Hindawi Publishing Corporation
International Journal of Mathematics and Mathematical Sciences
Volume 2015, Article ID 476926, 13 pages
http://dx.doi.org/10.1155/2015/476926

Research Article

Quantum Product of Symmetric Functions

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Received 22 September 2014; Accepted 25 February 2015

Academic Editor: Hernando Quevedo

We provide an explicit description of the quantum product of multisymmetric functions using the elementary multisymmetric functions introduced by Vaccarino.

1. Introduction

Fix a characteristic zero field $\mathbb{K}$. The algebrogeometric duality allows us to identify affine algebraic varieties with the $\mathbb{K}$-algebra of polynomial functions on it, and, reciprocally, a finitely generated algebra without nilpotent elements may be identified with its spectrum, provided with the Zarisky topology. Affine space $\mathbb{K}^n$ is thus identified with the algebra of polynomials in n-variables $\mathbb{K}[x_1,\dots,x_n]$. Consider the action of the symmetric group S_n on $\mathbb{K}^n$ by permutation of vector entries. The quotient space $\mathbb{K}^n/S_n$ is the configuration space of n unlabeled points with repetitions in $\mathbb{K}$. Polynomial functions on $\mathbb{K}^n/S_n$ may be identified with the algebra $\mathbb{K}[x_1,\dots,x_n]^{S_n}$ of S_n-invariant polynomials in $\mathbb{K}[x_1,\dots,x_n]$. A remarkable classical fact is that $\mathbb{K}^n/S_n$ is again a n-dimensional affine space [1]; indeed we have an isomorphism of algebras

$$\mathbb{K}[x_1,\dots,x_n]^{S_n} \simeq \mathbb{K}[e_1,\dots,e_n], \tag{1}$$

where, for $\alpha \in [n] = \{1,\dots,n\}$, e_α is the elementary symmetric polynomial given by

$$e_\alpha(x_1,\dots,x_n) = \sum_{|a|=\alpha} x^\alpha = \sum_{a\subseteq[n],\,|a|=\alpha}\prod_{j\in a} x_j. \tag{2}$$

The elementary symmetric polynomials are determined by the identity

$$\prod_{i=1}^{n}(1+x_i t) = \sum_{\alpha=0}^{n} e_\alpha(x_1,\dots,x_n)\,t^\alpha. \tag{3}$$

Using characteristic functions one shows for $\alpha_1,\dots,\alpha_m \in [n]$ that

$$e_{\alpha_1}\cdots e_{\alpha_m} = \sum_{a\in\mathbb{N}^n} c(\alpha_1,\dots,\alpha_m,a)\,x^a, \tag{4}$$

where $c(\alpha_1,\dots,\alpha_m,a)$ is the cardinality of the subset of matrices of format $n\times m$ with entries in $\{0,1\}$ such that

$$\sum_{j=1}^{n} A_{ij} = \alpha_i \quad \text{for } i\in[m], \qquad \sum_{i=1}^{m} A_{ij} = a_j \quad \text{for } j\in[n]. \tag{5}$$

A subtler situation arises when one considers the configuration space

$$\frac{(\mathbb{K}^d)^n}{S_n} \tag{6}$$

of n unlabeled points with repetitions in $\mathbb{K}^d$, for $d \geq 2$. In this case $(\mathbb{K}^d)^n/S_n$ is no longer an affine space; instead it is an affine algebraic variety. Polynomial functions on $(\mathbb{K}^d)^n/S_n$ are the so-called multisymmetric functions, also known as vector symmetric functions or MacMahon symmetric functions [1, 2], and they coincide with the algebra of invariant polynomials

$$\mathbb{K}[x_{11},\dots,x_{1d},\dots,x_{n1},\dots,x_{nd}]^{S_n}, \tag{7}$$

which admits a presentation of the following form:

$$\frac{\mathbb{K}\left[e_\alpha \mid |\alpha| \in [n]\right]}{I_{n,d}}, \tag{8}$$

where the elementary multisymmetric functions e_α, for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d$ a vector such that $|\alpha| = \alpha_1 + \cdots + \alpha_d \leq n$, are defined by the identity

$$\begin{aligned}\prod_{i=1}^{n}&\left(1 + x_{i1}t_1 + \cdots + x_{id}t_d\right) \\ &= \sum_{\alpha\in\mathbb{N}^d,\,|\alpha|\leq n} e_\alpha\left(x_{11}, \ldots, x_{1d}, \ldots, x_{n1}, \ldots, x_{nd}\right) t_1^{\alpha_1} \cdots t_d^{\alpha_d}.\end{aligned} \tag{9}$$

Explicitly, the multisymmetric function e_α is given by

$$\begin{aligned}e_\alpha&\left(x_{11}, \ldots, x_{1d}, \ldots, x_{n1}, \ldots, x_{nd}\right) \\ &= \sum_a x^a = \sum_{|a|=\alpha} \prod_{j\in[d]} \prod_{i\in a_j} x_{ij},\end{aligned} \tag{10}$$

where in the middle term we regard $a \in M_{n\times d}(\{0,1\})$ as a matrix such that

$$\begin{gathered}\sum_{i=1}^{n} a_{ij} = \alpha_j \quad \text{for } j \in [d], \qquad \sum_{j=1}^{d} a_{ij} \leq 1 \quad \text{for } i \in [n], \\ x^a = \prod_{i=1}^{n}\prod_{j=1}^{d} x_{ij}^{a_{ij}};\end{gathered} \tag{11}$$

and in the right-hand side term we let $a = (a_1, \ldots, a_d)$ be a d-tuple of disjoint sets $a_j \subseteq [n]$ such that

$$|a| = \left(\left|a_1\right|, \ldots, \left|a_d\right|\right) = \left(\alpha_1, \ldots, \alpha_d\right). \tag{12}$$

It is not difficult to check that any multisymmetric function can be written (not uniquely) as a linear combination of products of elementary multisymmetric functions. The nonuniqueness is controlled by the ideal $I_{n,d}$. For an explicit description of $I_{n,d}$ the reader may consult Dalbec [3] and Vaccarino [4].

One checks for $\alpha_1, \ldots, \alpha_m \in \mathbb{N}^d$ that

$$e_{\alpha_1} \cdots e_{\alpha_m} = \sum_{a\in M_{n\times d}(\mathbb{N})} c\left(\alpha_1, \ldots, \alpha_m, a\right) x^a, \tag{13}$$

where $c(\alpha_1, \ldots, \alpha_m, a)$ counts the number of cubical matrices

$$A = \left(A_{ijl}\right) \in \operatorname{Map}\left([m] \times [n] \times [d], \{0,1\}\right) \tag{14}$$

such that

$$\begin{gathered}\sum_{i=1}^{m} A_{ijl} = a_{jl} \quad \text{for } j \in [n],\ l \in [d], \\ \sum_{l=1}^{d} A_{ijl} \leq 1 \quad \text{for } i \in [m],\ j \in [n], \\ \sum_{j=1}^{n} A_{ijl} = \left(\alpha_i\right)_l \quad \text{for } i \in [m],\ l \in [d].\end{gathered} \tag{15}$$

Recall that an algebra may be analyzed by describing it by generators and relations or alternatively, as emphasized by Rota and his collaborators, by finding a suitable basis such that the structural coefficients are positive integers with preferably a nice combinatorial interpretation. The second approach for the case of multisymmetric functions was undertaken by Vaccarino [4] and his results will be reviewed in Section 2. The main goal of this work (see Section 5) is to generalize this combinatorial approach to multisymmetric functions from the classical to the quantum setting.

Quantum mechanics, the century old leading small distances physical theory, is still not quite fully understood by mathematicians. The transition from classical to quantum mechanics has been particularly difficult to grasp. An appealing approach to this problem is to characterize the process of quantization as a process of deformation of a commutative Poisson algebra into a noncommutative algebra [5]. In this approach classical phase space is replaced by quantum phase space, where an extra dimension parametrized by a formal variable $\hbar$ is added.

The classical phase space of a Lagrangian theory is naturally endowed with a closed two-form. In the nondegenerated case (i.e., in the symplectic case) this two-form can be inverted given rise to a Poisson bracket on the algebra of smooth functions on phase space. In a sense, the Poisson bracket may be regarded as a tangent vector in the space of deformations of the algebra of functions on phase space, that is, as an infinitesimal deformation. That this infinitesimal deformation can be integrated into a formal deformation is a result according to Fedosov [6] for the symplectic case and according to Kontsevich [7] for arbitrary Poisson manifolds.

Many Lagrangian physical theories are invariant under a continuous group of transformations; in that case the two-form on phase space is necessarily degenerated. Nevertheless, a Lagrangian theory might be invariant under a finite group and still retain its nondegenerated character. In the latter scenario all the relevant constructions leading to the quantum algebra of functions on phase space are equivariant and thus give rise to quantum algebra of invariant functions under the finite group. We follow this path along this work, being as explicit and calculative as possible. Our main aim is thus to provide foundations as well as practical tools for dealing with quantum symmetric functions.

2. Multisymmetric Functions

In this section we introduce Vaccarino's multisymmetric functions $e_\alpha(p)$ which are defined in analogy with the elementary multisymmetric functions of Introduction, as yet the definition is general enough to account for the symmetrization of arbitrary polynomial functions [4].

Fix $a, n, d \in \mathbb{N}^+$. Let $y_1, \ldots, y_d$ and $t_1, \ldots, t_a$ be a pair of sets of commuting independent variables over $\mathbb{K}$. For $\alpha = (\alpha_1, \ldots, \alpha_a) \in \mathbb{N}^a$ we set

$$|\alpha| = \sum_{i=1}^{a} \alpha_i, \qquad t^\alpha = \prod_{i=1}^{a} t_i^{\alpha_i}. \tag{16}$$

For $q \in \mathbb{K}[y_1, \ldots, y_d]$ and $i \in [n]$ we let $q(i) = q(x_{i1}, \ldots, x_{id})$ be the polynomial obtained by replacing each appearance of y_j in q by x_{ij}, for $j \in [d]$. For example, for $n = 2$ and $q = y_1 y_2 y_3 \in \mathbb{R}[y_1, y_2, y_3]$ we have

$$q(1) = x_{11}x_{12}x_{13}, \qquad q(2) = x_{21}x_{22}x_{23}. \tag{17}$$

Definition 1. Consider $\alpha \in \mathbb{N}^a$ such that $|\alpha| \le n$ and $p = (p_1, \ldots, p_a) \in \mathbb{K}[y_1, \ldots, y_d]^a$. The multisymmetric functions $e_\alpha(p) \in \mathbb{K}[(\mathbb{K}^d)^n]^{S_n}$ are determined by the identity

$$\prod_{i=1}^{n} \left(1 + p_1(i)t_1 + \cdots + p_a(i)t_a\right) = \sum_{|\alpha| \le n} e_\alpha(p)t^\alpha. \tag{18}$$

Example 2. For $n = 3$ and $p = (y_1 y_2, y_3 y_4)$, we have that $e_{(1,1)}(y_1 y_2, y_3 y_4)$ is equal to $x_{13}x_{14}x_{21}x_{22} + x_{21}x_{22}x_{33}x_{34} + x_{23}x_{24}x_{31}x_{32} + x_{11}x_{12}x_{33}x_{34} + x_{13}x_{14}x_{31}x_{32} + x_{11}x_{12}x_{23}x_{24}$ and $e_{(2,1)}(y_1 y_2, y_3 y_4) = x_{11}x_{12}x_{21}x_{22}x_{33}x_{34} + x_{11}x_{12}x_{23}x_{24}x_{31}x_{32} + x_{13}x_{14}x_{21}x_{22}x_{31}x_{32}$.

Example 3. For $p = (y_1, \ldots, y_d)$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| \in [n]$, the multisymmetric functions $e_\alpha(y_1, \ldots, y_d)$ are the elementary multisymmetric functions defined in Introduction.

Next couple of lemmas follow directly from Definition 1.

Lemma 4. *Let $\alpha \in \mathbb{N}^a$ be such that $|\alpha| \le n$ and let $p = (p_1, \ldots, p_a) \in \mathbb{K}[y_1, \ldots, y_d]^a$. The multisymmetric function $e_\alpha(p)$ is given by the combinatorial identity*

$$e_\alpha(p) = \sum_{c} \prod_{l=1}^{a} \prod_{i \in c_l} p_l(i), \tag{19}$$

where $c = (c_1, \ldots, c_a)$ is a tuple of disjoint subsets of $[n]$, with $|c| = (|c_1|, \ldots, |c_a|) = \alpha$.

Recall that the symmetrization map $\mathbb{K}[(\mathbb{K}^d)^n] \to \mathbb{K}[(\mathbb{K}^d)^n]^{S_n}$ sends f to

$$\sum_{\sigma \in S_n} f \circ \sigma, \tag{20}$$

where one regards $\sigma \in S_n$ as a map $\sigma : (\mathbb{K}^d)^n \to (\mathbb{K}^d)^n$.

Lemma 5. *Let $\alpha \in \mathbb{N}^a$ be such that $|\alpha| \le n$ and $p = (p_1, \ldots, p_a) \in \mathbb{K}[y_1, \ldots, y_d]^a$. The multisymmetric function $e_\alpha(p)$ is the symmetrization of the polynomial*

$$\begin{aligned} &p_1(1) \cdots p_1(\alpha_1) \cdots p_i\left(\sum_{l=1}^{i-1} \alpha_l + 1\right) \cdots p_i\left(\sum_{l=1}^{i} \alpha_l\right) \\ &\quad \cdots p_a\left(\sum_{l=1}^{a-1} \alpha_l + 1\right) \cdots p_a\left(\sum_{l=1}^{a} \alpha_l\right). \end{aligned} \tag{21}$$

Lemma 6. *Let $p = (p_1, \ldots, p_a) \in \mathbb{K}[y_1, \ldots, y_d]^a$ be expanded in monomials as*

$$p_1 = \sum_{j_1 \in [k_1]} c_{1j_1} m_{1j_1}, \ldots, p_a = \sum_{j_a \in [k_a]} c_{aj_a} m_{aj_a}. \tag{22}$$

Then

$$e_\alpha(p) = \sum_{\beta \in \mathbb{N}^{|k|},\, r(\beta) = \alpha} e_\beta(m) c^\beta, \tag{23}$$

where $m = (m_{11}, \ldots, m_{1k_1}, \ldots, m_{a1}, \ldots, m_{aj_a})$, $c = (c_{11}, \ldots, c_{1k_1}, \ldots, c_{a1}, \ldots, c_{ak_a})$, and for $\beta = (\beta_{11}, \ldots, \beta_{1k_1}, \ldots, \beta_{a1}, \ldots, \beta_{ak_a}) \in \mathbb{N}^{|k|}$ we set

$$r(\beta) = \left(\beta_{11} + \cdots + \beta_{1k_1}, \ldots, \beta_{a1} + \cdots + \beta_{ak_a}\right). \tag{24}$$

Proof. Let $|k| = k_1 + \cdots + k_a$ and $ct = (c_{11}t_1, \ldots, c_{1k_1}t_1, \ldots, c_{a1}t_a, \ldots, c_{ak_a}t_a)$. Then

$$\begin{aligned} \sum_{\alpha \in \mathbb{N}^a,\, |\alpha| \le n} e_\alpha(p)t^\alpha &= \prod_{i=1}^{n} \Bigg(1 + \sum_{j_1 \in [k_1]} c_{1j_1} m_{1j_1}(i)t_1 \\ &\qquad + \cdots + \sum_{j_a \in [k_a]} c_{aj_a} m_{aj_a}(i)t_a\Bigg) \\ &= \sum_{\beta \in \mathbb{N}^{|k|},\, |\beta| \le n} e_\beta(m)(ct)^\beta \\ &= \sum_{\beta \in \mathbb{N}^{|k|},\, |\beta| \le n} e_\beta(m)c^\beta t^{r(\beta)}. \end{aligned} \tag{25}$$

Thus we get

$$e_\alpha(p) = \sum_{\beta \in \mathbb{N}^{|k|},\, r(\beta) = \alpha} e_\beta(m)c^\beta. \tag{26}$$

□

The following result according to Vaccarino [4] provides an explicit formula for the product of multisymmetric functions. We include the proof since the same technique carries over to the more involved quantum case.

Theorem 7. *Fix $a, b, n \in \mathbb{N}^+$, $p \in \mathbb{K}[y_1, \ldots, y_d]^a$, and $q \in \mathbb{K}[y_1, \ldots, y_d]^b$. Let $\alpha \in \mathbb{N}^a$ and $\beta \in \mathbb{N}^b$ be such that $|\alpha|, |\beta| \le n$. Then one has*

$$e_\alpha(p)e_\beta(q) = \sum_{\gamma \in L(\alpha, \beta, n)} e_\gamma(p, q, pq), \quad \textit{where:} \tag{27}$$

(i) $(p, q, pq) = (p_1, \ldots, p_a, q_1, \ldots, q_b, p_1q_1, \ldots, p_1q_b, \ldots, p_aq_1, \ldots, p_aq_b)$,

(ii) *$L(\alpha, \beta, n)$ is the set of matrices $\gamma \in \mathrm{Map}([0, a] \times [0, b], \mathbb{N})$ such that*

$$\begin{gathered} \gamma_{00} = 0, \qquad |\gamma| = \sum_{l=0}^{a} \sum_{r=0}^{b} \gamma_{lr} \le n, \\ \sum_{r=0}^{b} \gamma_{lr} = \alpha_l \quad \textit{for } l \in [a], \\ \sum_{l=0}^{a} \gamma_{lr} = \beta_r \quad \textit{for } r \in [b]. \end{gathered} \tag{28}$$

Proof. Identify the matrix γ with the vector

$$\gamma = \left(0, \gamma_{01}, \ldots, \gamma_{0b}, \gamma_{10}, \ldots, \gamma_{a0}, \gamma_{20}, \ldots, \gamma_{2b}, \ldots, \gamma_{a0}, \ldots, \gamma_{ab}\right). \tag{29}$$

We have that

$$\sum_{|\alpha|, |\beta| \le n} e_\alpha(p)\, e_\beta(q)\, t^\alpha s^\beta \tag{30}$$

is equal to

$$\begin{aligned}
&= \left(\sum_{|\alpha|\le n} e_\alpha(p)\, t^\alpha\right)\left(\sum_{|\beta|\le n} e_\beta(q)\, s^\beta\right) \\
&= \prod_{i=1}^{n}\left(1 + \sum_{l=1}^{a} p_l(i)\, t_l\right)\prod_{i=1}^{n}\left(1 + \sum_{r=1}^{b} q_r(i)\, s_r\right) \\
&= \prod_{i=1}^{n}\left(1 + \sum_{l=1}^{a} p_l(i)\, t_l\right)\left(1 + \sum_{r=1}^{b} q_r(i)\, s_r\right) \\
&= \prod_{i=1}^{n}\left(1 + \sum_{l=1}^{a} p_l(i)\, t_l + \sum_{r=1}^{b} q_r(i)\, s_r\right. \\
&\qquad \left. + \sum_{l=1}^{a}\sum_{r=1}^{b} p_l(i)\, q_r(i)\, t_l s_r\right) \\
&= \prod_{i=1}^{n}\left(1 + \sum_{l=1}^{a} p_l(i)\, w_{l0} + \sum_{r=1}^{b} q_r(j)\, w_{0r}\right. \\
&\qquad \left. + \sum_{l=1}^{a}\sum_{r=1}^{b} p_l(i)\, q_r(i)\, w_{lr}\right) \\
&= \sum_{\gamma} e_\gamma(p, q, pq)\, w^\gamma,
\end{aligned} \tag{31}$$

where for $\gamma \in L(\alpha, \beta, n)$ we set

$$w^\gamma = \prod_{l=0}^{a}\prod_{r=0}^{b} w_{lr}^{\gamma_{lr}} = \prod_{l=0}^{a}\prod_{r=0}^{b} \left(t_l s_r\right)^{\gamma_{lr}} = \prod_{l=0}^{a}\prod_{r=0}^{b} t_l^{\gamma_{lr}} s_r^{\gamma_{lr}}, \tag{32}$$

using the conventions

$$t_0 = s_0 = 1, \qquad w_{lr} = t_l s_r \quad \text{for } l, r \ge 0. \tag{33}$$

For w^γ to be equal to $t^\alpha s^\beta$ we must have

$$\begin{aligned}
w^\gamma &= \prod_{l=0}^{a}\prod_{r=0}^{b} t_l^{\gamma_{lr}} s_r^{\gamma_{lr}} = \left(\prod_{l=1}^{a} t_l^{\sum_{r=0}^{b}\gamma_{lr}}\right)\left(\prod_{r=1}^{b} s_r^{\sum_{l=0}^{a}\gamma_{lr}}\right) \\
&= \left(\prod_{l=1}^{a} t_l^{\alpha_l}\right)\left(\prod_{k=1}^{b} s_r^{\beta_r}\right).
\end{aligned} \tag{34}$$

Thus we conclude that

$$\begin{aligned}
\sum_{r=0}^{b} \gamma_{lr} &= \alpha_l \quad \text{for } l \in [a], \\
\sum_{l=0}^{a} \gamma_{lr} &= \beta_r \quad \text{for } r \in [b].
\end{aligned} \tag{35}$$

□

Graphically, a matrix $\gamma \in L(\alpha, \beta, n)$ is represented as

$$\begin{array}{cccccccc}
0 & \gamma_{01} & \gamma_{02} & \gamma_{03} & \cdots & \gamma_{0b} & & \\
\gamma_{10} & \gamma_{11} & \gamma_{12} & \gamma_{13} & \cdots & \gamma_{1b} & \longrightarrow & \alpha_1 \\
\gamma_{20} & \gamma_{21} & \gamma_{22} & \gamma_{23} & \cdots & \gamma_{2b} & \longrightarrow & \alpha_2 \\
 & \vdots & \vdots & \vdots & & \vdots & & \vdots \\
\gamma_{a0} & \gamma_{a1} & \gamma_{a2} & \gamma_{a3} & \cdots & \gamma_{ab} & \longrightarrow & \alpha_a \\
 & \downarrow & \downarrow & \downarrow & & \downarrow & & \\
 & \beta_1 & \beta_2 & \beta_3 & \cdots & \beta_b, & &
\end{array} \tag{36}$$

where the horizontal and vertical arrows represent, respectively, row and column sums.

Example 8. For $n = 3$, $p = (y_1 y_2, y_1)$, and $q = (y_1 y_2, y_3)$ we have

$$\begin{aligned}
&e_{(1,1)}\left(y_1 y_2, y_1\right) e_{(2,1)}\left(y_1 y_2, y_3\right) \\
&\quad = \sum_{\gamma} e_\gamma\left(y_1 y_2, y_1, y_1 y_2, y_3, y_1^2 y_2^2, y_1 y_2 y_3, y_1^2 y_2, y_1 y_3\right),
\end{aligned} \tag{37}$$

where $\gamma = (\gamma_{10}, \gamma_{20}, \gamma_{01}, \gamma_{02}, \gamma_{11}, \gamma_{12}, \gamma_{21}, \gamma_{22}) \in \mathbb{N}^8$ is such that $|\gamma| \le 3$ and

$$\begin{aligned}
\gamma_{10} + \gamma_{11} + \gamma_{12} &= 1, \qquad & \gamma_{20} + \gamma_{21} + \gamma_{22} &= 1, \\
\gamma_{01} + \gamma_{11} + \gamma_{21} &= 2, \qquad & \gamma_{02} + \gamma_{12} + \gamma_{22} &= 1.
\end{aligned} \tag{38}$$

Looking at the solutions in $\mathbb{N}$ of the system of linear equations above we obtain

$$\begin{aligned}
&e_{(1,1)}\left(y_1 y_2, y_1\right) e_{(2,1)}\left(y_1 y_2, y_3\right) \\
&\quad = e_{(1,1,1)}\left(y_3, y_1^2 y_2^2, y_1^2 y_2\right) + e_{(1,1,1)}\left(y_1 y_2, y_1 y_2 y_3, y_1^2 y_2\right) \\
&\qquad + e_{(1,1,1)}\left(y_1 y_2, y_1^2 y_2^2, y_1 y_3\right).
\end{aligned} \tag{39}$$

Example 9. For $n = 4$, $p = (y_1^2 y_2, y_2^3 y_3, y_1 y_2 y_3)$, and $q = (y_1^3 y_2^2 y_3, y_1^2 y_3, y_2 y_3)$ we have

$$\begin{aligned}
&e_{(1,1,1)}\left(y_1^2 y_2, y_2^3 y_3, y_1 y_2 y_3\right) e_{(1,2,1)}\left(y_1^3 y_2^2 y_3, y_1^2 y_3, y_2 y_3\right) \\
&\quad = \sum_{\gamma} e_\gamma\left(y_1^2 y_2, y_2^3 y_3, y_1 y_2 y_3, y_1^3 y_2^2 y_3, y_1^2 y_3, y_2 y_3,\right. \\
&\qquad\qquad y_1^5 y_2^3, y_1^4 y_2 y_3, y_1^2 y_2^2 y_3, y_1^3 y_2^5 y_3^2, y_2^4 y_3^2, y_1 y_2^4 y_3, \\
&\qquad\qquad \left. y_1^4 y_2^3 y_3^2, y_1^3 y_2 y_3^2, y_1^2 y_2^2 y_3\right),
\end{aligned} \tag{40}$$

where $\gamma = (\gamma_{10}, \gamma_{20}, \gamma_{30}, \gamma_{01}, \gamma_{02}, \gamma_{03}, \gamma_{11}, \gamma_{12}, \gamma_{13}, \gamma_{21}, \gamma_{22}, \gamma_{23}, \gamma_{31}, \gamma_{32}, \gamma_{33}) \in \mathbb{N}^{15}$ is such that $|\gamma| \le 4$ and

$$\begin{aligned} &\gamma_{10} + \gamma_{11} + \gamma_{12} + \gamma_{13} = 1, \qquad &&\gamma_{20} + \gamma_{21} + \gamma_{22} + \gamma_{23} = 1, \\ &\gamma_{30} + \gamma_{31} + \gamma_{32} + \gamma_{33} = 1, \qquad &&\gamma_{01} + \gamma_{11} + \gamma_{21} + \gamma_{31} = 1, \\ &\gamma_{02} + \gamma_{12} + \gamma_{22} + \gamma_{32} = 2, \qquad &&\gamma_{03} + \gamma_{13} + \gamma_{23} + \gamma_{33} = 1. \end{aligned} \tag{41}$$

Looking at the solutions in $\mathbb{N}$ of the system of linear equations above we obtain

$$\begin{aligned} &e_{(1,1,1)}\left(y_1^2 y_2, y_2^3 y_3, y_1 y_2 y_3\right) e_{(1,2,1)}\left(y_1^3 y_2^2 y_3, y_1^2 y_3, y_2 y_3\right) \\ &\quad = e_{(1,1,1,1)}\left(y_2 y_3, y_1^4 y_2 y_3, y_2^4 y_3^2, y_1 y_2 y_3\right) \\ &\qquad + e_{(1,1,1,1)}\left(y_2 y_3, y_1^4 y_2 y_3, y_1^3 y_2^5 y_3^2, y_1^3 y_2 y_3^2\right) \\ &\qquad + e_{(1,1,1,1)}\left(y_2 y_3, y_1^5 y_2^3, y_2^4 y_3^2, y_1^3 y_2 y_3^2\right) \\ &\qquad + e_{(1,1,1,1)}\left(y_1^2 y_3, y_1^2 y_2^2 y_3, y_2^4 y_3^2, y_1^4 y_2^3 y_3^2\right) \\ &\qquad + e_{(1,1,1,1)}\left(y_1^2 y_3, y_1^2 y_2^2 y_3, y_1^3 y_2^5 y_3^2, y_1^3 y_2 y_3^2\right) \\ &\qquad + e_{(1,1,1,1)}\left(y_1^2 y_3, y_1^4 y_2 y_3, y_1 y_2^4 y_3, y_1^4 y_2^3 y_3^2\right) \\ &\qquad + e_{(1,1,1,1)}\left(y_1^2 y_3, y_1^4 y_2 y_3, y_1^3 y_2^5 y_3^2, y_1^2 y_2^2 y_3\right) \\ &\qquad + e_{(1,1,1,1)}\left(y_1^2 y_3, y_1^5 y_2^3, y_1 y_2^4 y_3, y_1^3 y_2 y_3^2\right) \\ &\qquad + e_{(1,1,1,1)}\left(y_1^2 y_3, y_1^5 y_2^3, y_2^4 y_3^2, y_1^2 y_2^2 y_3\right) \\ &\qquad + e_{(1,1,1,1)}\left(y_1^3 y_2^2 y_3, y_1^2 y_2^2 y_3, y_2^4 y_3^2, y_1^3 y_2 y_3^2\right) \\ &\qquad + e_{(1,1,1,1)}\left(y_1^3 y_2^2 y_3, y_1^4 y_2 y_3, y_1 y_2^3 y_3^2, y_1^2 y_2^2 y_3\right) \\ &\qquad + e_{(1,1,1,1)}\left(y_1^3 y_2^2 y_3, y_1^4 y_2 y_3, y_2^3 y_3^2, y_1^2 y_2^2 y_3\right). \end{aligned} \tag{42}$$

3. Review of Deformation Quantization

In this section we review a few needed notions on deformation quantization. We assume the reader to be somewhat familiar with Kontsevich's work [7], although that level of generality is not necessary to understand the applications to the quantization of canonical phase space. A Poisson bracket [8, 9] on a smooth manifold M is a $\mathbb{R}$-bilinear antisymmetric map

$$\{\ ,\ \} : C^{\infty}(M) \times C^{\infty}(M) \longrightarrow C^{\infty}(M), \tag{43}$$

where $C^{\infty}(M)$ is the space of real-valued smooth functions on M, and for $f, g, h \in C^{\infty}(M)$ the following identities hold:

$$\begin{aligned} \{f, gh\} &= \{f, g\} h + g \{f, h\}, \\ \{f, \{g, h\}\} &= \{\{f, g\}, h\} + \{g, \{f, h\}\}. \end{aligned} \tag{44}$$

A manifold equipped with a Poisson bracket is called a Poisson manifold. The Poisson bracket $\{\ ,\ \}$ is determined by an antisymmetric bilinear form α on T^*M, that is, by the Poisson bivector $\alpha \in \bigwedge^2 TM$ given in local coordinates $(x_1, x_2, \dots, x_d)$ on M by

$$\alpha_{ij} = \left\{x_i, x_j\right\}. \tag{45}$$

The bivector α determines the Poisson bracket as follows:

$$\{f, g\} = \alpha(df, dg) = \sum_{i,j \in [d]} \alpha_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}, \quad \text{for } f, g \in C^{\infty}(M). \tag{46}$$

If the Poisson bivector α_{ij} is nondegenerated (i.e., $\det(\alpha_{i,j}) \neq 0$) the Poisson manifold M is called symplectic.

Example 10. The space $\mathbb{R}^{2d}$ is a symplectic Poisson manifold with Poisson bracket given in the linear coordinates $(x_1, \dots, x_d, y_1, \dots, y_d)$ by

$$\{f, g\} = \sum_{i=1}^{d} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right), \quad \text{for } f, g \in C^{\infty}\left(\mathbb{R}^{2d}\right). \tag{47}$$

Equivalently, the Poisson bracket $\{\ ,\ \}$ on $C^{\infty}(\mathbb{R}^{2d})$ is determined by the identities

$$\left\{x_i, x_j\right\} = 0, \quad \left\{y_i, y_j\right\} = 0, \quad \left\{x_i, y_j\right\} = \delta_{ij}, \quad \text{for } i, j \in [d]. \tag{48}$$

This example is the so-called canonical phase space with n degrees of freedom.

Example 11. Let $(\mathfrak{g}, [\ ,\ ])$ be Lie algebra over $\mathbb{R}$ of dimension d. The dual vector space $\mathfrak{g}^*$ is a Poisson manifold with Poisson bracket given on $f, g \in C^{\infty}(\mathfrak{g}^*)$ by

$$\{f, g\}(\alpha) = \langle \alpha, [d_{\alpha} f, d_{\alpha} g] \rangle, \tag{49}$$

where $\alpha \in \mathfrak{g}^*$ and the differentials $d_{\alpha} f$ and $d_{\alpha} g$ are regarded as elements of $\mathfrak{g}$ via the identifications $T_{\alpha}^* \mathfrak{g}^* = \mathfrak{g}^{**} = \mathfrak{g}$. Choose a linear basis $e_1, \dots, e_d$ for $\mathfrak{g}$. The structural coefficients c_{ij}^k of $\mathfrak{g}$ are given, for $i, j, k \in [d]$, by

$$\left[e_i, e_j\right] = \sum_{k=1}^{d} c_{ij}^k e_k. \tag{50}$$

Let $(x_1, \dots, x_d)$ be the linear system of coordinates on $\mathfrak{g}^*$ relative to the basis $e_1, \dots, e_d$ of $\mathfrak{g}$. The Poisson bracket is determined by continuity and the identities

$$\left\{x_i, x_j\right\} = \sum_{k=1}^{d} c_{ij}^k x_k. \tag{51}$$

A formal deformation, or deformation quantization, of a Poisson manifold M is an associative product, called the star product,

$$\star : C^{\infty}(M)[[\hbar]] \otimes_{\mathbb{R}[[\hbar]]} C^{\infty}(M)[[\hbar]] \longrightarrow C^{\infty}(M)[[\hbar]], \tag{52}$$

defined on the space $C^{\infty}(M)[[\hbar]]$ of formal power series in $\hbar$ with coefficients in $C^{\infty}(M)$ such that the following conditions hold for $f, g \in C^{\infty}(M)$:

(i) $f \star g = \sum_{n=0}^{\infty} B_n(f,g)\hbar^n$, where the maps

$$B_n(\ ,\ ) : C^{\infty}(M) \times C^{\infty}(M) \longrightarrow C^{\infty}(M) \tag{53}$$

are bidifferential operators.

(ii) $f \star g = fg + (1/2)\{f,g\}\hbar + O(\hbar^2)$, where $O(\hbar^2)$ stand for terms of order 2 and higher in the variable $\hbar$.

Kontsevich in [7] constructed a $\star$-product for any finite dimensional Poisson manifold. For linear Poisson manifolds the Kontsevich $\star$-product goes as follows. Fix a Poisson manifold $(\mathbb{R}^d, \alpha)$; the Kontsevich $\star$-product is given on $f, g \in C^{\infty}(M)$ by

$$\begin{aligned} f \star g &= \sum_{n=0}^{\infty} B_n(f,g) \frac{\hbar^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{\Gamma \in \mathbb{G}_n} \omega_{\Gamma} B_{\Gamma}(f,g) \right) \frac{\hbar^n}{n!}, \quad \text{where} \end{aligned} \tag{54}$$

(i) $\mathbb{G}_n$ is a collection of graphs, called admissible graphs, each with $2n$ edges;

(ii) for each graph $\Gamma \in \mathbb{G}_n$, the constant $\omega_{\Gamma} \in \mathbb{R}$ is independent of d and α and it is computed through an integral in an appropriated configuration space;

(iii) $B_{\Gamma}(\ ,\ ) : C^{\infty}(\mathbb{R}^d) \times C^{\infty}(\mathbb{R}^d) \rightarrow C^{\infty}(\mathbb{R}^d)$ is a bidifferential operator associated with the graph $\Gamma \in \mathbb{G}_n$ and the Poisson bivector α. The definition of the operators $B_{\Gamma}(\ ,\ )$ is quite explicit and fairly combinatorial in nature.

Remark 12. Kontsevich himself has highlighted the fact that explicitly computing the integrals defining the constants ω_{Γ} is a daunting task currently beyond reach. One can however use the symbols ω_{Γ} as variables, and they will define a deformation quantization (with an extended ring of constants) as soon as these variables satisfy a certain system of quadratic equations [10].

We are going to use the Kontsevich $\star$-product in a slightly modified form

$$\text{Let } \mathbb{G} = \coprod_{n=0}^{\infty} \mathbb{G}_n, \quad \text{for } \Gamma \in \mathbb{G}, \text{ set } \overline{\Gamma} = n \text{ iff } \Gamma \in \mathbb{G}_n. \tag{55}$$

With this notation the Kontsevich $\star$-product is given on functions $f, g \in C^{\infty}(\mathbb{R}^d)$ by

$$f \star g = \sum_{\Gamma \in \mathbb{G}} \frac{\omega_{\Gamma}}{\overline{\Gamma}!} B_{\Gamma}(f,g)\hbar^{\overline{\Gamma}}. \tag{56}$$

Remark 13. The Kontsevich $\star$-product is defined over $\mathbb{R}$ since $c_{\Gamma} \in \mathbb{R}$. If α is a regular Poisson bivector, that is, the entries α_{ij} of the Poisson bivector are polynomial functions, then the $\star$-product on $C^{\infty}(\mathbb{R}^d)$ is restricted to a well-defined $\star$-product on the space $\mathbb{R}[x_1, \ldots, x_d]$ of polynomial functions on $\mathbb{R}^d$. We are interested in the quantization of symmetric polynomial functions; thus we assume that α is a regular Poisson bivector and work with quantum algebra $(\mathbb{R}[x_1, \ldots, x_d], \star)$.

4. Quantum Symmetric Functions

Let S_n be the symmetric group on n letters. For each subgroup $K \subseteq S_n$, consider the Polya functor $P_K : \mathbb{R}\text{-alg} \rightarrow \mathbb{R}\text{-alg}$ from the category of associative $\mathbb{R}$-algebras to itself, defined on objects as follows [10]. Let A be $\mathbb{R}$-algebra; the underlying vector space of $P_K A$ is given by

$$\begin{aligned} P_K A &= (A^{\otimes n})_K \\ &= \frac{A^{\otimes n}}{\langle a_1 \otimes \cdots \otimes a_n - a_{\sigma 1} \otimes \cdots \otimes a_{\sigma n} \mid a_i \in A,\ \sigma \in K \rangle}. \end{aligned} \tag{57}$$

Elements of $P_K A$ are written as $\overline{a_1 \otimes \cdots \otimes a_n}$. For $a_{ij} \in A$, the following identity determines the product on $P_K A$:

$$|K|^{m-1} \prod_{i=1}^{m} \left(\overline{\bigotimes_{j=1}^{n} a_{ij}} \right) = \sum_{\sigma \in \{1\} \times K^{m-1}} \overline{\bigotimes_{j=1}^{n} \left(\prod_{i=1}^{m} a_{i\sigma_i^{-1}(j)} \right)}. \tag{58}$$

The Polya functor P_K is also known as the coinvariants functor. The invariants functor

$$\begin{aligned} I^K : \mathbb{R}\text{-alg} &\longrightarrow \mathbb{R}\text{-alg}, \\ A &\longrightarrow (A^{\otimes n})^K \end{aligned} \tag{59}$$

is given on objects by

$$(A^{\otimes n})^K = \{a \in A^{\otimes n} \mid ga = a \text{ for } g \in K\}. \tag{60}$$

The product on $(A^{\otimes n})^K$ comes from the inclusion $(A^{\otimes n})^K \subset A^{\otimes n}$.

The functors I^K and P_K are naturally isomorphic to each other [10].

Suppose a finite group K acts on a Poisson manifold M and that the induced action of K on $(C^{\infty}(M)[[\hbar]], \star)$ is by algebra automorphisms; then we define the algebra of quantum K-symmetric functions on M as

$$(C^{\infty}(M)[[\hbar]], \star)_K \simeq (C^{\infty}(M)[[\hbar]], \star)^K. \tag{61}$$

Let $(\mathbb{R}^d, \alpha)$ be a regular Poisson manifold. The Cartesian product of Poisson manifolds is naturally endowed with the structure of a Poisson manifold; thus we get a regular Poisson manifold structure on $(\mathbb{R}^d)^n$. We use the following coordinates on the n-fold Cartesian product of $\mathbb{R}^d$ with itself:

$$\begin{aligned} (\mathbb{R}^d)^n = \{&(x_1, \ldots, x_n) \mid x_i = (x_{i1}, \ldots, x_{id}) \in \mathbb{R}^d, \\ &x_{ij} \in \mathbb{R},\ (i,j) \in [n] \times [d]\}. \end{aligned} \tag{62}$$

The ring of regular functions on $(\mathbb{R}^d)^n$ is the ring of polynomials on dn commutative variables:

$$\mathbb{R}\left[(\mathbb{R}^d)^n\right] = \mathbb{R}[x_{11}, \ldots, x_{1d}, \ldots, x_{n1}, \ldots, x_{nd}]. \tag{63}$$

Consider another set of commutative variables $y_1, \ldots, y_d$. Recall from Section 2 that for $f \in \mathbb{R}[y_1, \ldots, y_d]$ and $i \in [n]$ we set $f(i) = f(x_{i1}, \ldots, x_{id}) \in \mathbb{R}[x_{i1}, \ldots, x_{id}] \subseteq \mathbb{R}[(\mathbb{R}^d)^n]$.

The Poisson bracket on $(\mathbb{R}^d)^n$ is determined by the following identities:

$$\{x_{ki}, x_{lj}\} = \delta_{kl}\alpha_{ij}(k), \quad \text{for } i, j \in [d],\ k, l \in [n], \tag{64}$$

where the coordinates α_{ij} of the Poisson bivector $\alpha = \sum \alpha_{ij}\partial_i \wedge \partial_j$ are regarded as polynomials in $\mathbb{R}[y_1, \ldots, y_n]$. The Poisson bracket on $(\mathbb{R}^d)^n$ is S_n-invariant; indeed for $\sigma \in S_n$ we have

$$\{x_{\sigma ki}, x_{\sigma lj}\} = \delta_{\sigma k, \sigma l}\alpha_{ij}(\sigma k) = \sigma\left(\delta_{kl}\alpha_{ij}(k)\right). \tag{65}$$

Next results [10] provide a natural construction of groups acting as algebra automorphisms on the algebras $(\mathbb{R}[\mathbb{R}^d][[\hbar]], \star)$.

Theorem 14. *Let $(\mathbb{R}^d, \{\ ,\ \})$ be a regular Poisson manifold and let K be a subgroup of S_d such that the Poisson bracket $\{\ ,\ \}$ is K-equivariant. Then the action of K on $(\mathbb{R}[\mathbb{R}^d][[\hbar]], \star)$ is by algebra automorphisms.*

Corollary 15. *Let $(\mathbb{R}^d, \{\ ,\ \})$ be a regular Poisson manifold and consider a subgroup $K \subseteq S_n$. Then K acts by algebra automorphisms on $(\mathbb{R}[(\mathbb{R}^d)^n][[\hbar]], \star)$.*

Definition 16. Let $(\mathbb{R}^d, \{\ ,\ \})$ be a regular Poisson manifold. The algebra of quantum symmetric functions on $(\mathbb{R}^d)^n$ is given by

$$\left(\mathbb{R}\left[\left(\mathbb{R}^d\right)^n\right][[\hbar]], \star\right)_{S_n} \simeq \left(\mathbb{R}\left[\left(\mathbb{R}^d\right)^n\right][[\hbar]], \star\right)^{S_n}. \tag{66}$$

Example 17. Consider $\mathbb{R}^{2d}$ with its canonical symplectic Poisson structure; then $(\mathbb{R}^{2d})^n$ is also a symplectic Poisson manifold. Choose coordinates on $(\mathbb{R}^{2d})^n$ as follows:

$$\begin{aligned}\left(\mathbb{R}^{2d}\right)^n = \{(x_1, y_1, \ldots, x_n, y_n) \mid x_i &= (x_{i1}, \ldots, x_{id}), \\ y_i &= (y_{i1}, \ldots, y_{id}), \\ x_{ij}, y_{ij} &\in \mathbb{R}\}.\end{aligned} \tag{67}$$

The S_n-invariant Poisson bracket on $(\mathbb{R}^{2d})^n$ is given for $i, j \in [d]$ and $k, l \in [n]$ by

$$\{x_{ki}, x_{lj}\} = 0, \qquad \{y_{ki}, y_{lj}\} = 0, \qquad \{x_{ki}, y_{lj}\} = \delta_{kl}\delta_{ij}. \tag{68}$$

Example 18. Let $\mathfrak{g}$ be d-dimensional Lie algebra over $\mathbb{R}$ and let $\mathfrak{g}^*$ be its dual vector space. Then $\mathfrak{g}^*$ is a Poisson manifold, and therefore $(\mathfrak{g}^*)^n$ is also a Poisson manifold. The S_n-invariant Poisson bracket on $(\mathfrak{g}^*)^n$ is given, for $i, j \in [d]$ and $k, l \in [n]$, by

$$\{x_{ki}, x_{lj}\} = \delta_{kl}\sum_{m=0}^{d} c_{ij}^m x_{km}, \tag{69}$$

where c_{ij}^m are the structural coefficients of $\mathfrak{g}$.

Specializing Definition 16 we obtain the following natural notions. The algebra of quantum symmetric functions on $(\mathbb{R}^{2d})^n$ is given by

$$\left(\mathbb{R}\left[\left(\mathbb{R}^{2d}\right)^n\right][[\hbar]], \star\right)_{S_n} \simeq \left(\mathbb{R}\left[\left(\mathbb{R}^{2d}\right)^n\right][[\hbar]], \star\right)^{S_n}. \tag{70}$$

More generally, the algebra of quantum symmetric functions on $(\mathfrak{g}^*)^n$ is given by

$$\left(\mathbb{R}\left[(\mathfrak{g}^*)^n\right][[\hbar]], \star\right)_{S_n} \simeq \left(\mathbb{R}\left[(\mathfrak{g}^*)^n\right][[\hbar]], \star\right)^{S_n}. \tag{71}$$

5. $\star$-Product of Multisymmetric Functions

We are ready to state and proof the main result of this work which extends Theorem 7 from the classical to the quantum case: we provide an explicit formula for the $\star$-product of multisymmetric functions.

Recall that the $\star$-product can be expanded as a formal power series in $\hbar$ as

$$f \star g = \sum_{n=0}^{\infty} B_n(f, g)\frac{\hbar^n}{n!}. \tag{72}$$

Theorem 19. *Let $(\mathbb{R}^d, \{\ ,\ \})$ be a regular Poisson manifold and let $(\mathbb{R}[(\mathbb{R}^d)^n][[\hbar]], \star)^{S_n}$ be the algebra of quantum symmetric functions on $(\mathbb{R}^d)^n$. Fix $a, b, n \in \mathbb{N}^+$, $p \in R[y_1, \ldots, y_d]^a$, and $q \in R[y_1, \ldots, y_d]^b$. Let $\alpha \in \mathbb{N}^a$ and $\beta \in \mathbb{N}^b$ be such that $|\alpha|, |\beta| \le n$. The $\star$-product of $e_\alpha(p)$ and $e_\beta(q)$ is given by*

$$e_\alpha(p) \star e_\beta(q) = \sum_{m=0}^{\infty}\left(\sum_{\gamma \in Q(\alpha,\beta,n,m)} e_\gamma(B(p,q))\right)\hbar^m, \quad \textit{where} \tag{73}$$

(i) *$B(p, q) = (p, q, \ldots, B_k(p, q), \ldots)$ and*

$$\begin{aligned}B_k(p, q) = (B_k(p_1, q_1), \ldots, B_k(p_1, q_b), \ldots, \\ B_k(p_a, q_1), \ldots, B_k(p_a, q_b)),\end{aligned} \tag{74}$$

(ii) *$Q(\alpha, \beta, n, m)$ is the subset of* $\mathrm{Map}([0, a] \times [0, b] \times \mathbb{N}, \mathbb{N})$ *consisting of cubical matrices*

$$\gamma : [0, a] \times [0, b] \times \mathbb{N} \longrightarrow \mathbb{N} \quad \textit{such that} \tag{75}$$

(a) *$\gamma_{00k} = 0$ for $k \ge 0$; if either $l = 0$ or $r = 0$, then $\gamma_{lrk} = 0$ for $k \ge 1$,*

(b) *$|\gamma| = \sum_{l=0}^{a}\sum_{r=0}^{b}\sum_{k=0}^{\infty}\gamma_{lrk} \le n$, and $\sum_{l=0}^{a}\sum_{r=0}^{b}\sum_{k=0}^{\infty} k\gamma_{lrk} = m$,*

(c) *$\sum_{r=0}^{b}\sum_{k=0}^{\infty}\gamma_{lrk} = \alpha_l$ for $l \in [a]$, and $\sum_{l=0}^{a}\sum_{k=0}^{\infty}\gamma_{lrk} = \beta_r$ for $j \in [b]$.*

Proof. We have

$$\begin{aligned}
&\sum_{|\alpha|,|\beta|\le n}\sum_{m=0}^{\infty}B_m\left(e_\alpha(p),e_\beta(q)\right)t^\alpha s^\beta\hbar^m\\
&=\sum_{|\alpha|,|\beta|\le n}\left(e_\alpha(p)\star e_\beta(q)\right)t^\alpha s^\beta\\
&=\left(\sum_{|\alpha|\le n}e_\alpha(p)t^\alpha\right)\star\left(\sum_{|\beta|\le n}e_\beta(q)s^\beta\right)\\
&=\prod_{i=1}^{n}\left(1+\sum_{l=1}^{a}p_l(i)t_l\right)\star\prod_{i=1}^{n}\left(1+\sum_{r=1}^{b}q_r(i)s_r\right)\\
&=\prod_{i=1}^{n}\left(1+\sum_{l=1}^{a}p_l(i)t_l\right)\star\left(1+\sum_{r=1}^{b}q_r(i)s_r\right)\\
&=\prod_{i=1}^{n}\left(1+\sum_{l=1}^{a}p_l(i)t_l+\sum_{r=1}^{b}q_r(i)s_r\right.\\
&\qquad\left.+\sum_{l=1}^{a}\sum_{r=1}^{b}p_l(i)\star q_r(i)t_ls_r\right)\\
&=\prod_{i=1}^{n}\left(1+\sum_{l=1}^{a}p_l(i)t_l+\sum_{r=1}^{b}q_r(i)s_r\right.\\
&\qquad\left.+\sum_{l=1}^{a}\sum_{r=1}^{b}\sum_{k=0}^{\infty}B_k\left(p_l(i),q_r(i)\right)t_ls_r\hbar^k\right)\\
&=\prod_{i=1}^{n}\left(1+\sum_{l=1}^{a}p_l(i)w_{l00}+\sum_{r=1}^{b}q_r(i)w_{0r0}\right.\\
&\qquad\left.+\sum_{l=1}^{a}\sum_{r=1}^{b}\sum_{k=0}^{\infty}B_k\left(p_l(i),q_r(i)\right)w_{lrk}\right)\\
&=\sum_{\gamma\in Q(\alpha,\beta,n,m)}e_\gamma\left(B(p,q)\right)w^\gamma,\quad\text{where:}
\end{aligned}\tag{76}$$

$$\begin{aligned}
w^\gamma&=\prod_{l=0}^{a}\prod_{r=0}^{b}\prod_{k=0}^{\infty}w_{lrk}^{\gamma_{lrk}}=\prod_{l=0}^{a}\prod_{r=0}^{b}\prod_{k=0}^{\infty}\left(t_ls_r\hbar^k\right)^{\gamma_{lrk}}\\
&=\prod_{l=0}^{a}\prod_{r=0}^{b}\prod_{k=0}^{\infty}t_l^{\gamma_{lrk}}s_r^{\gamma_{lrk}}\hbar^{k\gamma_{lrk}},
\end{aligned}$$

and we are using the conventions

$$t_0=s_0=1,\qquad w_{ruk}=t_rs_u\hbar^k\quad\text{for } r,u,m\ge 0.\tag{77}$$

For w^γ to be equal to $t^\alpha s^\beta\hbar^m$ we must have

$$\begin{aligned}
&\left(\prod_{l=0}^{a}t_l^{\alpha_l}\right)\left(\prod_{r=0}^{b}s_r^{\beta_r}\right)\hbar^m\\
&=\prod_{l=0}^{a}\prod_{r=0}^{b}\prod_{k=0}^{\infty}t_l^{\gamma_{lrk}}s_r^{\gamma_{lrk}}\hbar^{k\gamma_{lrk}}\\
&=\left(\prod_{l=0}^{a}\prod_{r=0}^{b}\prod_{k=0}^{\infty}t_l^{\gamma_{lrk}}\right)\left(\prod_{l=0}^{a}\prod_{r=0}^{b}\prod_{k=0}^{\infty}s_r^{\gamma_{lrk}}\right)\\
&\quad\cdot\left(\prod_{l=0}^{a}\prod_{r=0}^{b}\prod_{k=0}^{\infty}\hbar^{k\gamma_{lrk}}\right)\\
&=\left(\prod_{l=1}^{a}t_l^{\sum_{r=0}^{b}\sum_{k=0}^{\infty}\gamma_{lrk}}\right)\left(\prod_{r=1}^{b}s_r^{\sum_{l=0}^{a}\sum_{k=0}^{\infty}\gamma_{lrk}}\right)\\
&\quad\cdot\hbar^{\sum_{l=0}^{a}\sum_{r=0}^{b}\sum_{k=0}^{\infty}k\gamma_{lrk}}
\end{aligned}\tag{78}$$

and thus we conclude that

$$\begin{aligned}
&\sum_{r=0}^{b}\sum_{k=0}^{\infty}\gamma_{lrk}=\alpha_l\quad\text{for } l\in[a],\\
&\sum_{l=0}^{a}\sum_{k=0}^{\infty}\gamma_{lrk}=\beta_r\quad\text{for } j\in[b],\\
&\sum_{l=0}^{a}\sum_{r=0}^{b}\sum_{k=0}^{\infty}k\gamma_{lrk}=m.
\end{aligned}\tag{79}$$

□

Corollary 20. *With the assumptions of Theorem 19, the Poisson bracket of the multisymmetric functions $e_\alpha(p)$ and $e_\beta(q)$ is given by*

$$\left\{e_\alpha(p),e_\beta(q)\right\}=2\sum_{\gamma\in Q(\alpha,\beta,n,1)}e_\gamma\left(B(p,q)\right).\tag{80}$$

Proof. It follows from Theorem 19 and the identity

$$\left\{e_\alpha(p),e_\beta(q)\right\}=2\frac{\partial}{\partial\hbar}\left(e_\alpha(p)\star e_\alpha(q)\right)|_{\hbar=0}.\tag{81}$$

□

For our next result we regard $(\mathbb{R}[(\mathbb{R}^d)^n][[\hbar]],\star)^{S_n}$ as topological algebra with topology induced by the inclusion

$$\left(\mathbb{R}\left[\left(\mathbb{R}^d\right)^n\right][[\hbar]],\star\right)^{S_n}\subseteq\left(\mathbb{R}\left[\left(\mathbb{R}^d\right)^n\right][[\hbar]],\star\right),\tag{82}$$

where a fundamental system of neighborhoods of $0\in\mathbb{R}[(\mathbb{R}^d)^n][[\hbar]]$ is given by the decreasing family of subalgebras

$$\begin{aligned}
&\mathbb{R}\left[\left(\mathbb{R}^d\right)^n\right][[\hbar]]\supseteq\hbar\mathbb{R}\left[\left(\mathbb{R}^d\right)^n\right][[\hbar]]\\
&\qquad\supseteq\cdots\supseteq\hbar^n\mathbb{R}\left[\left(\mathbb{R}^d\right)^n\right][[\hbar]]\supseteq\cdots.
\end{aligned}\tag{83}$$

Recall from Introduction that the elementary multisymmetric functions e_k, for $k\in\mathbb{N}^d$ with $|k|\le n$, are defined by the identity

$$\prod_{i=1}^{n}\left(1+x_{i1}t_1+\cdots+x_{id}t_d\right)=\sum_{k\in\mathbb{N}^d,\,|k|\le n}e_kt^k.\tag{84}$$

Similarly, the homogeneous multisymmetric functions h_k, for $k = (k_1, \dots, k_d) \in \mathbb{N}^d$, are defined by the identity

$$\prod_{i=1}^{n} \frac{1}{1 - x_{i1}t_1 - \cdots - x_{id}t_d} = \sum_{k \in \mathbb{N}^d} h_k t^k. \tag{85}$$

Let $\mathscr{M}_d$ be the set of (nontrivial) monomials in the variables $y_1, \dots, y_d$. The power sum symmetric function $e_1(m)$ is given, for $m \in \mathscr{M}_d$, by

$$e_1(m) = m(1) + \cdots + m(d). \tag{86}$$

Theorem 21. *The elementary multisymmetric functions e_k for $|k| \le n$, the homogeneous multisymmetric functions h_k for $|k| \le n$, and the power sum multisymmetric functions $e_1(m)$ with $m \in \mathscr{M}_d$ a monomial of degree less than or equal to n, together with $\hbar$ generate, respectively, the topological algebra $(\mathbb{R}[(\mathbb{R}^d)^n][[\hbar]], \star)^{S_n}$.*

Proof. It is known [1, 3, 4, 11, 12] that each of the aforementioned sets of multisymmetric functions generate the algebra of classical multisymmetric functions $\mathbb{R}[(\mathbb{R}^d)^n]^{S_n}$.

To go the quantum case the same argument is applied in each case, so we only consider the elementary symmetric functions. Take $f \in (\mathbb{R}[(\mathbb{R}^d)^n][[\hbar]], \star)^{S_n}$ and expand it as formal power series

$$f = \sum_{k=0}^{\infty} f_k \hbar^k \quad \text{with } f_k \in \mathbb{R}\left[\left(\mathbb{R}^d\right)^n\right]^{S_n}. \tag{87}$$

We can write f_0 as a linear combination of a product of elementary symmetric functions. For simplicity assume that $f_0 = e_{k_1} \cdots e_{k_m}$, then

$$f - e_{k_1} \star \cdots \star e_{k_m} \in O(\hbar). \tag{88}$$

Assume next that $(f - e_{k_1} \star \cdots \star e_{k_m})_1$ can be written as $e_{l_1} \cdots e_{l_r}$, then

$$f - e_{k_1} \star \cdots \star e_{k_m} - e_{l_1} \star \cdots \star e_{l_r} \hbar \in O\left(\hbar^2\right). \tag{89}$$

Proceeding by induction we see that f can be written as a formal power series in $\hbar$ with coefficients equal to the sum of the $\star$-product of elementary multisymmetric functions. □

Choose a variable t_m for each $m \in \mathscr{M}_d$, the set of monomials in the variables $y_1, \dots, y_d$, and set

$$\sum_{|\alpha| \le n} e_\alpha t^\alpha = \prod_{i=1}^{n} \left(1 + \sum_{m \in \mathscr{M}_d} m(i) t_m\right), \tag{90}$$

where $\alpha : \mathscr{M}_d \to \mathbb{N}$ of finite support and $t^\alpha = \prod_{m \in \mathscr{M}_d} t_m^{\alpha(m)}$.

Theorem 22. *The set $\{e_\alpha \hbar^k \mid |\alpha| \le n,\ k \ge 0\}$ is a topological basis for the topological algebra $(\mathbb{R}[(\mathbb{R}^d)^n][[\hbar]], \star)^{S_n}$. The product of basic elements is given by Lemma 6 and Theorem 19.*

Proof. It is well known that the symmetrization of monomials yields a basis for $\mathbb{R}[(\mathbb{R}^d)^n]^{S_n}$; thus from Lemma 5 we see that the set $\{e_\alpha \mid |\alpha| \le n\}$ is a basis for $\mathbb{R}[(\mathbb{R}^d)^n]^{S_n}$ as well. Thus forming the products $e_\alpha \hbar^k$ we obtain a topological basis for $(\mathbb{R}[(\mathbb{R}^d)^n][[\hbar]], \star)^{S_n}$. □

Next result describes the product of multisymmetric functions using the Kontsevich's $\star$-product. In this case one can give a more precise formula for the computation of the quantum higher corrections, that is, the coefficients that accompany the higher order powers in $\hbar$.

Theorem 23. *Let $(\mathbb{R}^d, \{\ ,\ \})$ be a regular Poisson manifold and let $(\mathbb{R}[(\mathbb{R}^d)^n][[\hbar]], \star)^{S_n}$ be the algebra of quantum symmetric functions on $(\mathbb{R}^d)^n$ with the Kontsevich $\star$-product. Fix $a, b, n \in \mathbb{N}^+$, $p \in R[y_1, \dots, y_d]^a$, and $q \in R[y_1, \dots, y_d]^b$. Let $\alpha \in \mathbb{N}^a$ and $\beta \in \mathbb{N}^b$ be such that $|\alpha|, |\beta| \le n$. The $\star$-product of $e_\alpha(p)$ and $e_\beta(q)$ is given by*

$$e_\alpha(p) \star e_\beta(q) = \sum_{m=0}^{\infty} \left(\sum_{\gamma \in K(\alpha,\beta,n,m)} e_\gamma(B(p,q)) \right) \hbar^m, \quad \textit{where} \tag{91}$$

(i) *$B(p,q) = (p, q, \dots, (\omega_\Gamma/\overline{\Gamma}!) B_\Gamma(p,q), \dots)$ and*

$$B_\Gamma(p,q) = (B_\Gamma(p_1,q_1), \dots, B_\Gamma(p_1,q_b), \dots, B_\Gamma(p_a,q_1), \dots, B_\Gamma(p_a,q_b)), \tag{92}$$

the polynomial $B_\Gamma(p_i,q_j)$ results of applying Kontsevich's bidifferential operator B_Γ to the pair (p_i,q_j);

(ii) *$K(\alpha,\beta,n,m)$ is the subset of $\mathrm{Map}([0,a] \times [0,b] \times G, \mathbb{N})$ consisting of maps*

$$\gamma : [0,a] \times [0,b] \times G \longrightarrow \mathbb{N} \quad \textit{such that} \tag{93}$$

(a) *$\gamma_{00\Gamma} = 0$; if either $l = 0$ or $r = 0$, then $\gamma_{lr\Gamma} = 0$ for $\overline{\Gamma} \ge 1$;*

(b) *$|\gamma| = \sum_{l=0}^{a} \sum_{r=0}^{b} \sum_{\Gamma \in G} \gamma_{lr\Gamma} \le n$, and $\sum_{l=0}^{a} \sum_{r=0}^{b} \sum_{\Gamma \in G} \overline{\Gamma} \gamma_{lr\Gamma} = m$,*

(c) *$\sum_{r=0}^{b} \sum_{\Gamma \in G} \gamma_{lr\Gamma} = \alpha_l$ for $l \in [a]$, and $\sum_{l=0}^{a} \sum_{\Gamma \in G} \gamma_{lr\Gamma} = \beta_r$ for $j \in [b]$.*

Proof. We have

$$\sum_{|\alpha|, |\beta| \le n} \sum_{m=0}^{\infty} B_m\left(e_\alpha(p), e_\beta(q)\right) t^\alpha s^\beta \hbar^m$$
$$= \sum_{|\alpha|, |\beta| \le n} \left(e_\alpha(p) \star e_\beta(q)\right) t^\alpha s^\beta$$
$$= \left(\sum_{|\alpha| \le n} e_\alpha(p) t^\alpha\right) \star \left(\sum_{|\beta| \le n} e_\beta(q) s^\beta\right)$$

$$= \prod_{i=1}^{n}\left(1+\sum_{l=1}^{a}p_l(i)\,t_l\right)\star\left(1+\sum_{r=1}^{b}q_r(i)\,s_r\right)$$
$$= \prod_{i=1}^{n}\left(1+\sum_{l=1}^{a}p_l(i)\,t_l+\sum_{r=1}^{b}q_r(i)\,s_r + \sum_{l=1}^{a}\sum_{r=1}^{b}p_l(i)\star q_r(i)\,t_l s_r\right)$$
$$= \prod_{i=1}^{n}\left(1+\sum_{l=1}^{a}p_l(i)\,t_l+\sum_{r=1}^{b}q_r(i)\,s_r + \sum_{l=1}^{a}\sum_{r=1}^{b}\sum_{\Gamma\in G}\frac{\omega_\Gamma}{\overline{\Gamma}!}B_\Gamma\left(p_l(i),q_r(i)\right)t_l s_r \hbar^{\overline{\Gamma}}\right)$$
$$= \prod_{i=1}^{n}\left(1+\sum_{l=1}^{a}p_l(i)\,w_{l0\emptyset}+\sum_{r=1}^{b}q_r(i)\,w_{0r\emptyset} + \sum_{l=1}^{a}\sum_{r=1}^{b}\sum_{\Gamma\in G}\frac{\omega_\Gamma}{\overline{\Gamma}!}B_\Gamma\left(p_l(i),q_r(i)\right)w_{rk\Gamma}\right)$$
$$= \sum_{\gamma\in K(\alpha,\beta,n,m)} e_\gamma\left(B(p,q)\right)w^\gamma, \quad \text{where:}$$
$$w^\gamma = \prod_{l=0}^{a}\prod_{r=0}^{b}\prod_{\Gamma\in G}w_{lr\Gamma}^{\gamma_{lr\Gamma}} = \prod_{l=0}^{a}\prod_{r=0}^{b}\prod_{\Gamma\in G}^{\infty}t_l^{\gamma_{lrk}}s_r^{\gamma_{lrk}}\hbar^{\overline{\Gamma}\gamma_{lr\Gamma}}, \tag{94}$$

and by convention $\emptyset$ stands for the unique graph in G with no edges (representing the classical product), and

$$t_0 = s_0 = 1, \qquad w_{ru\Gamma} = t_r s_u \hbar^{\overline{\Gamma}} \quad \text{for } r,u\ge 0,\ \Gamma\in\mathbb{G}. \tag{95}$$

For $t^\alpha s^\beta \hbar^m = w^\gamma$ we must have

$$\left(\prod_{l=0}^{a}t_l^{\alpha_l}\right)\left(\prod_{r=0}^{b}s_r^{\beta_r}\right)\hbar^m = \prod_{l=0}^{a}\prod_{r=0}^{b}\prod_{\Gamma\in G}t_l^{\gamma_{lrk}}s_r^{\gamma_{lrk}}\hbar^{\overline{\Gamma}\gamma_{lr\Gamma}} = \left(\prod_{l=1}^{a}t_l^{\sum_{r=0}^{b}\sum_{\Gamma\in G}\gamma_{lr\Gamma}}\right)\left(\prod_{r=1}^{b}s_r^{\sum_{l=0}^{a}\sum_{\Gamma\in G}\gamma_{lr\Gamma}}\right)\cdot\hbar^{\sum_{l=0}^{a}\sum_{r=0}^{b}\sum_{\Gamma\in G}\overline{\Gamma}\gamma_{lr\Gamma}}. \tag{96}$$

Thus we conclude that

$$\sum_{r=0}^{b}\sum_{\Gamma\in G}\gamma_{lr\Gamma} = \alpha_l \quad \text{for } l\in[a],$$
$$\sum_{l=0}^{a}\sum_{\Gamma\in G}\gamma_{lr\Gamma} = \beta_r \quad \text{for } j\in[b], \tag{97}$$
$$\sum_{l=0}^{a}\sum_{r=0}^{b}\sum_{\Gamma\in G}\overline{\Gamma}\gamma_{lr\Gamma} = m.$$

□

6. Symmetric Powers of the Weyl Algebras

In this section we study the case of two-dimensional canonical phase space, that is, the symplectic manifold $\mathbb{R}^2$ with the canonical Poisson bracket given as follows:

$$\{f,g\} = \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}. \tag{98}$$

Definition 24. The Weyl algebra is defined by generators and relations by

$$W = \frac{\mathbb{R}\langle x,y\rangle[[\hbar]]}{\langle yx-xy-\hbar\rangle}. \tag{99}$$

The deformation quantization of $(\mathbb{R}^2,\{\ ,\ \})$ is well known to be given by the Moyal product [13]. Moreover, one has the following result.

Theorem 25. *The Weyl algebra is isomorphic to the deformation quantization of polynomial functions on $\mathbb{R}^2$ with the canonical Poisson structure.*

Our goal in this section is to study the deformation quantization of the space $(\mathbb{R}^2)^n/S_n$, which can be identified with the algebra of quantum symmetric functions

$$\left(\mathbb{R}[x_1,\dots,x_n,y_1,\dots,y_n][[\hbar]],\star\right)^{S_n}, \tag{100}$$

or, equivalently, with the symmetric powers of the Weyl algebra

$$\frac{(W^{\otimes n})}{S_n}. \tag{101}$$

One shows by induction [10] that the following identity holds in the Weyl algebra:

$$\left(x^c y^d\right)\star\left(x^f y^g\right) = \sum_{k=0}^{\min}B_k\left(x^c y^d, x^f y^g\right)\hbar^k = \sum_{k=0}^{\min}\binom{d}{k}(f)_k\,x^{c+f-k}y^{d+g-k}\hbar^k, \quad \text{where}$$
$$\min = \min(d,f), \quad (f)_k = f(f-1)(f-2)(f-k+1). \tag{102}$$

Theorem 26. *Consider $\mathbb{R}^2$ with its canonical Poisson structure. Fix $a,b\in\mathbb{N}^+$ and let*

$$\left(x^{c_1}y^{d_1},\dots,x^{c_a}y^{d_a}\right)\in\mathbb{R}[x,y]^a,$$
$$\left(x^{f_1}y^{g_1},\dots,x^{f_b}y^{g_b}\right)\in\mathbb{R}[x,y]^b. \tag{103}$$

For $\alpha \in \mathbb{N}^a$ and $\beta \in \mathbb{N}^b$ the following identity holds:

$$e_\alpha\left(x^{c_1}y^{d_1},\ldots,x^{c_a}y^{d_a}\right)\star e_\beta\left(x^{f_1}y^{g_1},\ldots,x^{f_b}y^{g_b}\right) = \sum_{m=0}^{\infty}\left(\sum_{\gamma\in Q(\alpha,\beta,n,m)} e_\gamma\left(B\left(x^{c_1}y^{d_1},\ldots,x^{c_a}y^{d_a}, x^{f_1}y^{g_1},\ldots,x^{f_b}y^{g_b}\right)\right)\right)\hbar^m, \tag{104}$$

where

$$B\left(x^{c_1}y^{d_1},\ldots,x^{c_a},y^{d_a},x^{f_1}y^{g_1},\ldots,x^{f_b}y^{g_b}\right) = \left(x^{c_1}y^{d_1},\ldots,x^{c_a},y^{d_a},x^{f_1}y^{g_1},\ldots, x^{f_b}y^{g_b},\ldots,\binom{d_r}{k}(f_r)_k x^{c_l+f_r-k}y^{d_l+g_r-k},\ldots\right). \tag{105}$$

Proof. We have

$$\begin{aligned}
&\sum_{|\alpha|,|\beta|\le n}\sum_{m=0}^{\infty}B_m\left(e_\alpha\left(x^{c_1}y^{d_1},\ldots,x^{c_a}y^{d_a}\right), e_\beta\left(x^{f_1}y^{g_1},\ldots,x^{f_b}y^{g_b}\right)\right)t^\alpha s^\beta\hbar^m\\
&= \sum_{|\alpha|,|\beta|\le n}\left(e_\alpha\left(x^{c_1}y^{d_1},\ldots,x^{c_a}y^{d_a}\right)\star e_\beta\left(x^{f_1}y^{g_1},\ldots,x^{f_b}y^{g_b}\right)\right)t^\alpha s^\beta\\
&= \left(\sum_{|\alpha|\le n}e_\alpha\left(x^{c_1}y^{d_1},\ldots,x^{c_a}y^{d_a}\right)t^\alpha\right)\star\left(\sum_{|\beta|\le n}e_\beta\left(x^{f_1}y^{g_1},\ldots,x^{f_b}y^{g_b}\right)s^\beta\right)\\
&= \prod_{i=1}^{n}\left(1+\sum_{l=1}^{a}x_i^{c_l}y_i^{d_l}t_l\right)\star\left(1+\sum_{r=1}^{b}x_i^{f_r}y_i^{g_r}s_r\right)\\
&= \prod_{i=1}^{n}\left(1+\sum_{l=1}^{a}x_i^{c_l}y_i^{d_l}t_l+\sum_{r=1}^{b}x_i^{f_r}y_i^{g_r}s_r + \sum_{l=1}^{a}\sum_{r=1}^{b}x_i^{c_l}y_i^{d_l}\star x_i^{f_r}y_i^{g_r}t_ls_r\right)\\
&= \prod_{i=1}^{n}\left(1+\sum_{l=1}^{a}x_i^{c_l}y_i^{d_l}t_l+\sum_{r=1}^{b}x_i^{f_r}y_i^{g_r}s_r + \sum_{l=1}^{a}\sum_{r=1}^{b}\sum_{k=0}^{\min}\binom{d_r}{k}(f_r)_k x_i^{c_l+f_r-k}\cdot y_i^{d_l+g_r-k}t_ls_r\hbar^k\right)\\
&= \prod_{i=1}^{n}\left(1+\sum_{l=1}^{a}x_i^{c_l}y_i^{d_l}w_{l00}+\sum_{r=1}^{b}x_i^{f_r}y_i^{g_r}w_{0r0} + \sum_{l=1}^{a}\sum_{r=1}^{b}\sum_{k=0}^{\min}\binom{d_r}{k}(f_r)_k x_i^{c_l+f_r-k}\cdot y_i^{d_l+g_r-k}w_{lrk}\right)\\
&= \sum_{\gamma\in Q(\alpha,\beta,n,m)}e_\gamma\left(B\left(x^{c_1}y^{d_1},\ldots,x^{c_a},y^{d_a}, x^{f_1}y^{g_1},\ldots,x^{f_b}y^{g_b}\right)\right)w^\gamma,
\end{aligned} \tag{106}$$

where $\min = \min\{d_l, f_r\}$ and $w_{lrk} = t_ls_r\hbar^k$. □

Corollary 27. *With the assumptions of Theorem 26, the Poisson bracket of the multisymmetric functions $e_\alpha(p)$ and $e_\beta(q)$ is given by*

$$\left\{e_\alpha(p), e_\beta(q)\right\} = 2\sum_{\gamma\in Q(\alpha,\beta,n,1)}e_\gamma\left(B\left(x^{c_1}y^{d_1},\ldots,x^{c_a},y^{d_a}, x^{f_1}y^{g_1},\ldots,x^{f_b}y^{g_b}\right)\right). \tag{107}$$

Example 28. Let $p = y,\ q = x \in \mathbb{R}[x,y]$. We have

$$e_\alpha(y)\star e_\beta(x) = \sum_{m\ge 0}\sum_{\gamma\in Q(\alpha,\beta,n,m)}e_\gamma(y,x,xy,1)\hbar^m, \tag{108}$$

where the vectors $\gamma = (\gamma_{100},\gamma_{010},\gamma_{110},\gamma_{111}) \in \mathbb{N}^4$ is such that $|\gamma| \le n$ and

$$\gamma_{100}+\gamma_{110}+\gamma_{111} = \alpha, \qquad \gamma_{010}+\gamma_{110}+\gamma_{111} = \beta, \qquad \gamma_{111} = m. \tag{109}$$

For example, for $n = 3,\ \alpha = 2,\ \beta = 3$, we have

$$e_2(y)\star e_3(x) = e_{(1,2)}(x,xy) + e_{(1,1,1)}(x,xy,1)\hbar + e_{(1,2)}(x,1)\hbar^2 \tag{110}$$

since in this case $\gamma = (\gamma_{100},\gamma_{010},\gamma_{110},\gamma_{111}) \in \mathbb{N}^4$ is such that

$$\gamma_{100}+\gamma_{010}+\gamma_{110}+\gamma_{111} \le 3, \qquad \gamma_{100}+\gamma_{110}+\gamma_{111} = 2, \qquad \gamma_{010}+\gamma_{110}+\gamma_{111} = 3, \qquad \gamma_{111} = m. \tag{111}$$

Solving this equation for $m = 0, 1, 2$ we, respectively, obtain

$$\gamma = (0,1,2,0), \qquad \gamma = (0,1,1,1), \qquad \gamma = (0,1,0,2), \tag{112}$$

yielding the desired result.

On the other hand, from Definition 1 we get

$$e_2(y) = y_1y_2 + y_1y_3 + y_2y_3, \qquad e_3(x) = x_1x_2x_3, \tag{113}$$

and thus

$$\begin{aligned} e_2(y) \star e_3(x) &= (y_1y_2 + y_1y_3 + y_2y_3) \star (x_1x_2x_3) \\ &= (x_1x_2x_3y_1y_2 + x_1x_2x_3y_1y_3 + x_1x_2x_3y_2y_3) \\ &\quad + (x_1x_3y_1 + x_1x_2y_1 + x_2x_3y_2 + x_1x_2y_2 \\ &\qquad + x_2x_3y_3 + x_1x_3y_3)\hbar + (x_1 + x_2 + x_3)\hbar^2, \end{aligned} \tag{114}$$

which indeed is equal to

$$e_{(1,2)}(x, xy) + e_{(1,1,1)}(x, xy, 1)\hbar + e_{(1,2)}(x, 1)\hbar^2. \tag{115}$$

Note that $\{e_2(y), e_3(x)\} = 2e_{(1,1,1)}(x, xy, 1)$.

Example 29. Let $p = q = xy \in \mathbb{R}[x, y]$, then we have

$$\begin{aligned} &e_\alpha(xy) \star e_\beta(xy) \\ &\quad = \sum_{m\geq 0} \sum_{\gamma\in QL(\alpha,\beta,n,m)} e_\gamma(xy, xy, x^2y^2, xy)\hbar^m, \end{aligned} \tag{116}$$

where the vector $\gamma = (\gamma_{100}, \gamma_{010}, \gamma_{110}, \gamma_{111}) \in \mathbb{N}^4$ is such that $|\gamma| \leq n$ and

$$\begin{gathered} \gamma_{100} + \gamma_{110} + \gamma_{111} = \alpha, \qquad \gamma_{010} + \gamma_{110} + \gamma_{111} = \beta, \\ \gamma_{111} = m. \end{gathered} \tag{117}$$

Thus for $n = 2$, $\alpha = 2$, $\beta = 1$, we get

$$e_2(xy) \star e_1(xy) = e_{(1,1)}(xy, x^2y^2) + e_{(1,1)}(xy, xy)\hbar \tag{118}$$

since in this case $\gamma = (\gamma_{100}, \gamma_{010}, \gamma_{110}, \gamma_{111}) \in \mathbb{N}^4$ is such that

$$\begin{gathered} \gamma_{100} + \gamma_{010} + \gamma_{110} + \gamma_{111} \leq 2, \qquad \gamma_{100} + \gamma_{110} + \gamma_{111} = 2, \\ \gamma_{010} + \gamma_{110} + \gamma_{111} = 1, \qquad \gamma_{111} = m. \end{gathered} \tag{119}$$

Solving this equation for $m = 0, 1$ we, respectively, obtain

$$\gamma = (1, 0, 1, 0), \qquad \gamma = (1, 0, 0, 1), \tag{120}$$

yielding the desired result.

From Definition 1 we have $e_2(xy) = x_1y_1x_2y_2$, $e_1(xy) = x_1y_1 + x_2y_2$, and thus

$$\begin{aligned} e_2(xy) \star e_1(xy) &= (x_1y_1x_2y_2) \star (x_1y_1 + x_2y_2) \\ &= \left(x_1^2y_1^2x_2y_2 + x_1y_1x_2^2y_2^2\right) \\ &\quad + 2(x_1y_1x_2y_2)\hbar, \end{aligned} \tag{121}$$

which indeed is equal to $e_{(1,1)}(xy, x^2y^2) + e_{(1,1)}(xy, xy)\hbar$.

Note that $\{e_2(xy), e_1(xy)\} = 2e_{(1,1)}(xy, xy)$.

Example 30. Let $n = 2$, $\alpha = \beta = 2$; then

$$\begin{aligned} &e_2(x^ay) \star e_2(xy^b) \\ &\quad = e_2\left(x^{a+1}y^{b+1}\right) \\ &\qquad + e_{(1,1)}\left(x^{a+1}y^{b+1}, x^ay^b\right)\hbar + e_2\left(x^ay^b\right)\hbar^2. \end{aligned} \tag{122}$$

Using Theorem 19 we have

$$\begin{aligned} &e_2(x^ay) \star e_2(xy^b) \\ &\quad = \sum_{\gamma\in QL(2,2,2,m)} e_\gamma\left(x^ay, xy^b, x^{a+1}y^{b+1}, x^ay^b\right)\hbar^m, \end{aligned} \tag{123}$$

where $\gamma = (\gamma_{100}, \gamma_{010}, \gamma_{110}, \gamma_{111}) \in \mathbb{N}^4$ is such that

$$\begin{gathered} \gamma_{100} + \gamma_{010} + \gamma_{110} + \gamma_{111} \leq 2, \qquad \gamma_{100} + \gamma_{110} + \gamma_{111} = 2, \\ \gamma_{010} + \gamma_{110} + \gamma_{111} = 2, \qquad \gamma_{111} = m. \end{gathered} \tag{124}$$

Solving this equation for $m = 0, 1, 2$ we, respectively, obtain

$$\gamma = (0, 0, 2, 0), \qquad \gamma = (0, 0, 1, 1), \qquad \gamma = (0, 0, 0, 2). \tag{125}$$

Thus we get

$$\begin{aligned} &e_2(x^ay) \star e_2(xy^b) \\ &\quad = e_2\left(x^{a+1}y^{b+1}\right) + e_{(1,1)}\left(x^{a+1}y^{b+1}, x^ay^b\right)\hbar \\ &\qquad + e_2\left(x^ay^b\right)\hbar^2. \end{aligned} \tag{126}$$

On the other hand, from Definition 1 we have

$$e_2(x^ay) = x_1^ay_1x_2^ay_2, \qquad e_2(xy^b) = x_1y_1^bx_2y_2^b. \tag{127}$$

Computing directly the $\star$-product we obtain

$$\begin{aligned} &e_2(x^ay) \star e_2(xy^b) \\ &\quad = (x_1^ay_1x_2^ay_2) \star \left(x_1y_1^bx_2y_2^b\right) = x_1^{a+1}y_1^{b+1}x_2^{a+1}y_2^{b+1} \\ &\qquad + \left(x_1^ay_1^bx_2^{a+1}y_2^{b+1} + x_1^{a+1}y_1^{b+1}x_2^ay_2^b\right)\hbar \\ &\qquad + x_1^ay_1^bx_2^ay_2^b\hbar^2 \\ &\quad = e_2\left(x^{a+1}y^{b+1}\right) + e_{(1,1)}\left(x^{a+1}y^{b+1}, x^ay^b\right)\hbar \\ &\qquad + e_2\left(x^ay^b\right)\hbar^2. \end{aligned} \tag{128}$$

Note that $\{e_2(x^ay), e_2(xy^b)\} = 2e_{(1,1)}(x^{a+1}y^{b+1}, x^ay^b)$.

We close this work stating the main problem that our research opens.

Problem 31. Describe the relations in the algebra of quantum symmetric functions.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

Acknowledgments

The authors would like to thank Camilo Ortiz and Fernando Novoa for helping them to develop the software used in the examples. Eddy Pariguan thanks "Pontificia Universidad Javeriana" for providing funds for this research through Project ID-00006185 titled Descripción de Producto Cuántico para Funciones Multisimétricas.

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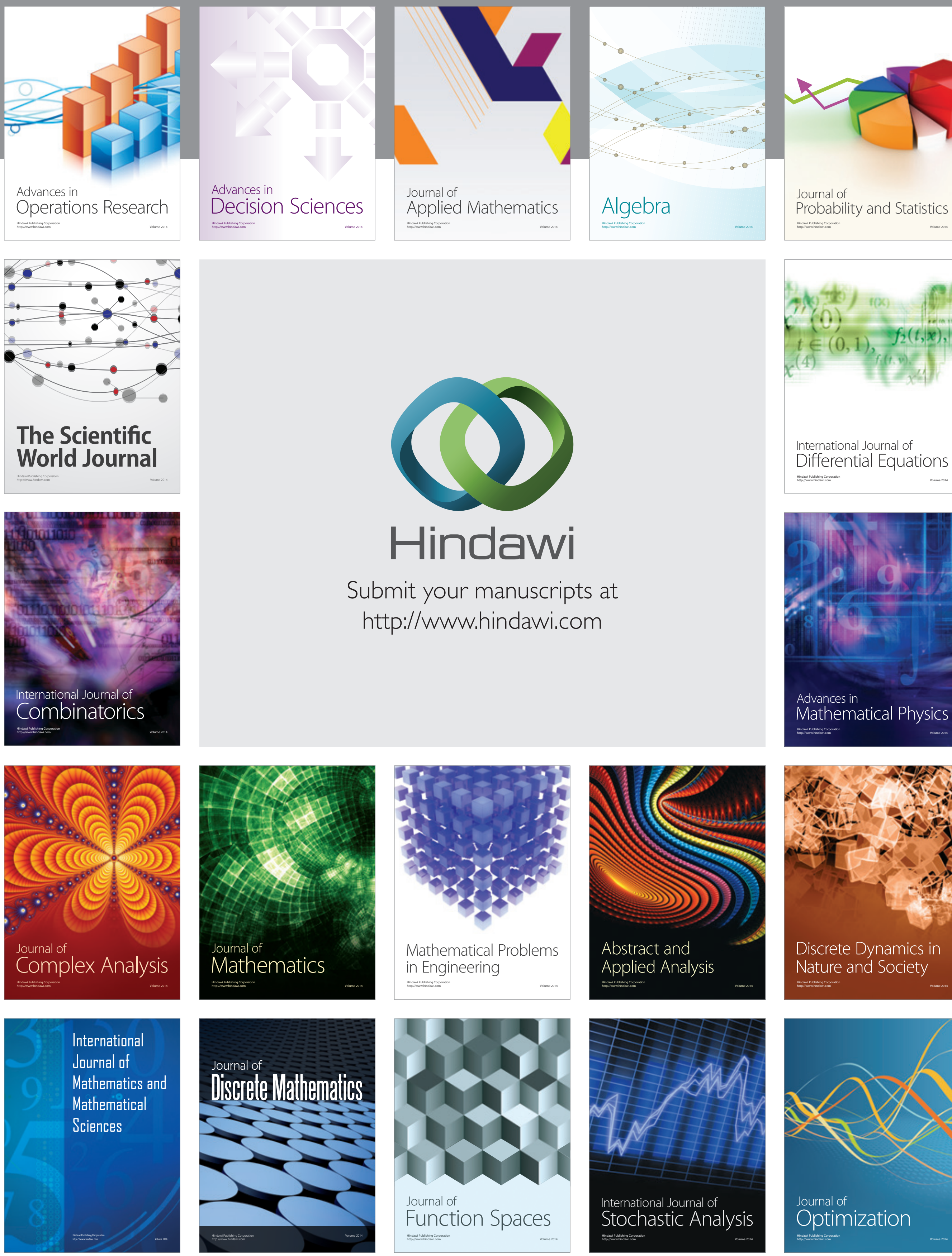

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