &\left.+\left\|L^n\right\|_{-1}+\left\|f(\cdot,t)-f^n\right\|+\left\|\nabla\cdot\left(a(x)\nabla\left(U_h^{n,1/2}-U_\tau\right)\right)\right\|_{-1}\right).\end{aligned} \tag{34}$$

Integrating the inequality from t_{n-1} to t_n $(2 \le n \le N)$, we have

$$Z(t_n) - Z(t_{n-1}) \le C \int_{t_{n-1}}^{t_n} \left(\|P_u\| + \|L^n\|_{-1} + \|f(\cdot,t) - f^n\| + \left\|\nabla \cdot \left(a(x)\nabla\left(U_h^{n,1/2} - U_\tau\right)\right)\right\|_{-1} \right) dt. \tag{35}$$

Using Lemma 1, we obtain

$$\begin{aligned}
&\int_{t_{n-1}}^{t_n} \left\|\nabla \cdot \left(a(x)\nabla\left(U_h^{n,1/2} - U_\tau\right)\right)\right\|_{-1} dt \\
&= \int_{t_{n-1}}^{t_n} \left\|\nabla \cdot \left(a(x)\nabla\left(U_h^{n,1/2} - U_h^n + U_h^n - U_\tau\right)\right)\right\|_{-1} dt \\
&\le C \left\|U_h^n - U_h^{n-1}\right\|_1 \int_{t_{n-1}}^{t_n} \left(1 - \frac{t - t_{n-1}}{\tau_n}\right) dt \\
&\quad + C\frac{\tau_n}{2} \left\|U_h^{n+1} - U_h^n\right\|_1 + C\frac{\tau_n}{2} \left\|U_h^n - U_h^{n-1}\right\|_1 \\
&\le C\tau_n \left\|U_h^{n+1} - U_h^n\right\|_1 + C\tau_n \left\|U_h^n - U_h^{n-1}\right\|_1, \\
&\int_{t_{n-1}}^{t_n} \|P_u(\cdot,t)\| \, dt = \left\|V_h^n - V_h^{n-1}\right\| \int_{t_{n-1}}^{t_n} \frac{t - t_{n-1}}{\tau_n} dt \\
&= \frac{\tau_n}{2} \left\|V_h^n - V_h^{n-1}\right\|.
\end{aligned} \tag{36}$$

By the definition of η_t^n, we get

$$Z(t_n) - Z(t_{n-1}) \le C\left(\tau_n \eta_t^n + \tau_n \|L^n\|_{-1} + \int_{t_{n-1}}^{t_n} \|f(\cdot,t) - f^n\| \, dt\right). \tag{37}$$

In order to estimate $\|L^n\|_{-1}$, we choose $v = \mathcal{I}_h \varphi$ in (24); then

$$\begin{aligned}
(L^n, \varphi) &= \left(f^n - \partial_t \bar{\partial} U_h^n, \varphi - \Pi_h v\right) - \left[a\left(U_h^{n,1/2}, \varphi\right) - a\left(U_h^{n,1/2}, \Pi_h v\right)\right] \\
&= \left(f^n - \partial_t \bar{\partial} U_h^n, \varphi - v\right) + \left(f^n - \partial_t \bar{\partial} U_h^n, v - \Pi_h v\right) \\
&\quad - \left[a\left(U_h^{n,1/2}, \varphi\right) - a\left(U_h^{n,1/2}, v\right)\right] \\
&\quad - \left[a\left(U_h^{n,1/2}, v\right) - a\left(U_h^{n,1/2}, \Pi_h v\right)\right] \\
&\triangleq \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3 + \mathcal{I}_4.
\end{aligned} \tag{38}$$

Using Green's formula, we have

$$\begin{aligned}
\mathcal{I}_3 &= -\left(a(x)\nabla U_h^{n,1/2}, \nabla(\varphi - v)\right) \\
&= -\sum_{K \in T_h} \left(a(x)\nabla U_h^{n,1/2}, \nabla(\varphi - v)\right) \\
&= \sum_{K \in T_h} \left(\nabla \cdot \left(a(x)\nabla U_h^{n,1/2}\right), \varphi - v\right)_{0,K} \\
&\quad - \sum_{E \in \mathcal{E}_h} \left(\left[a(x)\nabla U_h^{n,1/2}\right]_E, \varphi - v\right)_{0,E}.
\end{aligned} \tag{39}$$

By the definition of $\mathcal{R}_K^n, \mathcal{R}_E^n$, we get

$$\begin{aligned}
&\mathcal{I}_1 + \mathcal{I}_3 \\
&= \sum_{K \in T_h} \left(f^n - \partial_t \bar{\partial} U_h^n + \nabla \cdot \left(a(x)\nabla U_h^{n,1/2}\right), \varphi - v\right)_{0,K} \\
&\quad - \sum_{E \in \mathcal{E}_h} \left(\left[a(x)\nabla U_h^{n,1/2}\right]_E, \varphi - v\right)_{0,E} \\
&= \sum_{K \in T_h} \left(\mathcal{R}_K^n, \varphi - v\right)_{0,K} + \sum_{E \in \mathcal{E}_h} \left(\mathcal{R}_E^n, \varphi - v\right)_{0,E}.
\end{aligned} \tag{40}$$

From Cauchy-Schwarz inequality and Lemma 2, we can get

$$\begin{aligned}
\left|\mathcal{I}_1 + \mathcal{I}_3\right| &\le C \sum_{K \in T_h} \left\{h_K \left\|\mathcal{R}_K^n\right\|_{0,K} \|\varphi\|_{1,\omega_K}\right\} \\
&\quad + C \sum_{E \in \mathcal{E}_h} \left\{h_E^{1/2} \left\|\mathcal{R}_E^n\right\|_{0,E} \|\varphi\|_{1,\omega_E}\right\}.
\end{aligned} \tag{41}$$

For $\mathcal{I}_4$, since $\Pi_h v$ is a constant in $K \bigcap K_z^*$, $z \in Z_h(K)$, $K_z^* \in T_h^*$, we have

$$\begin{aligned}
&\int_K a(x)\nabla U_h^{n,1/2} \cdot \nabla v \, dx \\
&= \sum_{z \in Z_h(K)} \int_{K \bigcap K_z^*} a(x)\nabla U_h^{n,1/2} \cdot \nabla\left(v - \Pi_h v\right) dx \\
&= -\sum_{z \in Z_h(K)} \int_{K \bigcap K_z^*} \nabla \cdot \left(a(x)\nabla U_h^{n,1/2}\right) \cdot \left(v - \Pi_h v\right) dx \\
&\quad + \sum_{z \in Z_h(K)} \int_{\partial(K \bigcap K_z^*)} a(x)\nabla U_h^{n,1/2} \cdot \mathbf{n}\left(v - \Pi_h v\right) ds \\
&= -\int_K \nabla \cdot \left(a(x)\nabla U_h^{n,1/2}\right) \cdot \left(v - \Pi_h v\right) dx \\
&\quad + \int_{\partial K} a(x)\nabla U_h^{n,1/2} \cdot \mathbf{n}\left(v - \Pi_h v\right) ds \\
&\quad + \sum_{z \in Z_h(K)} \int_{K \bigcap \partial K_z^*} a(x)\nabla U_h^{n,1/2} \cdot \mathbf{n}\left(v - \Pi_h v\right) ds.
\end{aligned} \tag{42}$$

Since $a(x)\nabla U_h^n$ and v are continuous inside each element $K \in T_h$, we have

$$\sum_{z\in Z_h(K)} \int_{K\cap \partial K_z^*} a(x)\nabla U_h^n \cdot \mathbf{n} v\, ds = 0,$$
$$\sum_{z\in Z_h(K)} \int_{K\cap \partial K_z^*} a(x)\nabla U_h^{n,1/2} \cdot \mathbf{n} v\, ds = 0. \tag{43}$$

Thus,

$$\mathcal{I}_4 = \sum_{K\in T_h} \left(\nabla\cdot\left(a(x)\nabla U_h^{n,1/2}\right), v - \Pi_h v\right)_{0,K} - \sum_{E\in\mathcal{E}_h} \left(\left[a(x)\nabla U_h^{n,1/2}\right]_E, v - \Pi_h v\right)_{0,E}. \tag{44}$$

Then we get

$$\begin{aligned}\mathcal{I}_2 + \mathcal{I}_4 &= \sum_{K\in T_h} \left(f^n - \partial_t\bar{\partial} U_h^n + \nabla\cdot\left(a(x)\nabla U_h^{n,1/2}\right), v - \Pi_h v\right)_{0,K} - \sum_{E\in\mathcal{E}_h} \left(\left[a(x)\nabla U_h^{n,1/2}\right]_E, v - \Pi_h v\right)_{0,E} \\ &= \sum_{K\in T_h} (\mathcal{R}_K^n, v - \Pi_h v)_{0,K} + \sum_{E\in\mathcal{E}_h} (\mathcal{R}_E^n, v - \Pi_h v)_{0,E}.\end{aligned} \tag{45}$$

By (8) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left|\sum_{K\in T_h} (\mathcal{R}_K^n, v - \Pi_h v)_{0,K}\right| &\le C\sum_{K\in T_h} \left\{h_K \|\mathcal{R}_K^n\|_{0,K} \|v\|_{1,K}\right\} \\ &\le C\left(\sum_{K\in T_h} h_K^2 \|\mathcal{R}_K^n\|_{0,K}^2\right)^{1/2} \left(\sum_{K\in T_h} \|v\|_{1,K}^2\right)^{1/2} \\ &= C\left(\sum_{K\in T_h} h_K^2 \|\mathcal{R}_K^n\|_{0,K}^2\right)^{1/2} \|\mathcal{I}_h\varphi\|_{1,\Omega} \\ &\le C\left(\sum_{K\in T_h} h_K^2 \|\mathcal{R}_K^n\|_{0,K}^2\right)^{1/2} \|\varphi\|_1. \\ \left|\sum_{E\in\mathcal{E}_h} (\mathcal{R}_E^n, v - \Pi_h v)_{0,E}\right| &\le \sum_{E\in\mathcal{E}_h} \|\mathcal{R}_E^n\|_{0,E} \|v - \Pi_h v\|_{0,E} \\ &\le \left(\sum_{E\in\mathcal{E}_h} h_E \|\mathcal{R}_E^n\|_{0,E}^2\right)^{1/2} \cdot \left(\sum_{E\in\mathcal{E}_h} h_E^{-1} \|v - \Pi_h v\|_{0,E}^2\right)^{1/2}.\end{aligned} \tag{46}$$

Since $\Pi_h v$ is a piecewise constant function, by Lemma 3 and (8), we get

$$\begin{aligned}\sum_{E\in\mathcal{E}_h} h_E^{-1} \|v - \Pi_h v\|_{0,E}^2 &\le C\sum_{E\in\mathcal{E}_h} \left(h_E^{-2} \|v - \Pi_h v\|_{0,K}^2 + |v|_{1,K}^2\right) \le C\|v\|_1^2 \\ &\le C\|\varphi\|_1^2.\end{aligned} \tag{47}$$

Substituting the estimate of $\mathcal{I}_1$–$\mathcal{I}_4$ into (38) and by the definition of η_s^n, we have

$$(L^n, \varphi) \le C\eta_s^n \|\varphi\|_1; \tag{48}$$

hence

$$\frac{(L^n, \varphi)}{\|\varphi\|_1} \le C\eta_s^n, \qquad \|L^n\|_{-1} \le C\eta_s^n. \tag{49}$$

Substituting the estimation of $\|L^n\|_{-1}$ into (37), we get

$$Z(t_n) - Z(t_{n-1}) \le C\left(\tau_n\eta_t^n + \tau_n\eta_s^n + \int_{t_{n-1}}^{t_n} \|f(\cdot,t) - f^n\|\, dt\right). \tag{50}$$

Summing (50) from $n = 2$ to $n = m$, we obtain

$$Z(t_m) - Z(t_1) \le C\sum_{n=2}^m (\tau_n(\eta_t^n + \eta_s^n)) + C\sum_{n=2}^m \int_{t_{n-1}}^{t_n} \|f(\cdot,t) - f^n\|\, dt. \tag{51}$$

For $n = 1$, we have

$$Z(t_1) - Z(t_0) \le C\int_0^{t_1} \|f(\cdot,t) + \nabla\cdot(a(x)\nabla(u_0 + tv_0))\|\, dt. \tag{52}$$

Noting that $Z(t_0) = Z(0) = 0$, then

$$\begin{aligned}Z(t_m) &\le C\sum_{n=2}^m (\tau_n(\eta_t^n + \eta_s^n)) + C\sum_{n=2}^m \int_{t_{n-1}}^{t_n} \|f(\cdot,t) - f^n\|\, dt \\ &\quad + C\int_0^{t_1} \|f(\cdot,t) + \nabla\cdot(a(x)\nabla(u_0 + tv_0))\|\, dt.\end{aligned} \tag{53}$$

By the fact that $(1/\sqrt{2})(a + b) \le \sqrt{a^2 + b^2} \le a + b$ $(a, b > 0)$, we have

$$\begin{aligned}\|u^m - U_h^m\| + |v^m - V_h^m|_{-1} &\le C\sum_{n=2}^m (\tau_n(\eta_t^n + \eta_s^n)) + C\sum_{n=2}^m \int_{t_{n-1}}^{t_n} \|f(\cdot,t) - f^n\|\, dt \\ &\quad + C\int_0^{t_1} \|f(\cdot,t) + \nabla\cdot(a(x)\nabla(u_0 + tv_0))\|\, dt.\end{aligned} \tag{54}$$

In view of the definition of the operator $\mathcal{T}$, we have

$$\mathcal{T}\frac{\partial v}{\partial t}+u=\mathcal{T}f(\cdot,t), \tag{55}$$

$$\mathcal{T}\frac{\partial V_\tau}{\partial t}+U_h^{m,1/2}=\mathcal{T}f(\cdot,t^m),\quad t\in\left[t^{m-1},t^m\right]. \tag{56}$$

Subtracting (56) from (55), we get

$$\mathcal{T}\frac{\partial(v-V_\tau)}{\partial t}+\left(u-U_h^{m,1/2}\right)=\mathcal{T}\left(f(\cdot,t)-f(\cdot,t^m)\right), \tag{57}$$

$$\mathcal{T}\frac{\partial(v-V_\tau)}{\partial t}+\mathcal{T}\left(f(\cdot,t^m)-f(\cdot,t)\right)+\left(U_\tau-U_h^{m,1/2}\right)=\left(U_\tau-u\right). \tag{58}$$

Integrating (58) from t_{m-1} to t_m, we obtain

$$\begin{aligned}&\mathcal{T}\left(v^m-V_h^m\right)-\mathcal{T}\left(v^{m-1}-V_h^{m-1}\right)\\&+\int_{t_{m-1}}^{t_m}\mathcal{T}\left(f(\cdot,t^m)-f(\cdot,t)\right)dt\\&+\int_{t_{m-1}}^{t_m}\left(U_\tau-U_h^{m,1/2}\right)dt=\int_{t_{m-1}}^{t_m}\left(U_\tau-u\right)dt.\end{aligned} \tag{59}$$

Summing (59) from $k=1$ to $k=m$, we obtain

$$\begin{aligned}&\sum_{k=1}^{k=m}\int_{t_{k-1}}^{t_k}\left(U_\tau-u\right)dt\\&=\mathcal{T}\left(v^m-V_h^m\right)\\&\quad+\sum_{k=1}^{k=m}\int_{t_{k-1}}^{t_k}\mathcal{T}\left(f\left(\cdot,t^k\right)-f(\cdot,t)\right)dt\\&\quad+\sum_{k=1}^{k=m}\int_{t_{k-1}}^{t_k}\left(U_\tau-U_h^{k,1/2}\right)dt.\end{aligned} \tag{60}$$

Thus, we have

$$\begin{aligned}&\left\|\sum_{k=1}^{k=m}\int_{t_{k-1}}^{t_k}\left(U_\tau-u\right)dt\right\|_1\\&\le\left\|\mathcal{T}\left(v^m-V_h^m\right)\right\|_1\\&\quad+\left\|\sum_{k=1}^{k=m}\int_{t_{k-1}}^{t_k}\mathcal{T}\left(f\left(\cdot,t^k\right)-f(\cdot,t)\right)dt\right\|_1\\&\quad+\left\|\sum_{k=1}^{k=m}\int_{t_{k-1}}^{t_k}\left(U_\tau-U_h^{k,1/2}\right)dt\right\|_1\\&\le C\left\|v^m-V_h^m\right\|_{-1}\\&\quad+C\sum_{k=1}^{k=m}\int_{t_{k-1}}^{t_k}\left\|f\left(\cdot,t^k\right)-f(\cdot,t)\right\|_{-1}dt\\&\quad+\sum_{k=1}^{k=m}\int_{t_{k-1}}^{t_k}\left(\left\|U_\tau-U_h^k\right\|_1+\left\|U_h^k-U_h^{k,1/2}\right\|_1\right)dt.\end{aligned} \tag{61}$$

Then,

$$\begin{aligned}&\left\|\sum_{k=1}^{k=m}\int_{t_{k-1}}^{t_k}\left(U_\tau-u\right)dt\right\|_1\\&\le C\left\|v^m-V_h^m\right\|_{-1}\\&\quad+C\sum_{k=1}^{k=m}\int_{t_{k-1}}^{t_k}\left\|f\left(\cdot,t^k\right)-f(\cdot,t)\right\|dt\\&\quad+C\sum_{k=1}^{k=m}\tau_k\left\|U_h^k-U_h^{k-1}\right\|_1.\end{aligned} \tag{62}$$

By (62) and (54), we have

$$\begin{aligned}&\left\|u^m-U_h^m\right\|+\left\|\sum_{k=1}^{k=m}\int_{t_{k-1}}^{t_k}\left(U_\tau-u\right)dt\right\|_1\\&\le C\sum_{n=2}^{m}\left(\tau_n\left(\eta_t^n+\eta_s^n\right)\right)+C\sum_{n=2}^{m}\int_{t_{n-1}}^{t_n}\left\|f(\cdot,t)-f^n\right\|dt\\&\quad+C\int_0^{t_1}\left\|f(\cdot,t)+\nabla\cdot\left(a(x)\nabla\left(u_0+tv_0\right)\right)\right\|dt.\end{aligned} \tag{63}$$

□

3.2. Lower Bound. In order to derive the local lower bounds on the error, we will introduce some properties of the bubble functions. For each triangle $K\in T_h$, denote by $\lambda_{K,1},\lambda_{K,2},\lambda_{K,3}$ the barycentric coordinates. Define the element-bubble function ψ_K by

$$\begin{aligned}&\psi_K=27\lambda_{K,1}\lambda_{K,2}\lambda_{K,3},\quad\text{in } K;\\&\psi_K=0,\quad\text{in }\Omega\setminus K.\end{aligned} \tag{64}$$

Assume that K and K' share the edge $E\in\mathcal{E}_h$. Let the barycentric coordinates with respect to the end points of E be $\lambda_{E,1}$ and $\lambda_{E,2}$. Define the edge-bubble function ψ_E by

$$\begin{aligned}&\psi_E=4\lambda_{E,1}\lambda_{E,2},\quad\text{in }\omega_E=K\cup K';\\&\psi_E=0,\quad\text{in }\Omega\setminus\omega_E.\end{aligned} \tag{65}$$

For properties of the bubble functions, we have the following lemma [19].

Lemma 5. *For each of the elements $K \in T_h$ and $E \in \mathscr{E}_h$, functions ψ_K and ψ_E have the following properties:*

$$\begin{aligned} &\operatorname{supp}\psi_K \subset K,\\ &\max_{x\in K}\psi_K = 1,\\ &\int_K \psi_K dx = \frac{9}{20}|K| \sim h_K^2,\\ &\|\nabla\psi_K\|_{0,K} \le Ch_K^{-1}\|\psi_K\|_{0,K}.\\ &\qquad\qquad \psi_K \in [0,1],\\ &\operatorname{supp}\psi_E \subset \omega_E,\\ &\max_{x\in\omega_E}\psi_E = 1,\\ &\int_E \psi_E ds = \frac{2}{3}h_E,\\ &\int_{\omega_E}\psi_E dx = \frac{1}{3}|\omega_E| \sim h_E^2,\\ &\|\nabla\psi_E\|_{0,\omega_E} \le Ch_E^{-1}\|\psi_E\|_{0,\omega_E},\\ &\qquad\qquad \psi_E \in [0,1]. \end{aligned} \tag{66}$$

We define the average of $\mathscr{R}_K^n$ on K ($\overline{\mathscr{R}_K^n}$) and the average of $\mathscr{R}_E^n$ on E ($\overline{\mathscr{R}_E^n}$) by

$$\begin{aligned} \overline{\mathscr{R}_K^n} &= \frac{1}{|K|}\int_K \mathscr{R}_K^n dx,\\ \overline{\mathscr{R}_E^n} &= \frac{1}{h_E}\int_E \mathscr{R}_E^n ds. \end{aligned} \tag{67}$$

Then we have the following local lower bounds.

Theorem 6. *For any $K \in T_h$, $E \in \mathscr{E}_h$, the following local posteriori lower bounds on the error $u^n - U_h^n$ hold for a positive constant C independent of h_K and h_E:*

$$\begin{aligned} h_K\|\mathscr{R}_K^n\|_{0,K} \le C\Big(&\|u^n - U_h^n\|_{1,K} + h_K\|u_{tt}^n - \partial_t\bar{\partial}U_h^n\|_{0,K}\\ &+ \|U_h^{n+1} - U_h^n\|_{1,K} + \|U_h^n - U_h^{n-1}\|_{1,K}\\ &+ 2h_K\|\mathscr{R}_K^n - \overline{\mathscr{R}_K^n}\|_{0,K}\Big), \end{aligned} \tag{68}$$

$$\begin{aligned} h_E^{1/2}\|\mathscr{R}_E^n\|_{0,E} \le C\Big(&\|u^n - U_h^n\|_{1,\omega_E}\\ &+ h_E\|u_{tt}^n - \partial_t\bar{\partial}U_h^n\|_{0,\omega_E} + \|U_h^{n+1} - U_h^n\|_{1,\omega_E}\\ &+ \|U_h^n - U_h^{n-1}\|_{1,\omega_E} + h_E\|\mathscr{R}_K^n - \overline{\mathscr{R}_K^n}\|_{0,\omega_E}\\ &+ h_E^{1/2}\|\mathscr{R}_E^n - \overline{\mathscr{R}_E^n}\|_{0,E}\Big). \end{aligned} \tag{69}$$

Proof. By triangle inequality, we have

$$\|\mathscr{R}_K^n\|_{0,K} \le \|\overline{\mathscr{R}_K^n}\|_{0,K} + \|\mathscr{R}_K^n - \overline{\mathscr{R}_K^n}\|_{0,K}. \tag{70}$$

By the properties of ψ_K, the definition of $\mathscr{R}_K^n$, and Green's formulation, we have

$$\begin{aligned} \|\overline{\mathscr{R}_K^n}\|_{0,K}^2 &\sim \left(\overline{\mathscr{R}_K^n}, \psi_K\overline{\mathscr{R}_K^n}\right)\\ &= \left(\mathscr{R}_K^n, \psi_K\overline{\mathscr{R}_K^n}\right) - \left(\mathscr{R}_K^n - \overline{\mathscr{R}_K^n}, \psi_K\overline{\mathscr{R}_K^n}\right)\\ &= \left(f^n - \partial_t\bar{\partial}U_h^n + \nabla\cdot\left(a(x)\nabla U_h^{n,1/2}\right), \psi_K\overline{\mathscr{R}_K^n}\right)\\ &\quad - \left(\mathscr{R}_K^n - \overline{\mathscr{R}_K^n}, \psi_K\overline{\mathscr{R}_K^n}\right)\\ &= \left(f^n - u_{tt}^n, \psi_K\overline{\mathscr{R}_K^n}\right) + \left(u_{tt}^n - \partial_t\bar{\partial}U_h^n, \psi_K\overline{\mathscr{R}_K^n}\right)\\ &\quad + \left(\nabla\cdot\left(a(x)\nabla U_h^{n,1/2}\right), \psi_K\overline{\mathscr{R}_K^n}\right)\\ &\quad - \left(\mathscr{R}_K^n - \overline{\mathscr{R}_K^n}, \psi_K\overline{\mathscr{R}_K^n}\right)\\ &= a\left(u^n, \psi_K\overline{\mathscr{R}_K^n}\right)\\ &\quad - \int_K a(x)\nabla U_h^{n,1/2}\cdot\nabla\left(\psi_K\overline{\mathscr{R}_K^n}\right)dx\\ &\quad + \left(u_{tt}^n - \partial_t\bar{\partial}U_h^n, \psi_K\overline{\mathscr{R}_K^n}\right) - \left(\mathscr{R}_K^n - \overline{\mathscr{R}_K^n}, \psi_K\overline{\mathscr{R}_K^n}\right)\\ &= \int_K a(x)\nabla\left(u^n - U_h^{n,1/2}\right)\cdot\nabla\left(\psi_K\overline{\mathscr{R}_K^n}\right)dx\\ &\quad + \left(u_{tt}^n - \partial_t\bar{\partial}U_h^n, \psi_K\overline{\mathscr{R}_K^n}\right) - \left(\mathscr{R}_K^n - \overline{\mathscr{R}_K^n}, \psi_K\overline{\mathscr{R}_K^n}\right)\\ &\equiv \mathscr{P}_1 + \mathscr{P}_2 + \mathscr{P}_3. \end{aligned} \tag{71}$$

For $\mathscr{P}_1$, with Cauchy-Schwarz inequality and Lemma 5, we get

$$\begin{aligned} |\mathscr{P}_1| &\le C\left|u^n - U_h^{n,1/2}\right|_{1,K}\left\|\nabla\left(\psi_K\overline{\mathscr{R}_K^n}\right)\right\|_{0,K}\\ &= C\left|u^n - U_h^{n,1/2}\right|_{1,K}\|\nabla\psi_K\|_{0,K}\left|\overline{\mathscr{R}_K^n}\right|\\ &\le Ch_K^{-1}\left|u^n - U_h^{n,1/2}\right|_{1,K}\|\psi_K\|_{0,K}\left|\overline{\mathscr{R}_K^n}\right|\\ &= Ch_K^{-1}\left|u^n - U_h^{n,1/2}\right|_{1,K}\left\|\psi_K\overline{\mathscr{R}_K^n}\right\|_{0,K}\\ &\le Ch_K^{-1}\left|u^n - U_h^{n,1/2}\right|_{1,K}\left\|\overline{\mathscr{R}_K^n}\right\|_{0,K}. \end{aligned} \tag{72}$$

By Cauchy-Schwarz inequality and Lemma 5, we obtain

$$\begin{aligned} |\mathscr{P}_2| &\le \|u_{tt}^n - \partial_t\bar{\partial}U_h^n\|_{0,K}\left\|\psi_K\overline{\mathscr{R}_K^n}\right\|_{0,K}\\ &\le C\|u_{tt}^n - \partial_t\bar{\partial}U_h^n\|_{0,K}\left\|\overline{\mathscr{R}_K^n}\right\|_{0,K},\\ |\mathscr{P}_3| &\le \left\|\mathscr{R}_K^n - \overline{\mathscr{R}_K^n}\right\|_{0,K}\left\|\psi_K\overline{\mathscr{R}_K^n}\right\|_{0,K}\\ &\le \left\|\mathscr{R}_K^n - \overline{\mathscr{R}_K^n}\right\|_{0,K}\left\|\overline{\mathscr{R}_K^n}\right\|_{0,K}. \end{aligned} \tag{73}$$

Combining (71)–(73), we obtain

$$\begin{aligned} h_K \left\|\mathcal{R}_K^n\right\|_{0,K} &\le C\left(\left\|u^n - U_h^{n,1/2}\right\|_{1,K}\right. \\ &\quad \left.+ h_K\left\|u_{tt}^n - \partial_t\bar{\partial}U_h^n\right\|_{0,K} + 2h_K\left\|\mathcal{R}_K^n - \overline{\mathcal{R}_K^n}\right\|_{0,K}\right) \\ &\le C\left(\left\|u^n - U_h^n\right\|_{1,K} + h_K\left\|u_{tt}^n - \partial_t\bar{\partial}U_h^n\right\|_{0,K}\right. \\ &\quad + \left\|U_h^{n+1} - U_h^n\right\|_{1,K} + \left\|U_h^n - U_h^{n-1}\right\|_{1,K} \\ &\quad \left.+ 2h_K\left\|\mathcal{R}_K^n - \overline{\mathcal{R}_K^n}\right\|_{0,K}\right). \end{aligned} \tag{74}$$

For (69), by triangle inequality, similarly we have

$$h_E^{1/2}\left\|\mathcal{R}_E^n\right\|_{0,E} \le h_E^{1/2}\left\|\overline{\mathcal{R}_E^n}\right\|_{0,E} + h_E^{1/2}\left\|\mathcal{R}_E^n - \overline{\mathcal{R}_E^n}\right\|_{0,E}. \tag{75}$$

By Lemma 5 and Green's formulation, we get

$$\begin{aligned} \left\|\overline{\mathcal{R}_E^n}\right\|_{0,E}^2 &\sim \left(\overline{\mathcal{R}_E^n}, \psi_E\overline{\mathcal{R}_E^n}\right)_{0,E} \\ &= \left(\mathcal{R}_E^n, \psi_E\overline{\mathcal{R}_E^n}\right)_{0,E} + \left(\overline{\mathcal{R}_E^n} - \mathcal{R}_E^n, \psi_E\overline{\mathcal{R}_E^n}\right)_{0,E} \\ &= \left(a(x)\nabla U_h^{n,1/2}, \nabla\left(\psi_E\overline{\mathcal{R}_E^n}\right)\right)_{0,\omega_E} \\ &\quad + \left(\nabla\cdot\left(a(x)\nabla U_h^{n,1/2}\right), \psi_E\overline{\mathcal{R}_E^n}\right)_{0,\omega_E} \\ &\quad + \left(\overline{\mathcal{R}_E^n} - \mathcal{R}_E^n, \psi_E\overline{\mathcal{R}_E^n}\right)_{0,E} \\ &= \int_{\omega_E} a(x)\nabla U_h^{n,1/2}\cdot\nabla\left(\psi_E\overline{\mathcal{R}_E^n}\right)dx \\ &\quad - \int_{\omega_E} a(x)\nabla u^n\cdot\nabla\left(\psi_E\overline{\mathcal{R}_E^n}\right)dx \\ &\quad + \int_{\omega_E} a(x)\nabla u^n\cdot\nabla\left(\psi_E\overline{\mathcal{R}_E^n}\right)dx \\ &\quad + \left(\nabla\cdot\left(a(x)\nabla U_h^{n,1/2}\right), \psi_E\overline{\mathcal{R}_E^n}\right)_{0,\omega_E} \\ &\quad + \left(\overline{\mathcal{R}_E^n} - \mathcal{R}_E^n, \psi_E\overline{\mathcal{R}_E^n}\right)_{0,E} \\ &= \int_{\omega_E} a(x)\nabla\left(U_h^{n,1/2} - u^n\right)\cdot\nabla\left(\psi_E\overline{\mathcal{R}_E^n}\right)dx \\ &\quad + \left(\nabla\cdot\left(a(x)\nabla U_h^{n,1/2}\right), \psi_E\overline{\mathcal{R}_E^n}\right)_{0,\omega_E} \\ &\quad + \int_{\omega_E}\left(f^n - \partial_t\bar{\partial}U_h^n\right)\left(\psi_E\overline{\mathcal{R}_E^n}\right)dx \\ &\quad + \int_{\omega_E}\left(\partial_t\bar{\partial}U_h^n - u_{tt}^n\right)\left(\psi_E\overline{\mathcal{R}_E^n}\right)dx \\ &\quad + \left(\overline{\mathcal{R}_E^n} - \mathcal{R}_E^n, \psi_E\overline{\mathcal{R}_E^n}\right)_{0,E} \\ &= \int_{\omega_E} a(x)\nabla\left(U_h^{n,1/2} - u^n\right)\cdot\nabla\left(\psi_E\overline{\mathcal{R}_E^n}\right)dx \\ &\quad + \left(\mathcal{R}_K^n, \psi_E\overline{\mathcal{R}_E^n}\right)_{0,\omega_E} \\ &\quad + \left(\partial_t\bar{\partial}U_h^n - u_{tt}^n, \psi_E\overline{\mathcal{R}_E^n}\right) \\ &\quad + \left(\overline{\mathcal{R}_E^n} - \mathcal{R}_E^n, \psi_E\overline{\mathcal{R}_E^n}\right)_{0,E} \\ &\equiv \mathcal{O}_1 + \mathcal{O}_2 + \mathcal{O}_3 + \mathcal{O}_4. \end{aligned} \tag{76}$$

Now we will estimate the right-hand terms of (76). By Lemma 5 and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} \left|\mathcal{O}_1\right| &\le C\left|U_h^{n,1/2} - u^n\right|_{1,\omega_E}\left\|\nabla\left(\psi_E\overline{\mathcal{R}_E^n}\right)\right\|_{0,\omega_E} \\ &= C\left|U_h^{n,1/2} - u^n\right|_{1,\omega_E}\left\|\nabla\psi_E\right\|_{0,\omega_E}\left|\overline{\mathcal{R}_E^n}\right| \\ &\le Ch_E^{-1}\left|U_h^{n,1/2} - u^n\right|_{1,\omega_E}\left\|\psi_E\right\|_{0,\omega_E}\left|\overline{\mathcal{R}_E^n}\right| \\ &\le Ch_E^{-1/2}\left|U_h^{n,1/2} - u^n\right|_{1,\omega_E}\left\|\overline{\mathcal{R}_E^n}\right\|_{0,E}, \\ \left|\mathcal{O}_2\right| &\le \left\|\mathcal{R}_K^n\right\|_{0,\omega_E}\left\|\psi_E\overline{\mathcal{R}_E^n}\right\|_{0,\omega_E} \\ &= \left\|\mathcal{R}_K^n\right\|_{0,\omega_E}\left\|\psi_E\right\|_{0,\omega_E}\left|\overline{\mathcal{R}_E^n}\right| \le Ch_E\left\|\mathcal{R}_K^n\right\|_{0,\omega_E}\left|\overline{\mathcal{R}_E^n}\right| \\ &\le Ch_E^{1/2}\left\|\mathcal{R}_K^n\right\|_{0,\omega_E}\left\|\overline{\mathcal{R}_E^n}\right\|_{0,E}, \\ \left|\mathcal{O}_3\right| &\le \left\|\partial_t\bar{\partial}U_h^n - u_{tt}^n\right\|_{0,\omega_E}\left\|\psi_E\overline{\mathcal{R}_E^n}\right\|_{0,\omega_E} \\ &\le Ch_E^{1/2}\left\|u_{tt}^n - \partial_t\bar{\partial}U_h^n\right\|_{0,\omega_E}\left\|\overline{\mathcal{R}_E^n}\right\|_{0,E}, \\ \left|\mathcal{O}_4\right| &\le \left\|\overline{\mathcal{R}_E^n} - \mathcal{R}_E^n\right\|_{0,E}\left\|\psi_E\overline{\mathcal{R}_E^n}\right\|_{0,E} \\ &\le \left\|\overline{\mathcal{R}_E^n} - \mathcal{R}_E^n\right\|_{0,E}\left\|\overline{\mathcal{R}_E^n}\right\|_{0,E}. \end{aligned} \tag{77}$$

Combining (77) with (76), we get

$$\begin{aligned} \left\|\overline{\mathcal{R}_E^n}\right\|_{0,E} &\le Ch_E^{-1/2}\left|U_h^{n,1/2} - u^n\right|_{1,\omega_E} + Ch_E^{1/2}\left\|\mathcal{R}_K^n\right\|_{0,\omega_E} \\ &\quad + Ch_E^{1/2}\left\|u_{tt}^n - \partial_t\bar{\partial}U_h^n\right\|_{0,\omega_E} \\ &\quad + \left\|\overline{\mathcal{R}_E^n} - \mathcal{R}_E^n\right\|_{0,E}. \end{aligned} \tag{78}$$

With (74), we obtain

$$\begin{aligned} h_E^{1/2}\left\|\mathcal{R}_E^n\right\|_{0,E} &\le C\left\|u^n - U_h^{n,1/2}\right\|_{1,\omega_E} + Ch_E\left\|u_{tt}^n\right. \\ &\quad \left.- \partial_t\bar{\partial}U_h^n\right\|_{0,\omega_E} + Ch_E\left\|\mathcal{R}_K^n - \overline{\mathcal{R}_K^n}\right\|_{0,\omega_E} + h_E^{1/2}\left\|\mathcal{R}_E^n\right. \\ &\quad \left.- \overline{\mathcal{R}_E^n}\right\|_{0,E} \le C\left(\left\|u^n - U_h^n + U_h^n - U_h^{n,1/2}\right\|_{1,\omega_E}\right. \\ &\quad + h_E\left\|u_{tt}^n - \partial_t\bar{\partial}U_h^n\right\|_{0,\omega_E} + h_E\left\|\mathcal{R}_K^n - \overline{\mathcal{R}_K^n}\right\|_{0,\omega_E} \end{aligned}$$

Table 1: Error estimates for Case 1.

h	$\|u^N - U_h^N\|_0$	Rate	$\|u^N - U_h^N\|_1$	Rate	$\mathfrak{D}^N$	$\mathfrak{N}^N$	$\mathscr{R}$
$1/2^2$	$1.3839e-02$	—	$1.6047e-01$	—	0.4589	21.3382	46.4952
$1/2^3$	$4.0831e-03$	1.7610	$8.2097e-02$	0.9669	0.4417	21.1967	47.9920
$1/2^4$	$9.3149e-04$	2.1321	$4.1279e-02$	0.9919	0.4307	21.0618	48.9005
$1/2^5$	$2.0574e-04$	2.1787	$2.0670e-02$	0.9979	0.4247	20.9680	49.3734
$1/2^6$	$4.6977e-05$	2.1308	$1.0338e-02$	0.9996	0.4215	20.9138	49.6128

Table 2: Error estimates for Case 2.

h	$\|u^N - U_h^N\|_0$	Rate	$\|u^N - U_h^N\|_1$	Rate	$\mathfrak{D}^N$	$\mathfrak{N}^N$	$\mathscr{R}$
$1/2^2$	$1.2513e-01$	—	2.3532	—	6.6633	66.7292	10.0144
$1/2^3$	$3.2994e-02$	1.9232	1.1822	0.9931	6.3537	64.6603	10.1768
$1/2^4$	$8.1195e-03$	2.0227	$5.9245e-01$	0.9967	6.1806	63.5799	10.2870
$1/2^5$	$1.8443e-03$	2.1383	$2.9641e-01$	0.9991	6.0904	63.0179	10.3471
$1/2^6$	$4.0436e-04$	2.1894	$1.4822e-01$	0.9999	6.0444	62.7297	10.3782

$$+ h_E^{1/2} \left\| \mathscr{R}_E^n - \overline{\mathscr{R}_E^n} \right\|_{0,E} \Big) \le C \Big(\left\| u^n - U_h^n \right\|_{1,\omega_E} + h_E \left\| u_{tt}^n - \partial_t \bar{\partial} U_h^n \right\|_{0,\omega_E} + \left\| U_h^{n+1} - U_h^n \right\|_{1,\omega_E} + \left\| U_h^n - U_h^{n-1} \right\|_{1,\omega_E} + h_E \left\| \mathscr{R}_K^n - \overline{\mathscr{R}_K^n} \right\|_{0,\omega_E} + h_E^{1/2} \left\| \mathscr{R}_E^n - \overline{\mathscr{R}_E^n} \right\|_{0,E} \Big). \quad (79)$$

□

4. Numerical Examples

Now we present some numerical examples to show the performance of the proposed error estimator. We consider problem (1) in $\Omega \times [0,T] = [0,1;0,1] \times [0,1]$. We discretize Ω into N number of rectangles in each direction and then each rectangle is divided into two triangles, resulting in a mesh with size $h = \sqrt{2}/N$. Discretize time by taking time step $\tau_n = \Delta t = h$. We consider the following two cases.

Case 1. Consider

$$a(x,y) = 1 + \sin\left(\frac{\pi}{4}x\right) + \sin\left(\frac{\pi}{4}y\right) + e^{2x} + e^{2y}, \quad u(x,y,t) = x(1-x)\,y(1-y)\,e^t. \quad (80)$$

Case 2. Consider

$$a(x,y) = e^{(x+y)/2}, \quad u(x,y,t) = \sin(\pi x)\sin(\pi y)\,e^t. \quad (81)$$

Define

$$\mathfrak{D}^m = \sum_{n=2}^{m} \left\| u^n - U_h^n \right\|_1, \quad \mathfrak{N}^m = \sum_{n=2}^{m} \left(\eta_t^n + \eta_s^n\right), \quad \mathscr{R} = \frac{\mathfrak{N}^m}{\mathfrak{D}^m}. \quad (82)$$

We present the results of the above cases when $m = N$ at Tables 1 and 2.

From Tables 1 and 2 we can see that the global a posteriori error estimator can predict the exact global error. The error estimator is reliable as evidenced by the ratio $\mathscr{R}$ listed on the tables. This list shows that the ratio $\mathscr{R}$ is converging to a constant when the mesh size is decreased by half. This shows that the proposed global a posteriori error estimator is robust for predicting the error in the finite volume element method.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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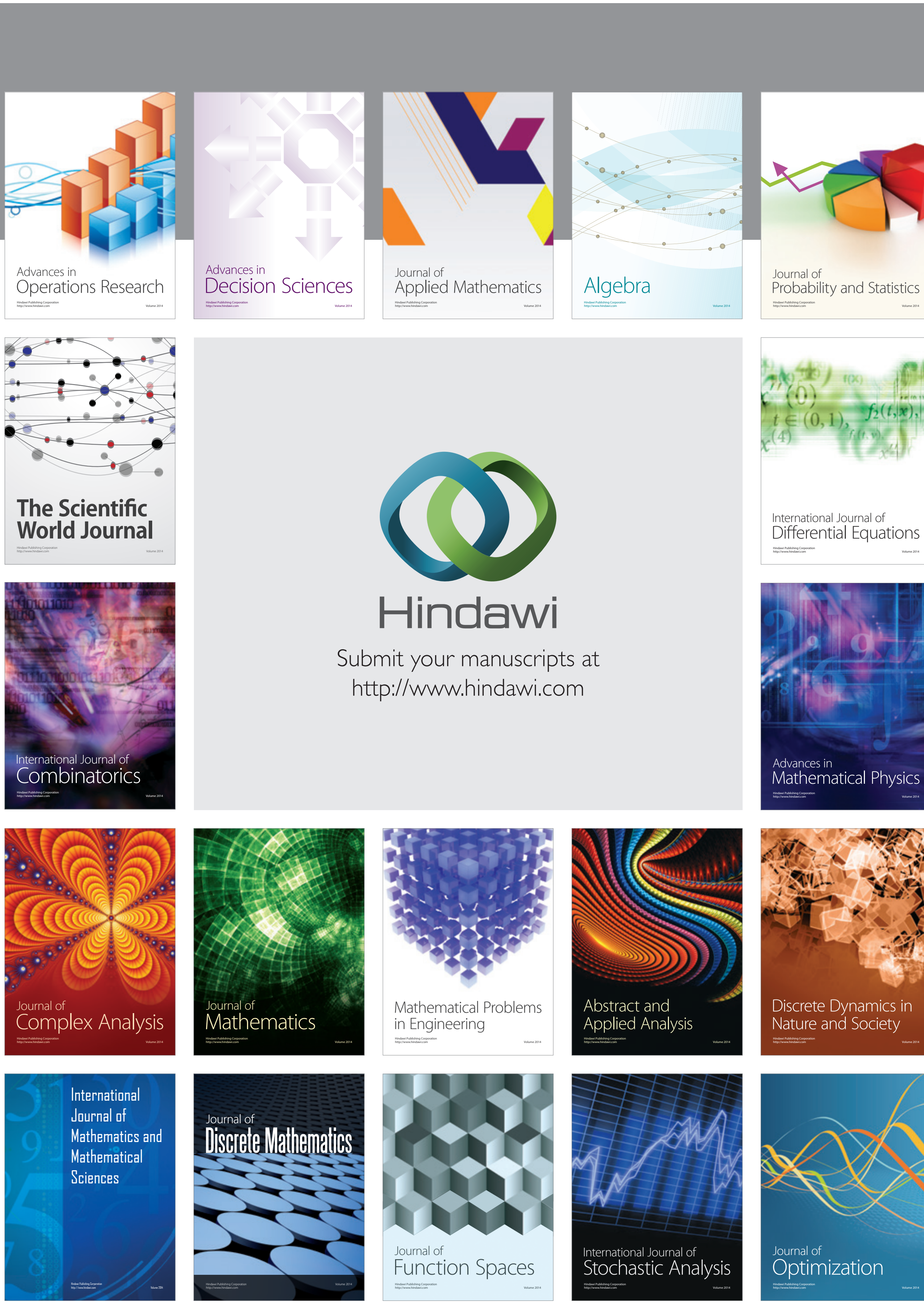
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