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Research Article

Properties of Third-Order Nonlinear Functional Differential Equations with Mixed Arguments

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The aim of this paper is to offer sufficient conditions for property (B) and/or the oscillation of the third-order nonlinear functional differential equation with mixed arguments $[a(t)[x''(t)]^\gamma]' = q(t)f(x[\tau(t)])+p(t)h(x[\sigma(t)])$. Both cases $\int^\infty a^{-1/\gamma}(s)\mathrm{d}s = \infty$ and $\int^\infty a^{-1/\gamma}(s)\mathrm{d}s < \infty$ are considered. We deduce properties of the studied equations via new comparison theorems. The results obtained essentially improve and complement earlier ones.

1. Introduction

We are concerned with the oscillatory and certain asymptotic behavior of all solutions of the third-order functional differential equations

$$[a(t)[x''(t)]^\gamma]' = q(t)f(x[\tau(t)]) + p(t)h(x[\sigma(t)]). \tag{E}$$

Throughout the paper, it is assumed that $a, q, p \in C([t_0,\infty))$, $\tau, \sigma \in C^1([t_0,\infty))$, $f, h \in C((-\infty,\infty))$, and

(H_1) γ is the ratio of two positive odd integers,

(H_2) $a(t)$, $q(t)$, $p(t)$ are positive,

(H_3) $\tau(t) \le t$, $\sigma(t) \ge t$, $\tau'(t) > 0$, $\sigma'(t) > 0$, $\lim_{t\to\infty}\tau(t) = \infty$,

(H_4) $f^{1/\gamma}(x)/x \ge 1$, $xh(x) > 0$, $f'(x) \ge 0$, and $h'(x) \ge 0$ for $x \ne 0$,

(H_5) $-f(-xy) \ge f(xy) \ge f(x)f(y)$ for $xy > 0$ and $-h(-xy) \ge h(xy) \ge h(x)h(y)$ for $xy > 0$.

By a solution of (E), we mean a function $x(t) \in C^2([T_x, \infty))$, $T_x \geq t_0$, which has the property $a(t)(x''(t))^\gamma \in C^1([T_x, \infty))$ and satisfies (E) on $[T_x, \infty)$. We consider only those solutions $x(t)$ of (E) which satisfy $\sup\{|x(t)| : t \geq T\} > 0$ for all $T \geq T_x$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x, \infty)$, and, otherwise, it is nonoscillatory. Equation (E) is said to be oscillatory if all its solutions are oscillatory.

Recently, (E) and its particular cases (see [1–17]) have been intensively studied. The effort has been oriented to provide sufficient conditions for every (E) to satisfy

$$\lim_{t \to \infty} |x(t)| = \infty \tag{1.1}$$

or to eliminate all nonoscillatory solutions. Following [6, 8, 13, 15], we say that (E) has property (B) if each of its nonoscillatory solutions satisfies (1.1).

We will discuss both cases

$$\int_{t_0}^{\infty} a^{-1/\gamma}(s)\mathrm{d}s < \infty, \tag{1.2}$$

$$\int_{t_0}^{\infty} a^{-1/\gamma}(s)\mathrm{d}s = \infty. \tag{1.3}$$

We will establish suitable comparison theorems that enable us to study properties of (E) regardless of the fact that (1.3) or (1.2) holds. We will compare (E) with the first-order advanced/delay equations, in the sense that the oscillation of these first-order equations yields property (B) or the oscillation of (E).

In the paper, we are motivated by an interesting result of Grace et al. [10], where the oscillation criteria for (E) are discussed. This result has been complemented by Baculíková et al. [5]. When studying properties of (E), the authors usually reduce (E) onto the corresponding differential inequalities

$$\begin{aligned} \left[a(t)\left[x''(t)\right]^\gamma\right]' &\geq q(t) f(x[\tau(t)]), \\ \left[a(t)\left[x''(t)\right]^\gamma\right]' &\geq p(t) h(x[\sigma(t)]), \end{aligned} \tag{E_σ}$$

and further study only properties of these inequalities. Therefore, the criteria obtained withhold information either from delay argument $\tau(t)$ and the corresponding functions $q(t)$ and $f(u)$ or from advanced argument $\sigma(t)$ and the corresponding functions $p(t)$ and $h(u)$. In the paper, we offer a technique for obtaining new criteria for property (B) and the oscillation of (E) that involve both arguments $\tau(t)$ and $\sigma(t)$. Consequently, our results are new even for the linear case of (E) and properly complement and extend earlier ones presented in [1–17].

Remark 1.1. All functional inequalities considered in this paper are assumed to hold eventually; that is, they are satisfied for all t large enough.

2. Main Results

The following results are elementary but useful in what comes next.

Lemma 2.1. *Assume that $A \geq 0$, $B \geq 0$, $\alpha \geq 1$. Then,*

$$(A+B)^{\alpha} \geq A^{\alpha} + B^{\alpha}. \tag{2.1}$$

Proof. If $A = 0$ or $B = 0$, then (2.1) holds. For $A \neq 0$, setting $x = B/A$, condition (2.1) takes the form $(1+x)^{\alpha} \geq 1 + x^{\alpha}$, which is for $x > 0$ evidently true. □

Lemma 2.2. *Assume that $A \geq 0$, $B \geq 0$, $0 < \alpha \leq 1$. Then,*

$$(A+B)^{\alpha} \geq \frac{A^{\alpha} + B^{\alpha}}{2^{1-\alpha}}. \tag{2.2}$$

Proof. We may assume that $0 < A < B$. Consider a function $g(u) = u^{\alpha}$. Since $g''(u) < 0$ for $u > 0$, function $g(u)$ is concave down; that is,

$$g\left(\frac{A+B}{2}\right) \geq \frac{g(A) + g(B)}{2} \tag{2.3}$$

which implies (2.2). □

The following result presents a useful relationship between an existence of positive solutions of the advanced differential inequality and the corresponding advanced differential equation.

Lemma 2.3. *Suppose that $p(t)$, $\sigma(t)$, and $h(u)$ satisfy (H_2), (H_3), and (H_4), respectively. If the first-order advanced differential inequality*

$$z'(t) - p(t)h(z(\sigma(t))) \geq 0 \tag{2.4}$$

has an eventually positive solution, so does the advanced differential equation

$$z'(t) - p(t)h(z(\sigma(t))) = 0. \tag{2.5}$$

Proof. Let $z(t)$ be a positive solution of (2.4) on $[t_1, \infty)$. Then, $z(t)$ satisfies the inequality

$$z(t) \geq z(t_1) + \int_{t_1}^{t} p(s)h(z(\sigma(s)))\mathrm{d}s. \tag{2.6}$$

Let

$$\begin{aligned} y_1(t) &= z(t), \\ y_n(t) &= z(t_1) + \int_{t_1}^{t} p(s)h\big(y_{n-1}(\sigma(s))\big)\mathrm{d}s, \qquad n = 2, 3 \ldots. \end{aligned} \tag{2.7}$$

It follows from the definition of $y_n(t)$ and (H_4) that the sequence $\{y_n\}$ has the property

$$z(t) = y_1(t) \geq y_2(t) \geq \cdots \geq z(t_1), \quad t \geq t_1. \tag{2.8}$$

Hence, $\{y_n\}$ converges pointwise to a function $y(t)$, where $z(t) \geq y(t) \geq z(t_1)$. Let $h_n(t) = p(t)h(y_n(\sigma(t)))$, $n = 1, 2, \ldots$, then $h_1(t) \geq h_2(t) \geq \cdots \geq 0$. Since $h_1(t)$ is integrable on $[t_1, t]$ and $\lim_{n\to\infty} h_n(t) = p(t)h(y(\sigma(t)))$, it follows by Lebesgue's dominated convergence theorem that

$$y(t) = z(t_1) + \int_{t_1}^{t} p(s)h(y(\sigma(s)))\mathrm{d}s. \tag{2.9}$$

Thus, $y(t)$ satisfies (2.5). □

We start our main results with the classification of the possible nonoscillatory solutions of (E).

Lemma 2.4. *Let $x(t)$ be a nonoscillatory solution of (E). Then, $x(t)$ satisfies, eventually, one of the following conditions*

(I)

$$x(t)x'(t) > 0, \qquad x(t)x''(t) > 0, \qquad x(t)\left[a(t)\left[x''(t)\right]^{\gamma}\right]' > 0, \tag{2.10}$$

(II)

$$x(t)x'(t) > 0, \qquad x(t)x''(t) < 0, \qquad x(t)\left[a(t)\left[x''(t)\right]^{\gamma}\right]' > 0, \tag{2.11}$$

and if (1.2) *holds, then also*

(III)

$$x(t)x'(t) < 0, \qquad x(t)x''(t) > 0, \qquad x(t)\left[a(t)\left[x''(t)\right]^{\gamma}\right]' > 0. \tag{2.12}$$

Proof. Let $x(t)$ be a nonoscillatory solution of (E), say $x(t) > 0$ for $t \geq t_0$. It follows from (E) that $[a(t)[x''(t)]^{\gamma}] > 0$, eventually. Thus, $x''(t)$ and $x'(t)$ are of fixed sign for $t \geq t_1$, t_1 large enough. At first, we assume that $x''(t) < 0$. Then, either $x'(t) > 0$ or $x'(t) < 0$, eventually. But $x''(t) < 0$ together with $x'(t) < 0$ imply that $x(t) < 0$. A contradiction, that is, Case (II) holds.

Now, we suppose that $x''(t) > 0$, then either Case (I) or Case (III) holds. On the other hand, if (1.3) holds, then Case (III) implies that $a(t)[x''(t)]^{\gamma} \geq c > 0$, $t \geq t_1$. Integrating from t_1 to t, we have

$$x'(t) - x'(t_1) \geq c^{1/\gamma} \int_{t_1}^{t} a^{-1/\gamma}(s)\mathrm{d}s, \tag{2.13}$$

which implies that $x'(t) \to \infty$ as $t \to \infty$, and we deduce that Case (III) may occur only if (1.2) is satisfied. The proof is complete. □

Remark 2.5. It follows from Lemma 2.4 that if (1.3) holds, then only Cases (I) and (II) may occur.

In the following results, we provide criteria for the elimination of Cases (I)–(III) of Lemma 2.4 to obtain property (B)/oscillation of (E).

Let us denote for our further references that

$$P(t)=\int_t^\infty a^{-1/\gamma}(u)\left(\int_u^\infty p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u, \tag{2.14}$$

$$Q(t)=\int_t^\infty a^{-1/\gamma}(u)\left(\int_u^\infty \frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u. \tag{2.15}$$

Theorem 2.6. *Let $0<\gamma\le 1$. Assume that $x(t)$ is a nonoscillatory solution of (E). If the first-order advanced differential equation*

$$z'(t)-P(t)e^{-\int_{t_1}^t Q(s)\mathrm{d}s}\,h^{1/\gamma}\left(e^{\int_{t_1}^{\sigma(t)} Q(s)\mathrm{d}s}\right)h^{1/\gamma}(z[\sigma(t)])=0 \tag{E_1}$$

is oscillatory, then Case (II) cannot hold.

Proof. Let $x(t)$ be a nonoscillatory solution of (E), satisfying Case (II) of Lemma 2.4. We may assume that $x(t)>0$ for $t\ge t_0$. Integrating (E) from t to ∞, one gets

$$-a(t)[x''(t)]^\gamma\ge\int_t^\infty q(s)f(x[\tau(s)])\mathrm{d}s+\int_t^\infty p(s)h(x[\sigma(s)])\mathrm{d}s. \tag{2.16}$$

On the other hand, the substitution $\tau(s)=u$ gives

$$\begin{aligned}\int_t^\infty q(s)f(x[\tau(s)])\mathrm{d}s&=\int_{\tau(t)}^\infty \frac{q(\tau^{-1}(u))}{\tau'(\tau^{-1}(u))}f(x(u))\mathrm{d}u\\&\ge\int_t^\infty \frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}f(x(s))\mathrm{d}s.\end{aligned} \tag{2.17}$$

Using (2.17) in (2.16), we find

$$-x''(t)\ge a^{-1/\gamma}(t)\left(\int_t^\infty \frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}f(x(s))\mathrm{d}s+\int_t^\infty p(s)h(x[\sigma(s)])\mathrm{d}s\right)^{1/\gamma}. \tag{2.18}$$

Taking into account the monotonicity of $x(t)$, it follows from Lemma 2.1 that

$$- x''(t) \geq \frac{f^{1/\gamma}(x(t))}{a^{1/\gamma}(t)}\left(\int_t^\infty \frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\mathrm{d}s\right)^{1/\gamma} + \frac{h^{1/\gamma}(x[\sigma(t)])}{a^{1/\gamma}(t)}\left(\int_t^\infty p(s)\mathrm{d}s\right)^{1/\gamma}, \tag{2.19}$$

where we have used (H_3) and (H_4). An integration from t to ∞ yields

$$\begin{aligned} x'(t) &\geq \int_t^\infty \frac{f^{1/\gamma}(x(u))}{a^{1/\gamma}(u)}\left(\int_u^\infty \frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u \\ &\quad + \int_t^\infty \frac{h^{1/\gamma}(x[\sigma(u)])}{a^{1/\gamma}(u)}\left(\int_u^\infty p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u \\ &\geq f^{1/\gamma}(x(t))Q(t) + h^{1/\gamma}(x[\sigma(t)])P(t). \end{aligned} \tag{2.20}$$

Regarding (H_4), it follows that $x(t)$ is a positive solution of the differential inequality

$$x'(t) - Q(t)x(t) \geq P(t)h^{1/\gamma}(x[\sigma(t)]). \tag{2.21}$$

Applying the transformation

$$x(t) = w(t)e^{\int_{t_1}^t Q(s)\mathrm{d}s}, \tag{2.22}$$

we can easily verify that $w(t)$ is a positive solution of the advanced differential inequality

$$w'(t) - P(t)e^{-\int_{t_1}^t Q(s)\mathrm{d}s}\, h^{1/\gamma}\left(e^{\int_{t_1}^{\sigma(t)} Q(s)\mathrm{d}s}\right)h^{1/\gamma}(w[\sigma(t)]) \geq 0. \tag{2.23}$$

By Lemma 2.3, we conclude that the corresponding differential equation (E_1) has also a positive solution. A contradiction. Therefore, $x(t)$ cannot satisfy Case (II). □

Remark 2.7. It follows from the proof of Theorem 2.8 that if at least one of the following conditions is satisfied:

$$\begin{gathered}
\int_{t_0}^{\infty} p(s)\mathrm{d}s = \infty, \\
\int_{t_0}^{\infty} \frac{q\left(\tau^{-1}(s)\right)}{\tau'\left(\tau^{-1}(s)\right)}\mathrm{d}s = \infty, \\
\int_{t_0}^{\infty} a^{-1/\gamma}(u)\left(\int_{u}^{\infty} p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u = \infty, \\
\int_{t_0}^{\infty} a^{-1/\gamma}(u)\left(\int_{u}^{\infty} \frac{q\left(\tau^{-1}(s)\right)}{\tau'\left(\tau^{-1}(s)\right)}\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u = \infty,
\end{gathered} \tag{2.24}$$

then any nonoscillatory solution $x(t)$ of (E) cannot satisfy Case (II). Therefore, we may assume that the corresponding integrals in (2.14)-(2.15) are convergent.

Now, we are prepared to provide new criteria for property (B) of (E) and also the rate of divergence of all nonoscillatory solutions.

Theorem 2.8. *Let* (1.3) *hold and* $0 < \gamma \leq 1$. *Assume that* (E_1) *is oscillatory. Then,* (E) *has property (B) and, what is more, the following rate of divergence for each of its nonoscillatory solutions holds:*

$$|x(t)| \geq c\int_{t_1}^{t} a^{-1/\gamma}(s)(t-s)\mathrm{d}s, \qquad c > 0. \tag{2.25}$$

Proof. Let $x(t)$ be a positive solution of (E). It follows from Lemma 2.4 and Remark 2.5 that $x(t)$ satisfies either Case (I) or (II). But Theorem 2.6 implies that the Case (II) cannot hold. Therefore, $x(t)$ satisfies Case (I), which implies (1.1); that is, (E) has property (B). On the other hand, there is a constant $c > 0$ such that

$$a(t)\left(x''(t)\right)^{\gamma} \geq c^{\gamma}. \tag{2.26}$$

Integrating twice from t_1 to t, we have

$$x(t) \geq c\int_{t_1}^{t}\left(\int_{t_1}^{u} a^{-1/\gamma}(s)\mathrm{d}s\right)\mathrm{d}u = c\int_{t_1}^{t} a^{-1/\gamma}(s)(t-s)\mathrm{d}s, \tag{2.27}$$

which is the desired estimate. □

Employing an additional condition on the function $h(x)$, we get easily verifiable criterion for property (B) of (E).

Corollary 2.9. *Let* $0 < \gamma \leq 1$ *and* (1.3) *hold. Assume that*

$$h^{1/\gamma}(x)/x \geq 1, \qquad |x| \geq 1, \tag{2.28}$$

$$\liminf_{t\to\infty} \int_t^{\sigma(t)} P(u) e^{\int_u^{\sigma(u)} Q(s)\mathrm{d}s} \mathrm{d}u > \frac{1}{e}. \tag{2.29}$$

Then, (E) *has property (B).*

Proof. First note that (2.29) implies

$$\int_{t_0}^{\infty} P(u) e^{\int_u^{\sigma(u)} Q(s)\mathrm{d}s} \mathrm{d}u = \infty. \tag{2.30}$$

By Theorem 2.8, it is sufficient to show that (E_1) is oscillatory. Assume the converse, let (E_1) have an eventually positive solution $z(t)$. Then, $z'(t) > 0$ and so $z(\sigma(t)) > c > 0$. Integrating (E_1) from t_1 to t, we have in view of (2.28)

$$\begin{aligned} z(t) &\geq \int_{t_1}^{t} P(u) e^{-\int_{t_1}^{u} Q(s)\mathrm{d}s} h^{1/\gamma}\left(e^{\int_{t_1}^{\sigma(u)} Q(s)\mathrm{d}s}\right) h^{1/\gamma}(z[\sigma(u)])\mathrm{d}u \\ &\geq h^{1/\gamma}(c) \int_{t_1}^{t} P(u) e^{\int_u^{\sigma(u)} Q(s)\mathrm{d}s} \mathrm{d}u. \end{aligned} \tag{2.31}$$

Using (2.30) in the previous inequalities, we get $z(t) \to \infty$ as $t \to \infty$. Therefore, $z(t) \geq 1$, eventually. Now, using (2.28) in (E_1), one can verify that $z(t)$ is a positive solution of the differential inequality

$$z'(t) - P(t) e^{\int_t^{\sigma(t)} Q(s)\mathrm{d}s} z(\sigma(t)) \geq 0. \tag{2.32}$$

But, by [14, Theorem 2.4.1], condition (2.29) ensures that (2.32) has no positive solutions. This is a contradiction, and we conclude that (E) has property (B). □

Example 2.10. Consider the third-order nonlinear differential equation with mixed arguments

$$\left(t^{1/3}\left(x''(t)\right)^{1/3}\right)' = \frac{a}{t^{4/3}} x^{1/3}(\lambda t) + \frac{b}{t^{4/3}} x^{\beta}(\omega t), \tag{E_{x1}}$$

where $a, b > 0$, $0 < \lambda < 1$, $\omega > 1$, and $\beta \geq 1/3$ is a ratio of two positive odd integers. Since

$$P(t) = \frac{27b^3}{t}, \qquad Q(t) = \frac{27a^3\lambda}{t}, \tag{2.33}$$

Corollary 2.9 implies that (E_{x1}) has property (B) provided that

$$b^3 \omega^{27a^3\lambda} \ln \omega > \frac{1}{27\mathrm{e}}. \tag{2.34}$$

Moreover, by Theorem 2.8, the rate of divergence of every nonoscillatory solution of (E_{x1}) is

$$|x(t)| \geq ct \ln t, \qquad c > 0. \tag{2.35}$$

For $\beta = 1/3$ and $\delta > 1$ satisfying $\delta^{1/3}(\delta-1)^{4/3} = 3a\lambda^{\delta/3} + 3b\omega^{\delta/3}$, one such solution is t^{δ}.

Now, we turn our attention to the case when $\gamma \geq 1$.

Theorem 2.11. *Let $\gamma \geq 1$. Assume that $x(t)$ is a nonoscillatory solution of (E). If the first-order advanced differential equation*

$$z'(t) - 2^{(1-\gamma)/\gamma} P(t) e^{[-2^{(1-\gamma)/\gamma}\int_{t_1}^{t} Q(s)\mathrm{d}s]} h^{1/\gamma}\left(e^{2^{(1-\gamma)/\gamma}\int_{t_1}^{\sigma(t)} Q(s)\mathrm{d}s}\right) h^{1/\gamma}(z[\sigma(t)]) = 0 \tag{E_2}$$

is oscillatory, then Case (II) cannot hold.

Proof. Let $x(t)$ be an eventually positive solution of (E), satisfying Case (II) of Lemma 2.4. Then, (2.18) holds. Lemma 2.2, in view of the monotonicity of $x(t)$, (H_3), and (H_4), implies

$$\begin{aligned} -x''(t) \geq{}& \frac{f^{1/\gamma}(x(t))}{2^{(\gamma-1)/\gamma} a^{1/\gamma}(t)} \left(\int_t^{\infty} \frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\mathrm{d}s\right)^{1/\gamma} \\ &+ \frac{h^{1/\gamma}(x[\sigma(t)])}{2^{(\gamma-1)/\gamma} a^{1/\gamma}(t)} \left(\int_t^{\infty} p(s)\mathrm{d}s\right)^{1/\gamma}. \end{aligned} \tag{2.36}$$

An integration from t to ∞ yields

$$\begin{aligned} x'(t) \geq{}& \int_t^{\infty} \frac{f^{1/\gamma}(x(u))}{2^{(\gamma-1)/\gamma} a^{1/\gamma}(u)} \left(\int_u^{\infty} \frac{q(\tau^{-1}(s))}{\tau'(\tau^{-1}(s))}\mathrm{d}s\right)^{1/\gamma} \mathrm{d}u \\ &+ \int_t^{\infty} \frac{h^{1/\gamma}(x[\sigma(u)])}{2^{(\gamma-1)/\gamma} a^{1/\gamma}(u)} \left(\int_u^{\infty} p(s)\mathrm{d}s\right)^{1/\gamma} \mathrm{d}u \\ \geq{}& f^{1/\gamma}(x(t)) 2^{(1-\gamma)/\gamma} Q(t) + h^{1/\gamma}(x[\sigma(t)]) 2^{(1-\gamma)/\gamma} P(t). \end{aligned} \tag{2.37}$$

Noting (H_4), we see that $x(t)$ is a positive solution of the differential inequality

$$x'(t) \geq 2^{(1-\gamma)/\gamma} Q(t) x(t) + 2^{(1-\gamma)/\gamma} P(t) h^{1/\gamma}(x[\sigma(t)]). \tag{2.38}$$

Setting

$$x(t) = w(t) e^{[2^{(1-\gamma)/\gamma}\int_{t_1}^{t} Q(s)\mathrm{d}s]}, \tag{2.39}$$

one can see that $w(t)$ is a positive solution of the advanced differential inequality

$$w'(t) - 2^{(1-\gamma)/\gamma} P(t) e^{\left[-2^{(1-\gamma)/\gamma} \int_{t_1}^{t} Q(s)\mathrm{d}s\right]} h^{1/\gamma}\left(e^{2^{(1-\gamma)/\gamma} \int_{t_1}^{\sigma(t)} Q(s)\mathrm{d}s}\right) h^{1/\gamma}(w[\sigma(t)]) \geq 0. \qquad (2.40)$$

By Lemma 2.3, we deduce that the corresponding differential equation (E_2) has also a positive solution. A contradiction. Therefore, $x(t)$ cannot satisfy Case (II). □

The following result is obvious.

Theorem 2.12. *Let* (1.3) *hold and* $\gamma \geq 1$. *Assume that* (E_2) *is oscillatory. Then,* (E) *has property (B) and, what is more, each of its nonoscillatory solutions satisfies* (2.25).

Now, we present easily verifiable criterion for property (B) of (E).

Corollary 2.13. *Let* (1.3) *and* (2.28) *hold and* $\gamma \geq 1$. *If*

$$\liminf_{t \to \infty} \int_{t}^{\sigma(t)} P(u) e^{\left[2^{(1-\gamma)/\gamma} \int_{u}^{\sigma(u)} Q(s)\mathrm{d}s\right]} \mathrm{d}u > \frac{2^{(\gamma-1)/\gamma}}{e}, \qquad (2.41)$$

then (E) *has property (B).*

Proof. The proof is similar to the proof of Corollary 2.9 and so it can be omitted. □

Remark 2.14. Theorems 2.6, 2.8, 2.11, and 2.12 and Corollaries 2.9 and 2.13 provide criteria for property (B) that include both delay and advanced arguments and all coefficients and functions of (E). Our results are new even for the linear case of (E).

Remark 2.15. It is useful to notice that if we apply the traditional approach to (E), that is, if we replace (E) by the corresponding differential inequality (E_σ), then conditions (2.29) of Corollary 2.9 and (2.41) of Corollary 2.13 would take the forms

$$\liminf_{t \to \infty} \int_{t}^{\sigma(t)} P(u)\mathrm{d}u > \frac{1}{e}, \qquad \liminf_{t \to \infty} \int_{t}^{\sigma(t)} P(u)\mathrm{d}u > \frac{2^{(\gamma-1)/\gamma}}{e}, \qquad (2.42)$$

respectively, which are evidently second to (2.29) and (2.41).

Example 2.16. Consider the third-order nonlinear differential equation with mixed arguments

$$\left(t(x''(t))^3\right)' = \frac{a}{t^6} x^3(\lambda t) + \frac{b}{t^6} x^{\beta}(\omega t), \qquad (E_{x2})$$

where $a, b > 0$, $0 < \lambda < 1$, $\beta \geq 3$ is a ratio of two positive odd integers and $\omega > 1$. It is easy to see that conditions (2.14) and (2.15) for (E_{x2}) reduce to

$$P(t) = \frac{b^{1/3}}{5^{1/3} t}, \qquad Q(t) = \frac{\lambda^{5/3} a^{1/3}}{5^{1/3} t}, \qquad (2.43)$$

respectively. It follows from Corollary 2.13 that (E_{x2}) has property (B) provided that

$$b^{1/3}\left[\omega^{\lambda^{5/3}a^{1/3}/2^{2/3}5^{1/3}}\right]\ln\omega \geq \frac{2^{2/3}5^{1/3}}{e}. \tag{2.44}$$

Moreover, (2.25) provides the following rate of divergence for every nonoscillatory solution of (E_{x2}):

$$|x(t)| \geq ct^{5/3}, \qquad c > 0. \tag{2.45}$$

Now, we eliminate Case (I) of Lemma 2.4, to get the oscillation of (E).

Theorem 2.17. *Let $x(t)$ be a nonoscillatory solution of (E). Assume that there exists a function $\xi(t) \in C^1([t_0,\infty))$ such that*

$$\xi'(t) \geq 0, \quad \xi(t) < t, \qquad \eta(t) = \sigma(\xi(\xi(t))) > t. \tag{2.46}$$

If the first-order advanced differential equation

$$z'(t) - \left\{\int_{\xi(t)}^{t} a^{-1/\gamma}(u)\left(\int_{\xi(u)}^{u} p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u\right\}h^{1/\gamma}(z[\eta(t)]) = 0 \tag{E_3}$$

is oscillatory, then Case (I) cannot hold.

Proof. Let $x(t)$ be an eventually positive solution of (E), satisfying Case (I). It follows from (E) that

$$\left[a(t)[x''(t)]^{\gamma}\right]' \geq p(t)h(x[\sigma(t)]). \tag{2.47}$$

Integrating from $\xi(t)$ to t, we have

$$\begin{aligned} a(t)[x''(t)]^{\gamma} - a(\xi(t))[x''(\xi(t))]^{\gamma} &\geq \int_{\xi(t)}^{t} p(s)h(x[\sigma(s)])\mathrm{d}s \\ &\geq h(x[\sigma(\xi(t))])\int_{\xi(t)}^{t} p(s)\mathrm{d}s. \end{aligned} \tag{2.48}$$

Therefore,

$$x''(t) \geq h^{1/\gamma}(x[\sigma(\xi(t))])a^{-1/\gamma}(t)\left(\int_{\xi(t)}^{t} p(s)\mathrm{d}s\right)^{1/\gamma}. \tag{2.49}$$

An integration from $\xi(t)$ to t yields

$$\begin{aligned} x'(t) &\geq \int_{\xi(t)}^{t} h^{1/\gamma}(x[\sigma(\xi(u))])a^{-1/\gamma}(u)\left(\int_{\xi(u)}^{u} p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u \\ &\geq h^{1/\gamma}(x[\eta(t)])\int_{\xi(t)}^{t} a^{-1/\gamma}(u)\left(\int_{\xi(u)}^{u} p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u. \end{aligned} \tag{2.50}$$

Consequently, $x(t)$ is a positive solution of the advanced differential inequality

$$x'(t) - \left\{\int_{\xi(t)}^{t} a^{-1/\gamma}(u)\left(\int_{\xi(u)}^{u} p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u\right\}h^{1/\gamma}(x[\eta(t)]) \geq 0. \tag{2.51}$$

Hence, by Lemma 2.3, we conclude that the corresponding differential equation (E_3) also has a positive solution, which contradicts the oscillation of (E_3). Therefore, $x(t)$ cannot satisfy Case (I). □

Combining Theorem 2.17 with Theorems 2.6 and 2.11, we get two criteria for the oscillation of (E).

Theorem 2.18. *Let* (1.3) *hold and* $0 < \gamma \leq 1$. *Assume that both of the first-order advanced equations* (E_1) *and* (E_3) *are oscillatory, then* (E) *is oscillatory.*

Proof. Assume that (E) has a nonoscillatory solution. It follows from Remark 2.5 that $x(t)$ satisfies either Case (I) or (II). But both cases are excluded by the oscillation of (E_1) and (E_3). □

Corollary 2.19. *Let* $0 < \gamma \leq 1$. *Assume that* (1.3), (2.28), (2.29), *and* (2.46) *hold. If*

$$\liminf_{t\to\infty}\int_{t}^{\eta(t)}\left\{\int_{\xi(v)}^{v} a^{-1/\gamma}(u)\left(\int_{\xi(u)}^{u} p(s)\mathrm{d}s\right)^{1/\gamma}\mathrm{d}u\right\}\mathrm{d}v > \frac{1}{e}, \tag{2.52}$$

then (E) *is oscillatory.*

Proof. Conditions (2.29) and (2.52) guarantee the oscillation of (E_1) and (E_3), respectively. The assertion now follows from Theorem 2.18. □

Example 2.20. We consider once more the third-order differential equation (E_{x1}) with the same restrictions as in Example 2.10. We set $\xi(t) = \alpha_0 t$, where $\alpha_0 = (1+\sqrt{\omega})/2\sqrt{\omega}$. Then condition (2.52) takes the form

$$b^3\frac{(1-\alpha_0)\left(1-\alpha_0^{1/3}\right)^3}{\alpha_0^2}\ln\left(\omega\alpha_0^2\right) > \frac{1}{27e}, \tag{2.53}$$

which by Corollary 2.19, implies the oscillation of (E_{x1}).

The following results are obvious.

Theorem 2.21. *Let* (1.3) *hold and* $\gamma \geq 1$. *Assume that both of the first-order advanced equations* (E_2) *and* (E_3) *are oscillatory, then* (E) *is oscillatory.*

Corollary 2.22. *Let* $\gamma \geq 1$. *Assume that* (1.3), (2.28), (2.41), (2.46), *and* (2.52) *hold. Then* (E) *is oscillatory.*

Example 2.23. We recall again the differential equation (E_{x2}) with the same assumptions as in Example 2.16. We set $\xi(t) = \alpha_0 t$ with $\alpha_0 = (1+\sqrt{\omega})/2\sqrt{\omega}$. Then condition (2.52) reduces to

$$b^{1/3}\frac{(1-\alpha_0)\left(1-\alpha_0^5\right)^{1/3}}{\alpha_0^{8/3}}\ln\left(\omega\alpha_0^2\right) > \frac{5^{1/3}}{e}, \tag{2.54}$$

which, by Corollary 2.22, guarantees the oscillation of (E_{x2}).

The following result is intended to exclude Case (III) of Lemma 2.4.

Theorem 2.24. *Let* $x(t)$ *be a nonoscillatory solution of* (E). *Assume that* (1.2) *holds. If the first-order delay differential equation*

$$z'(t) + \left(\int_{t_1}^{t} q(s)\mathrm{d}s\right)^{1/\gamma}\left(\int_{t}^{\infty} a^{-1/\gamma}(s)\mathrm{d}s\right) f^{1/\gamma}(z[\tau(t)]) = 0. \tag{E_4}$$

is oscillatory, then Case (III) cannot hold.

Proof. Let $x(t)$ be a positive solution of (E), satisfying Case (III) of Lemma 2.4. Using that $a(t)[x''(t)]^{\gamma}$ is increasing, we find that

$$\begin{aligned} -x'(t) &\geq \int_{t}^{\infty} x''(s)\mathrm{d}s = \int_{t}^{\infty}\left(a^{1/\gamma}(s)x''(s)\right)a^{-1/\gamma}(s)\mathrm{d}s \\ &\geq a(t)^{1/\gamma}x''(t)\int_{t}^{\infty} a^{-1/\gamma}(s)\mathrm{d}s. \end{aligned} \tag{2.55}$$

Integrating the inequality $[a(t)[x''(t)]^{\gamma}]' \geq q(t)f(x[\tau(t)])$ from t_1 to t, we have

$$a(t)\left[x''(t)\right]^{\gamma} \geq \int_{t_1}^{t} q(s)f(x[\tau(s)]\mathrm{d}s) \geq f(x[\tau(t)])\int_{t_1}^{t} q(s)\mathrm{d}s. \tag{2.56}$$

Thus,

$$a^{1/\gamma}(t)x''(t) \geq f^{1/\gamma}(x[\tau(t)])\left(\int_{t_1}^{t} q(s)\mathrm{d}s\right)^{1/\gamma}. \tag{2.57}$$

Combining (2.57) with (2.55), we find

$$0 \geq x'(t) + \left(\int_{t_1}^{t} q(s) \mathrm{d}s \right)^{1/\gamma} \left(\int_{t}^{\infty} a^{-1/\gamma}(s) \mathrm{d}s \right) f^{1/\gamma}(x[\tau(t)]). \tag{2.58}$$

It follows from [16, Theorem 1] that the corresponding differential equation (E_4) also has a positive solution. A contradiction. For that reason, $x(t)$ cannot satisfy Case (III). □

The following results are immediate.

Theorem 2.25. *Let* (1.2) *hold and* $0 < \gamma \leq 1$. *Assume that both of the first-order advanced equations* (E_1) *and* (E_4) *are oscillatory, then* (E) *has property (B).*

Theorem 2.26. *Let* (1.2) *hold and* $0 < \gamma \leq 1$. *Assume that all of the three first-order advanced equations* (E_1), (E_3), *and* (E_4) *are oscillatory, then* (E) *is oscillatory.*

Theorem 2.27. *Let* (1.2) *hold and* $\gamma \geq 1$. *Assume that both of the first-order advanced equations* (E_2) *and* (E_4) *are oscillatory, then* (E) *has property (B).*

Theorem 2.28. *Let* (1.2) *hold and* $\gamma \geq 1$. *Assume that all of the three first-order advanced equations* (E_2), (E_3), *and* (E_4) *are oscillatory, then* (E) *is oscillatory.*

3. Summary

In this paper, we have presented new comparison theorems for deducing the property (B)/oscillation of (E) from the oscillation of a set of the suitable first-order delay/advanced differential equation. We were able to present such criteria for studied properties that employ all coefficients and functions included in studied equations. Our method essentially simplifies the examination of the third-order equations, and, what is more, it supports backward the research on the first-order delay/advanced differential equations. Our results here extend and complement latest ones of Grace et al. [10], Agarwal et al. [1–3], Cecchi et al. [6], Parhi and Pardi [15], and the present authors [4, 8]. The suitable illustrative examples are also provided.

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