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Locally rotationally symmetric Bianchi type I cosmology in $f(R, T)$ gravity

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Abstract This manuscript is devoted to the investigation of the Bianchi type I universe in the context of $f(R, T)$ gravity. For this purpose, we explore the exact solutions of locally rotationally symmetric Bianchi type I spacetime. The modified field equations are solved by assuming an expansion scalar θ proportional to the shear scalar σ, which gives $A = B^n$, where A, B are the metric coefficients and n is an arbitrary constant. In particular, three solutions have been found and physical quantities are calculated in each case.

1 Introduction

Recent observations resulting in astrophysical data have unfolded an amazing picture of the expanding universe. The cosmic acceleration is well supported by high redshift supernovae, the cosmic microwave background anisotropy, and galaxy clustering [1–6]. The universe seems to be filled with an exotic cosmic fluid known as dark energy having a strong negative pressure. It constitutes almost 70 % of the total energy budget of our universe. We can describe dark energy with an equation of state (EoS) parameter $\omega = p/\rho$, where ρ and p represent the energy density and pressure of dark energy. It has been proved that the expansion of the universe is accelerating when $w \approx -1$ [7–9]. Phantom-like dark energy is found in the region where $\omega < -1$. The universe with phantom dark energy ends up with a finite time future singularity known as cosmic doomsday or the big rip [10–14]. Modified theories of gravity seem attractive to explain the phenomena of dark energy and late-time acceleration. It is now expected that the issues of cosmic acceleration and quintessence can be addressed using higher order theories of gravity [15].

Many generalizations of Einstein field equations have been proposed in the last few decades. The $f(T)$ theory of gravity is an example which has been recently developed.

This theory is a generalized version of teleparallel gravity in which the Weitzenböck connection is used instead of Levi-Civita connection. The theory seems interesting as it may explain the current cosmic acceleration without involving the dark energy. Some researchers have done a considerable amount of work in this theory so far [16–25]. Another extended theory, known as $f(R)$ theory of gravity, has also attracted attention of the researchers in recent years. The $f(R)$ theory is actually an extension of standard Einstein–Hilbert action involving a function of the Ricci scalar R. The cosmic acceleration may be justified by involving the term $1/R$, which is required at small curvatures. The $f(R)$ theory seems to be most appropriate due to important $f(R)$ models in cosmological contexts.

Some viable $f(R)$ gravity models [26] have been suggested which justify the unification of early-time inflation and late-time acceleration. The dark matter problems can also be addressed using viable $f(R)$ gravity models [27–30]. Starobinsky [31] gave a first viable complete inflationary $R + R^2$ model in agreement with the observational data. The cosmological model without additional singularities can be constructed using $f(R)$ gravity in which both inflation in the early universe and dark energy in the present universe is described [32]. Hendi and Momeni [33] investigated black hole solutions in a $f(R)$ theory of gravity with conformal anomaly. Jamil et al. explored a $f(R)$ tachyon model using Noether symmetries [34]. Capozziello et al. [35] have proved that dust matter and dark energy phases can be achieved by finding the exact solutions using a power law $f(R)$ cosmological model. A reasonable amount of work has been done so far in this theory [36–42]. Some review articles [43–46] may be useful to have a better understanding of the theory.

Recently, Harko et al. [47] proposed a new generalization known as $f(R, T)$ theory of gravity in which R is the scalar curvature and T denotes the trace of the energy-momentum tensor. Jamil et al. [48] reconstructed some cosmological models in $f(R, T)$ gravity where it was proved

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that the dust fluid reproduced ΛCDM. Sharif and Zubair [49] gave the reconstruction and stability conditions of $f(R, T)$ gravity with a Ricci and modified Ricci dark energy. The same authors [50] discussed the laws of thermodynamics in this theory. However, it has been established that the first law of black hole thermodynamics is violated for $f(R, T)$ gravity [51]. Santos [52] investigated a Gödel type universe in the context of $f(R, T)$ modified theories of gravity. Houndjo [53] reconstructed $f(R, T)$ gravity by taking $f(R, T) = f_1(R) + f_2(T)$ and it was shown that $f(R, T)$ gravity allowed for a transition of matter from a dominated phase to an acceleration phase. Harko and Lake [54] investigated cylindrically symmetric interior string like solutions in $f(R, L_m)$ theory of gravity. In a recent paper [55], we explored the exact solutions of a cylindrically symmetric spacetime in $f(R, T)$ gravity and recovered two solutions, which corresponded to an exterior metric of cosmic string and a non-null electromagnetic field.

Isotropization is an important issue as one discusses whether the universe can result in isotropic solutions without the need of fine tuning the model parameters [56]. The universe seems to have an isotropic and homogeneous geometry at the end of the inflationary era [57]. However, the class of anisotropic geometries has gained popularity in the light of the recently announced Planck Probe results [58]. Further, it is believed that the early universe may not have been exactly uniform. Therefore, inhomogeneous and anisotropic models of universe have an important role to play in theoretical cosmology. This prediction motivates us to describe the early stages of the universe with the models having an anisotropic background. Thus, the existence of anisotropy in the early phases of the universe is an interesting phenomenon to investigate. Bianchi type models are among the simplest models with anisotropic background. Many authors [59–64] investigated Bianchi type spacetimes in different contexts. Kumar and Singh [65] solved the field equations in the presence of a perfect fluid using a Bianchi type I spacetime in general relativity (GR). Moussiaux et al. [66] investigated the exact solution for a vacuum Bianchi type III model in the presence of the cosmological constant. A Bianchi type III string cosmology with bulk viscosity has been studied by Xing-Xiang [67]. He assumed an expansion scalar proportional to the shear scalar to find the solutions. Wang [68] explored string cosmological models in Kantowski–Sachs spacetime. Magnetized Bianchi type III massive string cosmological models in GR have been investigated by Upadhaya [69]. Hellaby [70] gave a review of some recent developments in inhomogeneous models and it was concluded that the universe is inhomogeneous on many scales.

The investigation of Bianchi type models in modified or alternative theories of gravity is another interesting topic of discussion. Perfect fluid solutions using a Bianchi type I spacetime in scalar tensor theory have been explored by

Kumar and Singh [71]. Singh and Agrawal [72] studied Bianchi type III cosmological models in scalar tensor theory. Adhav et al. [73] found an exact solution of the vacuum Brans–Dicke field equations for a spatially homogeneous and anisotropic model. FRW cosmologies in $f(R)$ gravity have been investigated by Paul et al. [74]. A Bianchi type I model in $f(R)$ gravity was studied where it was shown how to integrate anisotropic degrees of freedom explicitly and to reduce the problem to a single differential equation for the volume factor [75]. Vacuum and non-vacuum solutions of Bianchi types I and V spacetimes in metric $f(R)$ gravity have been explored [76,77]. Sharif and Kausar [78] investigated non-vacuum solutions of a Bianchi type VI universe by considering the isotropic and anisotropic fluids as the source of dark matter and energy. In a recent paper, we have explored a Bianchi type I cosmology in $f(R, T)$ gravity with some interesting results [79]. It was concluded that the EoS parameter $w \to -1$ as $t \to \infty$, which suggested an accelerated expansion of the universe. Thus it is hoped that $f(R, T)$ gravity may explain the resent phase of cosmic acceleration of our universe. This theory can be used to explore many issues and may provide some satisfactory results.

In this paper, we explore the exact solutions of locally rotationally symmetric (LRS) Bianchi type I spacetime in $f(R, T)$ gravity. The field equations are solved by assuming expansion scalar θ proportional to the shear scalar σ, which gives $A = B^n$, where A, B are the metric coefficients and n is an arbitrary constant. The plan is as follows: the field equations in $f(R, T)$ gravity are briefly introduced in Sect. 2. In Sect. 3, the solutions of the field equations are investigated along with some important physical parameters. The last section is used to summarize and conclude the results.

2 $f(R, T)$ Gravity formalism

The action for $f(R, T)$ theory of gravity is given by [47]

$$S = \int \sqrt{-g} \left(\frac{1}{2\kappa} f(R, T) + L_m \right) \mathrm{d}^4 x, \tag{1}$$

where g is the determinant of the metric tensor $g_{\mu\nu}$ and L_m is the usual matter Lagrangian. It would be worthwhile to mention that if we replace $f(R, T)$ with $f(R)$, we get the action for $f(R)$ gravity and the replacement of $f(R, T)$ with R leads to the action of GR. The $f(R, T)$ gravity field equations are obtained by varying the action S in Eq. (1) with respect to the metric tensor $g_{\mu\nu}$,

$$f_R(R, T)R_{\mu\nu} - \frac{1}{2} f(R, T)g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)f_R(R, T)$$
$$= \kappa T_{\mu\nu} - f_T(R, T)(T_{\mu\nu} + \Theta_{\mu\nu}), \tag{2}$$

where ∇_μ denotes the covariant derivative and

$$\Box \equiv \nabla^\mu \nabla_\mu, \quad f_R(R, T) = \frac{\partial f_R(R, T)}{\partial R},$$

$$f_T(R, T) = \frac{\partial f_R(R, T)}{\partial T}, \quad \Theta_{\mu\nu} = g^{\alpha\beta} \frac{\delta T_{\alpha\beta}}{\delta g^{\mu\nu}}.$$

Contraction of Eq. (2) yields

$$f_R(R, T)R + 3\Box f_R(R, T) - 2f(R, T)$$
$$= \kappa T - f_T(R, T)(T + \Theta), \tag{3}$$

where $\Theta = \Theta_\mu{}^\mu$. This is an important equation because it provides a relationship between the Ricci scalar R and the trace T of the energy-momentum tensor. Using the matter Lagrangian L_m, the standard matter energy-momentum tensor is derived as

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu}, \tag{4}$$

satisfying the EoS

$$p = w\rho, \tag{5}$$

where $u_\mu = \sqrt{g_{00}}(1, 0, 0, 0)$ is the four-velocity in co-moving coordinates and ρ and p denote the energy density and the pressure of the fluid, respectively. Perfect fluid problems involving energy density and pressure are not easy to deal with. Moreover, there does not exist a unique definition for the matter Lagrangian. Thus we can assume the matter Lagrangian to be $L_m = -p$, which gives

$$\Theta_{\mu\nu} = -pg_{\mu\nu} - 2T_{\mu\nu}, \tag{6}$$

and consequently the field equations (2) take the form

$$f_R(R, T)R_{\mu\nu} - \frac{1}{2}f(R, T)g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)f_R(R, T)$$
$$= \kappa T_{\mu\nu} + f_T(R, T)(T_{\mu\nu} + pg_{\mu\nu}). \tag{7}$$

It is mentioned here that these field equations depend on the physical nature of the matter field. Many theoretical models corresponding to different matter contributions for $f(R, T)$ gravity are possible. However, Harko et al. [47] gave three classes of these models,

$$f(R, T) = \begin{cases} R + 2f(T), \\ f_1(R) + f_2(T), \\ f_1(R) + f_2(R)f_3(T). \end{cases}$$

In this paper, we consider the first and second class only to explore the exact LRS Bianchi I solutions.

3 Exact LRS Bianchi type I solutions

Here we first develop some important cosmological parameters and field equations for a LRS Bianchi type I spacetime and then find the exact solutions of field equations for constant and non-constant curvature case.

3.1 LRS Bianchi type I spacetime

The line element of LRS Bianchi type I spacetime is given by

$$ds^2 = dt^2 - A^2(t)dx^2 - B^2(t)[dy^2 + dz^2], \tag{8}$$

where A and B are cosmic scale factors. The corresponding Ricci scalar turns out to be

$$R = -2\left[\frac{\ddot{A}}{A} + \frac{2\ddot{B}}{B} + \frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2}\right], \tag{9}$$

where dot denotes derivative with respect to t. The average scale factor a and the volume scale factor V are defined as

$$a = \sqrt[3]{AB^2}, \quad V = a^3 = AB^2. \tag{10}$$

The average Hubble parameter H is given in the form

$$H = \frac{1}{3}\left(\frac{\dot{A}}{A} + \frac{2\dot{B}}{B}\right). \tag{11}$$

The expansion scalar θ and the shear scalar σ are defined as follows:

$$\theta = u^\mu_{;\mu} = \frac{\dot{A}}{A} + 2\frac{\dot{B}}{B}, \tag{12}$$

$$\sigma^2 = \frac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} = \frac{1}{3}\left[\frac{\dot{A}}{A} - \frac{\dot{B}}{B}\right]^2, \tag{13}$$

where

$$\sigma_{\mu\nu} = \frac{1}{2}(u_{\mu;\alpha}h^\alpha_\nu + u_{\nu;\alpha}h^\alpha_\mu) - \frac{1}{3}\theta h_{\mu\nu}, \tag{14}$$

$h_{\mu\nu} = g_{\mu\nu} - u_\mu u_\nu$ is the projection tensor. Now we explore the solutions of the field equations for two classes of $f(R, T)$ models.

3.2 $f(R, T) = R + 2f(T)$

For the model $f(R, T) = R + 2f(T)$, the field equations become

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \kappa T_{\mu\nu} + 2f_T(T)T_{\mu\nu}$$
$$+ [f(T) + 2pf_T(T)]g_{\mu\nu}. \tag{15}$$

Here we find the most basic possible solution of this theory in spite of the complicated nature of the field equations. For the sake of simplicity, we use a natural system of units ($G = c = 1$) and $f(T) = \lambda T$, where λ is an arbitrary constant. In

this case, the gravitational field equations take a form similar to GR,

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} - \lambda(T + 2p)g_{\mu\nu} = (8\pi + 2\lambda)T_{\mu\nu}, \quad (16)$$

where the term $\lambda(T + 2p)$ may play the role of the cosmological constant Λ of the GR field equations. It would be worthwhile to mention here that the dependence of the cosmological constant Λ on the trace of energy momentum tensor T has already been proposed by Poplawski [80] and the cosmological constant in the gravitational Lagrangian is a function of T. Consequently the model was named "$\Lambda(T)$ gravity". It has been proved that recent astrophysical data favor a variable cosmological constant, which is consistent with $\Lambda(T)$ gravity. $\Lambda(T)$ gravity has been shown to be more general than Palatini $f(R)$ gravity [81]. Now using Eq. (16), we obtain a set of differential equations for the LRS Bianchi type I spacetime,

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} = (8\pi + 3\lambda)\rho - \lambda p, \quad (17)$$

$$-\frac{2\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} = (8\pi + 3\lambda)p - \lambda\rho, \quad (18)$$

$$-\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} = (8\pi + 3\lambda)p - \lambda\rho. \quad (19)$$

Thus we have three differential equations with four unknowns, namely A, B, p and ρ. Adding Eqs. (17) and (18) gives

$$\frac{2\dot{A}\dot{B}}{AB} - \frac{2\ddot{B}}{B} = (8\pi + 2\lambda)(\rho + p). \quad (20)$$

Similarly, addition of Eqs. (17) and (19) yields

$$\frac{\dot{A}\dot{B}}{AB} - \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} = (8\pi + 2\lambda)(\rho + p). \quad (21)$$

Subtracting Eqs. (20) and (21), it follows that

$$\frac{\dot{A}\dot{B}}{AB} + \frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} = 0. \quad (22)$$

Now we are left with only one differential equation and two unknowns. Therefore, we need an additional constraint to address Eq. (22). Here we use the physical condition that the expansion scalar θ is proportional to the shear scalar σ, which leads to

$$A = B^n, \quad (23)$$

where n is an arbitrary real number and we consider $n \neq 0, 1$ for non-trivial solutions. The physical reason for this assumption is justified as the observations of the velocity redshift relation for extragalactic sources suggest that the Hubble expansion of the universe may achieve isotropy when $\frac{\sigma}{\theta}$ is constant [82]. Collins [83] showed the physical significance of this condition for a perfect fluid with a barotropic EoS. In

the literature [67,84–88], many authors have proposed this condition to find the exact solutions of the field equations.

Thus, using Eq. (23), Eq. (22) takes the form

$$\frac{\ddot{B}}{B} + (n + 1)\frac{\dot{B}^2}{B^2} = 0, \quad (24)$$

which yields a solution

$$B = c_1[(n + 2)t + c_2]^{\frac{1}{n+2}}, \quad (25)$$

where c_1 and c_2 are constants of integration. Thus, the solution metric takes the form

$$ds^2 = dt^2 - c_1^{2n}[(n + 2)t + c_2]^{\frac{2n}{n+2}}dx^2$$
$$- c_1^2[(n + 2)t + c_2]^{\frac{2}{n+2}}[dy^2 + dz^2]. \quad (26)$$

The volume scale factor turns out to be

$$V = a^3 = c_1^{n+2}[(n + 2)t + c_2]. \quad (27)$$

The expansion scalar and the shear scalar become

$$\theta = \frac{n+2}{(n+2)t + c_2}, \quad \sigma^2 = \frac{1}{3}\left[\frac{n-1}{(n+2)t + c_2}\right]^2. \quad (28)$$

It is mentioned here that the isotropy condition, i.e., $\frac{\sigma^2}{\theta} \to 0$ as $t \to \infty$, is satisfied in this case. It can be observed from Eqs. (27) and (28) that the spatial volume is zero at $t = 0$, while the expansion scalar is infinite, which suggests that the universe starts evolving with zero volume at $t = 0$, i.e. we have the big bang scenario. It is further observed that the average scale factor is zero at the initial epoch $t = 0$ and hence the model has a point type singularity [89]. The energy density and pressure of the universe take the form

$$\rho = p = \frac{(1 + 2n)}{2(\lambda + 4\pi)[(n + 2)t + c_2]^2}, \quad (29)$$

suggesting the EoS parameter $\omega = 1$ corresponding to a stiff fluid universe. The average Hubble parameter turns out to be

$$H = \frac{n+2}{3[(n+2)t + c_2]}. \quad (30)$$

Therefore

$$\frac{H}{H_0} = \frac{(n+2)t_0 + c_2}{(n+2)t + c_2}, \quad (31)$$

where H_0 is the present value of the Hubble parameter. The redshift for a distant source is directly related to the scale factor of the universe at the time when the photons were emitted from the source. The scale factor a and the redshift z are related through the equation

$$a = \frac{a_0}{1 + z}, \quad (32)$$

where a_0 is the present value of the scale factor. Thus we obtain

$$\frac{a_0}{a} = 1 + z = \left[\frac{(n+2)t_0 + c_2}{(n+2)t + c_2} \right]^{\frac{1}{3}}. \tag{33}$$

Using Eqs. (31) and (33), we obtain the value of the Hubble parameter in terms of the redshift parameter

$$H = H_0(1 + z)^3. \tag{34}$$

According to the Hubble law, the distance of a given galaxy is proportional to the recessional velocity as measured by the Doppler red shift. Thus, the value of the Hubble parameter in terms of the redshift has much importance in an astrophysical context.

The deceleration parameter q in cosmology is a measure of the cosmic acceleration of the universe's expansion and is defined as

$$q = -\frac{\ddot{a}a}{\dot{a}^2}. \tag{35}$$

It is mentioned here that the behavior of the models of the universe depend upon the sign of q. A positive deceleration parameter provides a decelerating model, while a negative value corresponds to inflation. For this solution, the value of the deceleration parameter turns out to be $q = 2$, which suggests a decelerating model of the universe.

Models of the universe close to ΛCDM can be described using the cosmic jerk parameter j, a dimensionless third derivative of the scale factor with respect to the cosmic time [90]. The value of the jerk parameter is constant for a flat ΛCDM model. The jerk parameter is defined as

$$j = \frac{1}{H^3} \frac{\dddot{a}}{a}. \tag{36}$$

The expression for the jerk parameter in terms of the deceleration parameter turns out to be

$$j = q + 2q^2 - \frac{\dot{q}}{H}. \tag{37}$$

Thus we obtain $j = 10$ in the case of our solution. It would be worthwhile to mention here that this solution gives $R = 0$ for $n = -\frac{1}{2}$. In this case, the solution metric takes the form

$$ds^2 = dt^2 - \left[c_1 \left(\frac{3}{2}t + c_2 \right) \right]^{-\frac{2}{3}} dx^2$$
$$- \left[c_1 \left(\frac{3}{2}t + c_2 \right) \right]^{\frac{4}{3}} [dy^2 + dz^2]. \tag{38}$$

Without loss of generality, we take $c_2 = 0$ and re-define the parameters, i.e., $\sqrt[3]{\frac{2x^3}{3c_1^{3/2}}} \longrightarrow \tilde{x}$, $\sqrt[3]{\frac{9c_1^3 y^3}{4}} \longrightarrow \tilde{y}$, and $\sqrt[3]{\frac{9c_1^3 z^3}{4}} \longrightarrow \tilde{z}$; the above metric takes the form

$$ds^2 = dt^2 - t^{-\frac{2}{3}} d\tilde{x}^2 - t^{\frac{4}{3}} (d\tilde{y}^2 + d\tilde{z}^2), \tag{39}$$

which is exactly the same as the well-known Kasner metric [91].

Now we discuss the possibility of solutions for a nonlinear form of $f(T)$. We assume $f(T) = \lambda T^2$ so that Eq. (15) takes the form

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = (8\pi + 4\lambda T) T_{\mu\nu} + \lambda T (T + 4p) g_{\mu\nu}. \tag{40}$$

Now using Eq. (40), we obtain a set of independent differential equations for the LRS Bianchi type I spacetime,

$$\frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} = 8\pi\rho + 5\lambda\rho^2 - 14\lambda\rho p - 3\lambda p^2, \tag{41}$$

$$-\frac{2\ddot{B}}{B} - \frac{\dot{B}^2}{B^2} = 8\pi p - 9\lambda p^2 + 6\lambda p\rho - \lambda\rho^2, \tag{42}$$

$$-\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} - \frac{\dot{A}\dot{B}}{AB} = 8\pi p - 9\lambda p^2 + 6\lambda p\rho - \lambda\rho^2. \tag{43}$$

These equations yield the same solution metric as given by Eq. (38). However, in this case the energy density can be obtained by solving the equation

$$\rho^2 + \frac{3 - \omega}{2\lambda(1 - 3\omega)}\rho - \frac{1 + 2n}{8\lambda(1 - 3\omega)[(n+2)t + c_2]^2} = 0. \tag{44}$$

The quadratic equation is due to the nonlinear form of $f(T)$.

3.3 $f(R, T) = f_1(R) + f_2(T)$

Now we explore the solutions with a more general class. Here the field equations for the model $f(R, T) = f_1(R) + f_2(T)$ become

$$f_{1R}(R) R_{\mu\nu} - \frac{1}{2} f_1(R) g_{\mu\nu} - (\nabla_\mu \nabla_\nu - g_{\mu\nu} \Box) f_{1R}(R)$$
$$= [\kappa + f_{2T}(T)] T_{\mu\nu} + \left[p f_{2T}(T) + \frac{1}{2} f_2(T) \right] g_{\mu\nu}. \tag{45}$$

Contracting the field equations (45), we obtain

$$R f_{1R}(R) - 2 f_1(R) + 3\Box f_{1R}(R) = \kappa T + 2 f_2(T)$$
$$+ [T + 4p] f_{2T}(T). \tag{46}$$

Using this, we can write

$$f_1(R) = \frac{3\Box f_{1R}(R) + R f_{1R}(R) - \kappa T - 2 f_2(T) - [T + 4p] f_{2T}(T)}{2}. \tag{47}$$

Inserting this in Eq. (45), we get

$$\frac{f_{1R}(R) R_{\mu\nu} - \nabla_\mu \nabla_\nu f_{1R}(R) - (\kappa + f_{2T}(T)) T_{\mu\nu}}{g_{\mu\nu}}$$
$$= \frac{R f_{1R}(R) - \Box f_{1R}(R) - \kappa T - T f_{2T}(T)}{4}. \tag{48}$$

Since the metric (8) depends only on t, one can view Eq. (48) as the set of differential equations for $f_{1R}(t)$, $f_{2T}(t)$, A, B, ρ and p. It follows from Eq. (48) that the combination

$$A_\mu \equiv \frac{f_{1R}(R)R_{\mu\mu} - \nabla_\mu \nabla_\mu f_{1R}(R) - (\kappa + f_{2T}(T))T_{\mu\nu}}{g_{\mu\mu}} \tag{49}$$

is independent of the index μ and hence $A_\mu - A_\nu = 0$ for all μ and ν. Thus $A_0 - A_1 = 0$ yields

$$-\frac{2\ddot{B}}{B} + \frac{2\dot{A}\dot{B}}{AB} + \frac{\dot{A}\dot{f}_{1R}(R)}{Af_{1R}(R)} - \frac{\ddot{f}_{1R}(R)}{f_{1R}(R)}$$
$$-\left[\frac{\kappa + f_{2T}(T)}{f_{1R}(R)}\right](\rho + p) = 0. \tag{50}$$

Also, $A_0 - A_2 = 0$ leads to

$$-\frac{\ddot{A}}{A} - \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} + \frac{\dot{B}^2}{B^2} + \frac{\dot{B}\dot{f}_{1R}(R)}{Bf_{1R}(R)} - \frac{\ddot{f}_{1R}(R)}{f_{1R}(R)}$$
$$-\left[\frac{\kappa + f_{2T}(T)}{f_{1R}(R)}\right](\rho + p). \tag{51}$$

Now we have two differential equations with six unknowns, namely A, B, $f_1(R)$, $f_2(T)$, p, and ρ. Here we also use $A = B^n$ so that Eqs. (50) and (51) take the form

$$-\frac{2\ddot{B}}{B} + 2n\frac{\dot{B}^2}{B^2} + n\frac{\dot{B}\dot{f}_{1R}(R)}{Bf_{1R}(R)} - \frac{\ddot{f}_{1R}(R)}{f_{1R}(R)}$$
$$-\left[\frac{\kappa + f_{2T}(T)}{f_{1R}(R)}\right](\rho + p) = 0, \tag{52}$$

$$(n+1)\frac{\ddot{B}}{B} + (n^2 - 2n - 1)\frac{\dot{B}^2}{B^2} - \frac{\dot{B}\dot{f}_{1R}(R)}{Bf_{1R}(R)} + \frac{\ddot{f}_{1R}(R)}{f_{1R}(R)}$$
$$+\left[\frac{\kappa + f_{2T}(T)}{f_{1R}(R)}\right](\rho + p) = 0. \tag{53}$$

Case I: Exponential law solutions It has been proved that the dark matter and dark energy phases can be achieved by finding the exact solutions using a power law $f(R)$ model [35]. So it would be interesting to assume $f_1(R)$ to be in power law form to solve the field equations. We follow the approach of Nojiri and Odintsov [92] and take the assumption $f_{1R}(R) \propto f_0 R^m$, where f_0 is an arbitrary constant. So in this case the addition of Eqs. (52) and (53) yields

$$(n^3 + 2n^2 + 2n + 2)\dot{B}^4 + (2n^2 + 4n + 3)B\dot{B}^2\ddot{B}$$
$$+ (n+2)B^2\ddot{B}^2 - 2m(n^2 + n + 1)\dot{B}^4$$
$$+ m(2n^2 + n)B\dot{B}^2\ddot{B} + m(n+2)B^2\ddot{B}\dddot{B} = 0. \tag{54}$$

It is interesting to notice that this equation admits an exponential solution of the form

$$B(t) = e^{c_3 t + c_4}, \tag{55}$$

where c_3 and c_4 are arbitrary constants. The exponential solution is satisfied with the constraint equation

$$n^3 + 4n^2 + 7n + 6 = 0. \tag{56}$$

The solutions of this equation turn out to be

$$n = -2, \quad -1 \pm i\sqrt{2}. \tag{57}$$

It is mentioned here that the real value of n gives a constant Ricci scalar, while we obtain a non-constant Ricci scalar for the complex values of n. We discard the imaginary case and consider the real value of n to get a physical solution

$$ds^2 = dt^2 - e^{-4(c_3 t + c_4)}dx^2 - e^{2(c_3 t + c_4)}(dy^2 + dz^2). \tag{58}$$

The average Hubble parameter turns out to be zero here. All other dynamical quantities like the volume scale factor of universe, the expansion scalar θ, and the shear scalar σ are constant for this solution. The energy density and the pressure of the universe are related by the equation

$$\rho + p = \frac{(-6)^{m+1}f_0 c_3^2}{\kappa + f_{2T}(T)}. \tag{59}$$

Many expressions for pressure and energy density can be evaluated for different choices of $f_2(T)$. For example when $f_2(T) = \lambda T^2$ and using Eq. (5), we obtain

$$\rho^2 + \frac{\kappa}{2\lambda(1 - 3\omega)}\rho - \frac{(-6)^{m+1}f_0 c_3^2}{2\lambda(1 + \omega)(1 - 3\omega)} = 0, \tag{60}$$

which is quadratic in ρ and one can work out its roots to get the energy density.

Case II: Power law solutions Here we assume that the solution is in power law form, i.e. $B(t) = (c_5 t + c_6)^k$, where c_5, c_6, and k are arbitrary real constants with $k \neq 0$. Using Eq. (54), we obtain a constraint equation,

$$(n^3 + 4n^2 + 7n + 6)k^2 - [2m(n^2 + 2n + 3) + 2n^2 + 6n$$
$$+ 7]k + (1 + 2m)(n + 2) = 0. \tag{61}$$

This equation is important because it will be used to reconstruct different forms of $f_1(R)$ models with suitable solutions of the field equations. For example, here we investigate the solution for $n = -1$. In this case, Eq. (61) gives

$$k = 2m + 1, \quad m \neq 0, 1. \tag{62}$$

So the solution metric takes the form

$$ds^2 = dt^2 - (c_5 t + c_6)^{-2(2m+1)}dx^2$$
$$- (c_5 t + c_6)^{2(2m+1)}(dy^2 + dz^2). \tag{63}$$

The volume scale factor and average Hubble parameter become

$$V = a^3 = (c_5 t + c_6)^{2(2m+1)}, \quad H = \frac{c_5(2m+1)}{3(c_5 t + c_6)}. \tag{64}$$

The expansion scalar and the shear scalar turn out to be

$$\theta = \frac{c_5(2m+1)}{c_5 t + c_6}, \quad \sigma^2 = \frac{4}{3} \left[\frac{c_5(2m+1)}{c_5 t + c_6} \right]^2. \qquad (65)$$

The isotropy condition $\frac{\sigma^2}{\theta} \to 0$ as $t \to \infty$ is also satisfied in this case. Using Eq. (64), we get

$$\frac{H}{H_0} = \frac{c_5 t_0 + c_6}{c_5 t + c_6}, \quad \frac{a_0}{a} = 1 + z = \left[\frac{c_5 t_0 + c_6}{c_5 t + c_6} \right]^{\frac{2(2m+1)}{3}}. \qquad (66)$$

Thus the value of the Hubble parameter in terms of the redshift parameter turns out to be

$$H = H_0 (1+z)^{\frac{3}{2(2m+1)}}. \qquad (67)$$

The deceleration parameter in this case becomes

$$q = \frac{1 - 4m}{2(2m+1)}, \qquad (68)$$

while the jerk parameter is given by

$$j = \frac{8(4m^2 - 5m + 1)}{(2m+1)^2}. \qquad (69)$$

By observing Eqs. (67)–(69), it is clear that singularity occurs at $m = -\frac{1}{2}$. Further, the Ricci scalar turns out to be

$$R = -\frac{2 c_5^2 (1 + 2m)(1 + 4m)}{(c_5 t + c_6)^2}. \qquad (70)$$

For $m = -\frac{1}{2}$ or $m = -\frac{1}{4}$, the Ricci scalar turns out to be constant, i.e. $R = 0$. $m = -\frac{1}{2}$ corresponds to Minkwoski spacetime, while for $m = -\frac{1}{4}$ the metric takes the form

$$ds^2 = dt^2 - (c_5 t + c_6)^{-1} dx^2 - (c_5 t + c_6)(dy^2 + dz^2). \qquad (71)$$

This solution gives a point singularity at $t = -\frac{c_6}{c_5}$. The Ricci scalar remains non-constant for $m \neq -\frac{1}{2}, -\frac{1}{4}$. For the special case when $m = -1$, $f_1(R)$ takes the logarithmic form,

$$f_1(R) \propto f_0 \ln |R| + c_7, \qquad (72)$$

where c_7 is an integration constant. Here the energy density and the pressure of the universe are related by the equation

$$\rho + p = -\frac{f_0}{\kappa + f_{2T}(T)}. \qquad (73)$$

Here we can also calculate many expressions for the energy density depending upon the value of $f_2(T)$.

4 Concluding remarks

This paper is devoted to the study of a Bianchi type cosmology in $f(R,T)$ gravity. We explore the exact solutions of the field equations for a LRS Bianchi type I spacetime. Since the field equations are highly nonlinear and complicated, we use the assumption that the expansion scalar θ is proportional to the shear scalar σ to solve them. It gives $A = B^n$, where A, B are the metric coefficients and n is an arbitrary constant. Mainly we have explored three solutions of modified field equations using different assumptions.

The first solution is obtained for the model $f(R,T) = R + 2f(T)$. The isotropy condition, i.e. $\frac{\sigma^2}{\theta} \to 0$ as $t \to \infty$, is satisfied for the solution. The spatial volume is zero at $t = 0$ and the expansion scalar is infinite, which suggests that the universe starts evolving with zero volume at $t = 0$, i.e. we have the big bang scenario. The average scale factor turns out to be zero at the initial epoch $t = 0$ and hence the model has a point type singularity [89]. The expressions for energy density and pressure suggest that the EoS parameter $\omega = 1$, which corresponds to a stiff fluid universe. The deceleration parameter q turns out to be $q = 2$, which suggests a decelerating model of the universe. We have also calculated the jerk parameter $j = 10$ in the case of our solution. It is worth mentioning here that this solution gives $R = 0$ for $n = -\frac{1}{2}$, corresponding to the well-known Kasner solution already available in GR [91].

We have also explored the more general solutions of the field equation by considering the model $f(R,T) = f_1(R) + f_2(T)$. Moreover, we have not used any conventional assumption like a constant deceleration parameter or a variation law of the Hubble parameter to investigate the solutions in this case. In particular, the exponential law and power law solutions have been investigated for this model. For the exponential law case, the average Hubble parameter turns out to be zero, and all other dynamical quantities like the volume scale factor of universe, the expansion scalar θ, and the shear scalar σ are constant. It is worthwhile to mention here that when $f_2(T) = 0$, this class corresponds to the $f(R)$ gravity model. For $f_2(T) \neq 0$, different expressions for the energy density can be generated with different choices of the $f_2(T)$ models. A power law solution provides a non-constant scalar curvature and thus many important $f(R)$ models can be reconstructed. As a special case, we have developed an important logarithmic $f(R)$ model.

References

1. D.N. Spergel et al., Astrophys. J. Suppl. **148**, 175 (2003)
2. D.N. Spergel et al., Astrophys. J. Suppl. **170**, 377 (2007)
3. A.G. Riess et al., (Supernova Search Team), Astron. J. **116**, 1009 (1998)
4. C.L. Bennett et al., Astrophys. J. Suppl. **148**, 1 (2003)
5. A.G. Riess et al., Astrophys. J. **607**, 665 (2004)
6. M. Tegmartk et al., Phys. Rev. D **69**, 103501 (2004)
7. J. Hogan, Nature **448**, 240 (2007)
8. P.S. Corasaniti et al., Phys. Rev. D **70**, 083006 (2004)
9. J. Weller, A.M. Lewis, Mon. Not. Astron. Soc. **346**, 987 (2003)
10. A.A. Starobinsky, Gravit. Cosmol. **6**, 157 (2000)
11. R.R. Caldwell, Phys. Lett. B **545**, 23 (2002)
12. R.R. Caldwell, M. Kamionkowski, N.N. Weinberg, Phys. Rev. Lett. **91**, 071301 (2003)
13. S. Nojiri, S.D. Odintsov, Phys. Rev. D **70**, 103522 (2004)
14. I. Brevik, V.V. Obukhov, A.V. Timoshkin, Y. Rabochaya, Astrophys. Space Sci. **346**, 267 (2013)
15. S. Capozziello, Int. J. Mod. Phys. D **11**, 483 (2002)
16. K. Bamba, C.Q. Geng, C.C. Lee, L.W. Luo, JCAP **1101**, 021 (2011)
17. B. Li, T.P. Sotiriou, J.D. Barrow, Phys. Rev. D **83**, 104017 (2011)
18. M. Jamil, D. Momeni, R. Myrzakulov, Eur. Phys. J. C **72**, 1959 (2012)
19. M. Jamil, D. Momeni, R. Myrzakulov, Eur. Phys. J. C **72**, 2075 (2012)
20. M. Jamil, D. Momeni, R. Myrzakulov, Eur. Phys. J. C **72**, 2122 (2012)
21. M. Jamil, D. Momeni, R. Myrzakulov, Eur. Phys. J. C **72**, 2137 (2012)
22. M. Jamil, D. Momeni, R. Myrzakulov, Eur. Phys. J. C **73**, 2267 (2013)
23. M. Jamil, D. Momeni, R. Myrzakulov, P. Rudra, J. Phys. Soc. Jpn. **81**, 114004 (2012)
24. M. Jamil, K. Yesmakhanova, D. Momeni, R. Myrzakulov, Cent. Eur. J. Phys. **10**, 1065 (2012)
25. M. Jamil, D. Momeni, R. Myrzakulov, Gen. Relativ. Gravit. **45**, 263 (2013)
26. S. Nojiri, S.D. Odintsov, Problems of modern theoretical physics, a volume in honour of prof. Buchbinder, I.L. in the occasion of his 60th birthday (TSPU Publishing, Tomsk), pp. 266–285. arXiv:0807.0685
27. W. Hu, I. Sawicki, Phys. Rev. D **76**, 064004 (2007)
28. S.A. Appleby, R.A. Battye, Phys. Lett. B **654**, 7 (2007)
29. A.A. Starobinsky, JETP Lett. **86**, 156 (2007)
30. S. Nojiri, S.D. Odintsov, Int. J. Geom. Methods Mod. Phys. **115**, 4 (2007)
31. A.A. Starobinsky, Phys. Lett. B **91**, 99 (1980)
32. S.A. Appleby, R.A. Battye, A.A. Starobinsky, JCAP **1006**, 005 (2010)
33. S.H. Hendi, D. Momeni, Eur. Phys. J. C **71**, 1823 (2011)
34. M. Jamil, F.M. Mahomed, D. Momeni, Phys. Lett. B **702**, 315 (2011)
35. S. Capozziello, P. Martin-Moruno, C. Rubano, Phys. Lett. B **664**, 12 (2008)
36. K. Bamba, S. Nojiri, S.D. Odintsov, D. Saez-Gomez, Phys. Lett. B **730**, 136 (2014)
37. K. Bamba, A.N. Makarenko, A.N. Myagky, S. Nojiri, S.D. Odintsov, JCAP **01**, 008 (2014)
38. K. Bamba, S. Nojiri, S.D. Odintsov, Phys. Lett. B **698**, 451 (2011)
39. S. Capozziello, S. Vignolo, Int. J. Geom. Methods Mod. Phys. **8**, 167 (2011)
40. S. Capozziello, F. Darabi, D. Vernieri, Mod. Phys. Lett. A **26**, 65 (2011)
41. T. Multamäki, I. Vilja, Phys. Rev. D **74**, 064022 (2006)

42. A. Azadi, D. Momeni, M. Nouri-Zonoz, Phys. Lett. B **670**, 210 (2008)
43. K. Bamba, S. Capozziello, S. Nojiri, S.D. Odintsov, Astrophys. Space Sci. **342**, 155 (2012)
44. A.D. Felice, S. Tsujikawa, Living Rev. Relativ. **13**, 3 (2010)
45. T.P. Sotiriou, V. Faraoni, Rev. Mod. Phys. **82**, 451 (2010)
46. T. Clifton, P.G. Ferreira, A. Padilla, C. Skordis, Phys. Rep. **513**, 1 (2012)
47. T. Harko, F.S.N. Lobo, S. Nojiri, S.D. Odintsov, Phys. Rev. D **84**, 024020 (2011)
48. M. Jamil, D. Momeni, M. Raza, R. Myrzakulov, Eur. Phys. J. C **72**, 1999 (2012)
49. M. Sharif, M. Zubair, Astrophys. Space Sci. **349**, 529 (2014)
50. M. Sharif, M. Zubair, JCAP **03**, 028 (2012)
51. M. Jamil, D. Momeni, R. Myrzakulov, Chin. Phys. Lett. **29**, 109801 (2012)
52. A.F. Santos, Mod. Phys. Lett. A **28**, 1350141 (2013)
53. M.J.S. Houndjo, Int. J. Mod. Phys. D **21**, 1250003 (2012)
54. T. Harko, M.J. Lake, Eur. Phys. J. C **75**:60 (2015)
55. M.F. Shamir, Z. Raza, Astrophys. Space Sci. **356**, 111 (2015)
56. C.R. Fadragas, G. Leon, E.N. Saridakis, Class. Quantum Gravity **31**, 075018 (2014)
57. A.D. Linde, Lect. Notes Phys. **738**, 1 (2008)
58. P.A.R. Ade et al., Astron. Astrophys. **571**, A16 (2014)
59. D. Lorenz-Petzold, Astrophys. Space Sci. **85**, 59 (1982)
60. J.L. Hanquin, J. Demaret, Class. Quantum Gravity **1**, 291 (1984)
61. I. Yavuz, I. Yilmaz, Astrophys. Space Sci. **245**, 131 (1996)
62. S. Chakraborty, N.C. Chakraborty, U. Debnath, Int. J. Mod. Phys. D **12**, 325 (2003)
63. T. Christodoulakis, P.A. Terzis, Class. Quantum Gravity **24**, 875 (2007)
64. A. Bagora, Astrophys. Space Sci. **319**, 155 (2009)
65. S. Kumar, C.P. Singh, Astrophys. Space Sci. **312**, 57 (2007)
66. A. Moussiaux, P. Tombal, J. Demaret, J. Phys. A Math. Gen. **14**, 277 (1981)
67. W. Xing-Xiang, Chin. Phys. Lett. **22**, 29 (2005)
68. X. Wang, Astrophys. Space Sci. **298**, 433 (2005)
69. R.D. Upadhaya, S. Dave, Braz. J. Phys. **38**, 4 (2008)
70. C. Hellaby, Proceedings of Science, 5th International School on Field Theory and Gravitation, April 20–24 (2009). arXiv:0910.0350
71. S. Kumar, C.P. Singh, Int. J. Theor. Phys. **47**, 1722 (2008)
72. T. Singh, A.K. Agrawal, Astrophys. Space Sci. **182**, 289 (1991)
73. K.S. Adhav, M.R. Ugale, C.B. Kale, M.P. Bhende, Int. J. Theor. Phys. **48**, 178 (2009)
74. B.C. Paul, P.S. Debnath, S. Ghose, Phys. Rev. D **79**, 083534 (2009)
75. V.T. Gurovich, A.A. Starobinsky, JETP **50**, 844 (1979)
76. M. Sharif, M.F. Shamir, Class. Quantum Gravity **26**, 235020 (2009)
77. M. Sharif, M.F. Shamir, Gen. Relativ. Gravit. **42**, 2643 (2010)
78. M. Sharif, H.R. Kausar, Astrophys. Space Sci. **332**, 463 (2011)
79. M.F. Shamir, JETP **146**, 281 (2014)
80. N.J. Poplawski (2006). arXiv:gr-qc/0608031
81. N.J. Poplawski, Class. Quantum Gravity **23**, 4819 (2006)
82. R. Kantowski, R.K. Sachs, J. Math. Phys. **7**, 443 (1966)
83. C.B. Collins, Phys. Lett. A **60**, 397 (1977)
84. K.S. Thorne, Astrophys. J. **148**, 51 (1967)
85. C.B. Collins, S.W. Hawking, Astrophys. J. **180**, 317 (1973)
86. S.R. Roy, S.K. Banerjee, Class. Quantum Gravity **11**, 1943 (1995)
87. R. Bali, P. Kumawat, Phys. Lett. B **665**, 332 (2008)
88. M. Sharif, M. Zubair, Astrophys. Space Sci. **330**, 399 (2010)
89. M.A.H. MacCallum, Commun. Math. Phys. **18**, 2116 (1971)
90. M. Visser, Gen. Relativ. Gravit. **37**, 1541 (2005)
91. M. Cataldo, S.D. Campo, Phys. Rev. D **61**, 128301 (2000)
92. S. Nojiri, S.D. Odintsov, Phys. Rep. **505**, 59 (2011)