Research Article

Adaptive Exponential Synchronization of Coupled Complex Networks on General Graphs

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We investigate the synchronization in complex dynamical networks, where the coupling configuration corresponds to a weighted graph. An adaptive synchronization method on general coupling configuration graphs is given. The networks may synchronize at an arbitrarily given exponential rate by enhancing the updated law of the variable coupling strength and achieve synchronization more quickly by adding edges to original graphs. Finally, numerical simulations are provided to illustrate the effectiveness of our theoretical results.

1. Introduction

In general, complex networks consist of a large number of nodes and links among them, in which a node is a fundamental cell with specific activity, hence, complex networks and graphs closely contact each other. The dynamics on complex networks is one on graphs, though the graphs may have different characteristic, for example, classical random graph model [1], small-world model [2–4], scale-free model [5], or others that are closely related to natural structure.

Synchronization of coupled complex dynamical networks is a universal phenomenon in various fields of science and society. All kinds of synchronization, including adaptive synchronization, global synchronization, antisynchronization, phase synchronization, projection synchronization, and generalized synchronization have been studied [6–14], in particular on the synchronization of an array of linearly coupled identical systems [15–18]. Researches [15, 19] imply that the structural properties of a network must have inevitable effect on the ability and speed of synchronization, but such work still does not see more. In addition, as pointed in [18], the coupling strength may be self-adaptive due to the spontaneousness of updated law, not be calculated numerically in many other works.

In the paper [18], an adaptive synchronization method is introduced, and the networks can synchronize by enhancing the coupling strength automatically under a simple updated law. However, their work is limited to tree-like networks and cannot be applied to general networks. In fact, a tree is a graph without cycles; in this paper, we try to extend the work in [18] to general networks or graphs. We find that their method is in fact effective to general graphs by a rigorous proof, and the networks can also achieve synchronization at an arbitrarily given exponential rate by increasing coupling strength. Based on the knowledge of spectral graph theory, networks can synchronize more quickly by increasing the algebraic connectivity of the graphs, which can be realized by adding edges to original graphs. Finally, numerical simulations are provided to illustrate the effectiveness of theoretical results.

2. Preliminaries

In this section, we now introduce some notations and preliminaries.

Consider the complex dynamical network consisting of N linearly and diffusively coupled identical nodes with full diagonal coupling, and each node is an n-dimensional dynamical oscillator which may be chaotic. The state equations of the network are

$$\dot{x}_i = f(x_i) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j, \quad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^T \in \mathbb{R}^n$ is a state vector of node i, $f(x_i) = (f_1(x_i), f_2(x_i), \ldots, f_n(x_i))^T : \mathbb{R}^n \to \mathbb{R}^n$ is a given

nonlinear vector valued function describing the dynamics of the nodes, $c > 0$ represents coupling strength, and the inner coupling link matrix is a diagonal matrix $\Gamma = \text{diag}\{r_1, r_2, \ldots, r_n\}$ with $r_i > 0$. The coupling configuration matrix $A = (a_{ij})_{N \times N}$ is a zero row sums matrix with nonnegative off-diagonal entries, representing the topological structure of the network.

There is a weighted graph corresponding to the coupling configuration matrix A, called the coupling configuration graph, defined as a graph G on vertices $1, 2, \ldots, N$ which contains an edge ij ($i \neq j$) with weight a_{ij} if and only if $a_{ij} > 0$. Giving an arbitrary orientation of the edges of G, so that each edge has a head and a tail, and a labeling of edges as $e_1, e_2, \ldots, e_E$, where E denotes the number of edges of G. We obtain an edge-vertex incidence matrix of G, denoted as $M := M(A) = (m_{ij})_{E \times N}$, which is defined as $m_{ij} = \sqrt{a_{ij}}$ (resp., $m_{ij} = -\sqrt{a_{ij}}$) if the the edge e_i has the vertex j as a head (resp., a tail), and $m_{ij} = 0$ otherwise. The Laplacian matrix of G is defined as $L = M^T M = D - B$, where D is a diagonal matrix, and its ith diagonal entry of which is exactly the degree of the vertex i, that is, $d_i = \sum_{j \in N(i)} a_{ij}$, where $N(i)$ is the set of neighbors of i in the graph G (or the vertices joining i by edges), and $B = (b_{ij})$ is a weighted adjacency matrix of G such that $b_{ij} = a_{ij}$ if ij is an edge of G and $b_{ij} = 0$ otherwise. One can find that $L = -A$ and is symmetric and positive semidefinite, so its eigenvalues can be arranged as

$$0 = \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{N-1}, \tag{2}$$

where $\lambda_0 = 0$ as L has zero row sums, and $\lambda_1 > 0$ if and only if G is connected and is called the algebraic connectivity of G by Fiedler [20] in the case of G is simple (i.e., all edges have weight 1). If G is connected, the corresponding eigenvector of the eigenvalue 0 is an all one vector (up to a scalar multiple), denoted by **1**. One can refer to Chung [21] and Merris [22] for the details of Laplacian matrices of graphs.

If we denote $x = (x_1^T, x_2^T, \ldots, x_N^T)^T, F(x) = (f^T(x_1), f^T(x_2), \ldots, f^T(x_N))^T$ and substitute $-L$ for A, (1) is transformed as

$$\dot{x} = F(x) - cL \otimes \Gamma x. \tag{3}$$

Taking the transformation $y = xe^{at}(a > 0)$, (3) is written as

$$\dot{y} = e^{at} F\left(e^{-at} y\right) - \left(cL \otimes \Gamma - aI_{Nn}\right) y, \tag{4}$$

where $I_{Nn} = I_N \otimes I_n$, and I_m denotes an identity matrix of order m.

In this paper, we adopt the l_2-norm for vectors and the induced spectral norm for matrices. We always suppose that the function F in (3) is Lipschitz continuous, or equivalently f in (1) is Lipschitz continuous, that is, there exists a constant $l > 0$ such that for any $x, y \in R^n$,

$$\left\| f(x) - f(y) \right\| \leq l \cdot \left\| x - y \right\|. \tag{5}$$

Denote by $\mathbf{M} = M \otimes I_n$, one can obtain the following lemma.

Lemma 1 (see [15]). *Suppose that the coupling configuration graph corresponding to $A = -L$ is connected. Then, the dynamical networks (4) achieve synchronization if and only if $\lim_{t \to \infty} \| \mathbf{M}y \| = 0$.*

3. Main Results

If the network (4) achieves synchronization, surely the network (3) or (1) achieves synchronization with exponential rate a. So, we directly discuss (4), the main result of this paper is stated as follows.

Theorem 2. *Suppose that F is Lipschitz continuous, and the coupling configuration graph corresponding to $A = -L$ is connected. Then, the network (4) achieves synchronization, and in particular the network (1) achieves exponential synchronization with rate a, when the coupling strength c is adapted duly according to the following updated law:*

$$\dot{c} = k y^T(t) \left(L^2 \otimes \Gamma\right) y(t), \tag{6}$$

where $k > 0$ is an arbitrary constant.

Proof. Construct a Lyapunov function

$$V(t) = \frac{1}{2} \left\| \mathbf{M}y(t) \right\|^2 + \frac{1}{2k}(c - h)^2, \tag{7}$$

where h is a sufficiently large constant.

Noting that $\mathbf{M}^T \mathbf{M} = L \otimes I_n$, and substituting (6) for $\dot{c}$, we get the derivative of $V(t)$ along the trajectories of (4) as follows:

$$\begin{aligned}
\dot{V} &= (\mathbf{M}y)^T (\mathbf{M}\dot{y}) + \frac{1}{k}(c - h)\dot{c} \\
&= (\mathbf{M}y)^T \mathbf{M}\left[e^{at} F\left(e^{-at} y\right) - (cL \otimes \Gamma - aI_{Nn})y\right] \\
&\quad + (c - h) y^T \left(L^2 \otimes \Gamma\right) y \\
&= e^{at}(\mathbf{M}y)^T \mathbf{M}F\left(e^{-at} y\right) - h y^T \left(L^2 \otimes \Gamma\right) y + a y^T (L \otimes I_n) y \\
&\quad - c y^T \mathbf{M}^T \mathbf{M}(L \otimes \Gamma) y + c y^T \left(L^2 \otimes \Gamma\right) y.
\end{aligned} \tag{8}$$

Noting that

$$\mathbf{M}^T \mathbf{M}(L \otimes \Gamma) = (L \otimes I_n)(L \otimes \Gamma) = L^2 \otimes \Gamma, \tag{9}$$

we have

$$\dot{V} = e^{at}(\mathbf{M}y)^T \mathbf{M}F\left(e^{-at} y\right) - h y^T \left(L^2 \otimes \Gamma\right) y + a y^T (L \otimes I_n) y. \tag{10}$$

Let G be the coupling configuration graph corresponding to $A = -L$. Let $\vec{G}$ be the directed graph after an arbitrary orientation of the edges of G, and let $\mathscr{E}(\vec{G})$ be the edge set of $\vec{G}$. An oriented edge in $\mathscr{E}(\vec{G})$ is denoted by $e = i \to j$, where j is the head of e and i is the tail of e.

Observe that

$$
(\mathbf{M}y)^T \mathbf{M} F\left(e^{-at}y\right)
$$

$$
= \sum_{e=i \to j \in \mathscr{E}(\vec{G})} m_{ij}\left(y_j - y_i\right)^T \cdot m_{ij}\left[f\left(e^{-at}y_j\right) - f\left(e^{-at}y_i\right)\right]
$$

$$
\leq e^{-at} \cdot l \cdot \sum_{e=i \to j \in \mathscr{E}(\vec{G})} m_{ij}^2 \left\|y_j - y_i\right\|^2
$$

$$
= le^{-at}(\mathbf{M}y)^T (\mathbf{M}y) = le^{-at}y^T \left(L \otimes I_n\right) y.
$$

(11)

Hence,

$$
\dot{V} \leq (l+a)\, y^T \left(L \otimes I_n\right) y - h y^T \left(L^2 \otimes \Gamma\right) y
$$

$$
\leq (l+a)\, y^T \left(L \otimes I_n\right) y - h r y^T \left(L^2 \otimes I_n\right) y,
$$

(12)

where $r = \min_{1 \leq i \leq n} r_i > 0$.

Note that L is a real and symmetric matrix, so we may let $z_0 := (1/\sqrt{n})\mathbf{1}, z_1, \ldots, z_{N-1}$ as unit orthogonal eigenvectors corresponding to eigenvalues $\lambda_0 = 0, \lambda_1, \ldots, \lambda_{N-1}$ of the matrix L, respectively. Let $\overline{z}_0, \overline{z}_1, \ldots, \overline{z}_{n-1}$ be an arbitrary orthogonal base of $\mathbb{R}^n$. Thus, $z_i \otimes \overline{z}_j$, $i = 0, 1, \ldots, N-1$, $j = 0, 1, \ldots, n-1$, form an orthogonal base of $\mathbb{R}^{Nn}$, which are also the unit orthogonal eigenvectors of $L \otimes I_n$ and $L^2 \otimes I_n$. As the associated graph of L or A is connected, $\lambda_1 > 0$, we now write

$$
y(t) = \sum_{i,j} \alpha_{ij}(t)\, z_i \otimes \overline{z}_j.
$$

(13)

Then, we obtain

$$
\dot{V} \leq (l+a) \sum_{i \geq 0, j} \lambda_i \alpha_{ij}(t)^2 - hr \sum_{i \geq 0, j} \lambda_i^2 \alpha_{ij}(t)^2
$$

$$
= (l+a) \sum_{i \geq 1, j} \lambda_i \alpha_{ij}(t)^2 - hr \sum_{i \geq 1, j} \lambda_i^2 \alpha_{ij}(t)^2
$$

$$
\leq (l+a) \sum_{i \geq 1, j} \lambda_i \alpha_{ij}(t)^2 - hr \sum_{i \geq 1, j} \lambda_i \lambda_1 \alpha_{ij}(t)^2
$$

$$
= (l+a-hr\lambda_1) \sum_{i \geq 1, j} \lambda_i \alpha_{ij}(t)^2
$$

$$
= (l+a-hr\lambda_1)\, y^T \left(L \otimes I_n\right) y
$$

$$
= (l+a-hr\lambda_1)(\mathbf{M}y)^T (\mathbf{M}y).
$$

(14)

Hence, we can choose a sufficiently large h such that $\beta := l + a - hr\lambda_1 < 0$. Thus,

$$
\dot{V} \leq \beta(\mathbf{M}y(t))^T (\mathbf{M}y(t)).
$$

(15)

It is obvious that $\dot{V} = 0$ if and only if $\mathbf{M}y = 0$. In addition, if $\mathbf{M}y = 0$, then $y = \mathbf{1} \otimes y_1$, and $\dot{V} = 0$ by (10). Therefore, the set $E = \{y \mid \mathbf{M}y = 0\}$ is the largest invariant set contained in $\dot{V} = 0$ for the system (4). Then, we obtain $\lim_{t \to \infty} \| \mathbf{M}y \| = 0$ based on the LaSalle invariant principle of differential equations. Hence, by Lemma 1, the complex dynamical network (4) is synchronized under the updated
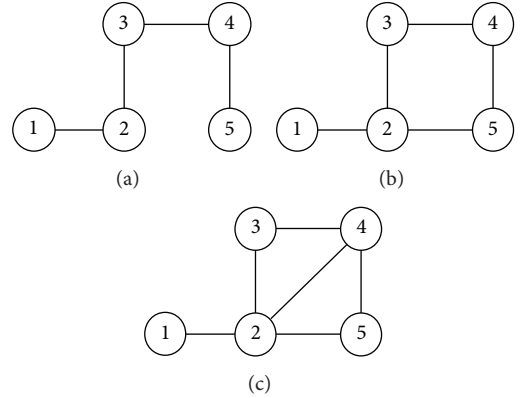

Figure 1: Three small networks.

law (6) of coupling strength c. In particular, the complex dynamical network (1) is exponentially synchronized with rate a. □

Remark 3. In [18], the updated law of coupling strength c is given as follows:

$$
\dot{c} = Kx(t)^T \mathbf{M}^T \mathbf{M}\mathbf{M}^T \mathbf{M}\Gamma x(t) = Kx(t)^T L^2 \otimes \Gamma x(t),
$$

(16)

where $\Gamma = I_N \otimes \Gamma$; the synchronization speed of complex dynamical network (1) or (3) can be controlled by the constant K in (16). Here, we take $k = Ke^{2at}$; the synchronization is realized with exponential rate a. So, we make a further illustration of the above statement in [18] from (6) and Theorem 2.

Remark 4. From (15) in the proof of Theorem 2, if β is taken smaller, $V(t)$ will decrease quickly, and then the synchronization of the network will be attained quickly. This will be done if the Laplacian matrix of the coupling configuration graph has a larger eigenvalue λ_1. Based on the knowledge of spectral graph theory (or see [23]), λ_1 is not decreased by adding edges to original graph. Therefore, we can add edges to the coupling configuration graph such that the complex network achieves synchronization more quickly.

Remark 5. As the coupling configuration graph is connected and Lemma 1 holds, the dynamical network (4) achieves synchronization. If the graph is disconnected, the network only can synchronize in each connected component of the graph but not in the whole graph.

4. Numerical Simulations

In this section, we present several numerical examples to show the theoretical results. In particular, we consider complex dynamical networks with the Lorenz system as each node

$$
\begin{bmatrix} \dot{x}_{i1} \\ \dot{x}_{i2} \\ \dot{x}_{i3} \end{bmatrix} = \begin{bmatrix} \alpha\left(x_{i2} - x_{i1}\right) \\ \gamma x_{i1} - x_{i1}x_{i3} - x_{i2} \\ x_{i1}x_{i2} - \beta x_{i3} \end{bmatrix}, \quad i = 1, 2, 3, 4, 5,
$$

(17)

which is chaotic when $\alpha = 10, \beta = 8/3, \gamma = 28$. Three small networks with five nodes are shown in Figure 1, where the
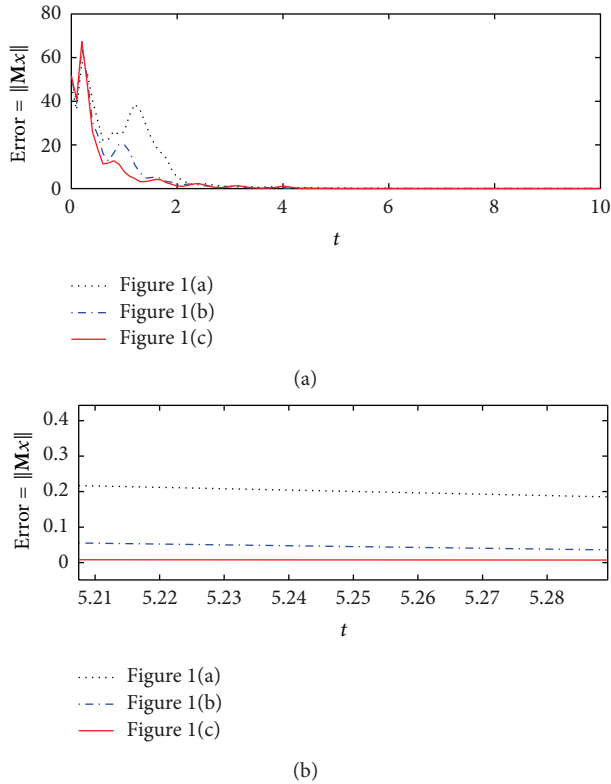
(a)

(b)

Figure 2: Synchronization errors ∥ **M**x ∥ of three networks in Figure 1.

weight of every edge in each network is 1. Choose the inner coupling link matrix to be $\Gamma = \text{diag}\{1, 2, 3\}$.

The corresponding coupling configuration matrices are listed in the same order as follows:

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 1 & 0 & 1 & -2 \end{bmatrix},$$

(18)

$$\begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & -4 & 1 & 1 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & 1 & -3 & 1 \\ 0 & 1 & 0 & 1 & -2 \end{bmatrix}.$$

According to Theorem 2, the updated law of coupling strength c is chosen as $\dot{c} = Kx^T(t)(L^2 \otimes \Gamma)x(t)$ (i.e., $k = K$, $a = 0$ in (6)). Here $K = 0.0001$, the initial coupling strength $c = 0$, and the initial value of x is taken as $[-8, 2, 35, 1, -9, -3, -4, 5, 1, 2, -4, 1, -7, 8, 4]^T$. The synchronization errors ∥ **M**x ∥ of three networks are shown in Figure 2, respectively, where the figure listed below is a local magnification of the above figure, and Figure 2 also shows that the method in [18] is effective to general networks. Numerical simulations show that synchronization can be reached more quickly by increasing the algebraic connectivity of the graph, which can be realized by adding edges to original graph.
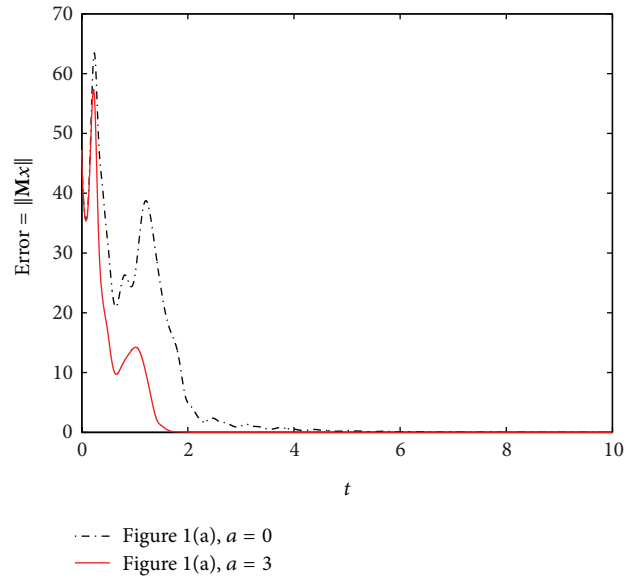

Figure 3: Exponential Synchronization errors ∥**M**x∥ of the network in Figure 1(a).

Figure 3 shows that adaptive exponential synchronization is achieved for complex dynamical network in Figure 1(a), where $k = 0.0001$, $a = 0$, and $a = 3$ in (6). The same results also hold for the networks in Figures 1(b) and 1(c).

5. Conclusions

In this paper, an adaptive synchronization method on general networks or graphs is obtained. According to our method, networks may synchronize at an arbitrarily given exponential rate by increasing coupling strength and synchronize more quickly by increasing the algebraic connectivity of the corresponding graphs, which can be achieved by adding edges to original graphs. The obtained results also extend the work in [18] as a tree is a graph without cycles. Finally, numerical simulations are provided to illustrate the effectiveness of our theoretical results.

Acknowledgments

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