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On the probability of ruin in the compound Poisson risk model with potentially delayed claims

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Abstract In this paper, we consider the compound Poisson risk model involving two types of dependent claims, namely main claims and by-claims. The by-claim is induced by the main claim with a certain probability and the occurrence of a by-claim may be delayed depending on associated main claim amount. Using Rouché's theorem, both of the survival probability with zero initial surplus and the Laplace transform of the survival probability are obtained from an integro-differential equations system. Then, using the Laplace transform, we derive a defective renewal equation satisfied by the survival probability. An exact representation for the solution of this equation is derived through an associated compound geometric distribution. For exponential claim sizes, we present an explicit formula for the survival probability. We also illustrate the influence of model parameters in the dependent risk model on the survival probability by numerical examples.

Mathematics Subject Classification 60J65 · 91B30

المخلص

نعتبر في هذه الورقة نموذج بواسون المركب للمجازفة والذي ينطوي على نوعين من الادعاءات التابعة، وبشكل أكثر تحديداً الادعاءات الرئيسية والادعاءات الجانبية. ينجم الادعاء الجانبي عن الادعاء الرئيس باحتمالية معينة، ويمكن أن يتم تأخير وقوع الادعاء الجانبي بالاعتماد على مقدار الادعاء الرئيس المرتبط به. باستخدام مبرهنة روشي، يتم الحصول على كل من الاحتمالية البقاء مع فائض ابتدائي صفري وعلى تحويل لابلاس لاحتمالية البقاء من نظام معادلات تكاملية – تفاضلية. ثم، وباستخدام تحويل لابلاس، نشق معادلة تجديد قاصرة تحققها احتمالية البقاء. يتم اشتقاق تمثيل مضبوط لحل هذه المعادلة من خلال توزيع هندسي مركب مرتبط. بالنسبة لأحجام الادعاءات الأسيّة، نقدم صيغة صريحة لاحتمالية البقاء. نرسخ أيضاً تأثير الوسائط في نموذج المجازفة الاعتمادي على احتمالية البقاء بوساطة أمثلة عددية.

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1 Introduction

In fact, insurance claims may be delayed due to various reasons. This phenomenon may happen in reality. For a catastrophe such as an earthquake or a rain-storm, it is very likely that there exist other insurance claims after the immediate ones. Since the work by [9], risk models with this special feature have been discussed by many authors. For example, “Boogaert and Haezendonck (1989) studied the mathematical properties of a liability process with settling delay within the framework of an economics environment” (See [3]). [13] studied a compound binomial model with delayed claims and obtained recursive formulas for finite time survival probabilities. [10] also studied this risk model. They derived an upper bound for the ruin probability. [18] studied an extension to the risk model proposed in [13]. [14] studied a risk model with delayed claims, in which the time of delay for the occurrence of a by-claim is assumed to be exponentially distributed. [8] presented a sample path large deviation principle for the delayed claims risk model presented in [14]. [11] first considered the risk model with delayed claims and a constant dividend barrier in a financial market driven by a time-homogeneous Markov chain. The expected discounted dividend payments prior to ruin were derived. [12] considered the compound Poisson risk model with delayed claims and proved that the ruin probability for this risk model decreases as the probability of the delay of by-claims increases. In addition, the dependence between other components in risk models was also studied. For instance, [1] considered the case in which the distribution of a claim interval is controlled by the previous claim size through a mechanism that if the claim size exceeds a random level the next claim interval will follow one type of distribution, if not, it will follow another type of distribution. Their model depicts common sense that when a certain kind of catastrophe is big enough, people will pay more attention to it and so the time until the next occurrence is longer. [2] and [4] considered a particular dependence structure among the inter-claim time and subsequent claim size. Furthermore, [16] have studied risk models with dependence between inter-claim times and claim sizes.

Note that all risk models described in the paragraph above rely on the assumption that each main claim induces a by-claim with certainty. Motivated by the fact that each main claim induces a by-claim randomly and the probability of delay of each claim is not independent of claim amounts, in this paper, we consider the compound Poisson risk model with two types of individual claims where the two types of claim have different distributions of severity. In this risk model, there will be a main claim, at every jump time of the number process. Let the aggregate main claims process be a compound Poisson process and let $\{N(t); t \geq 0\}$ be the corresponding Poisson claim number process, with intensity λ . Its jump times are denoted by $\{T_i\}_{i \geq 1}$ with $T_0 = 0$. The main claim amounts $\{Y_i\}_{i \geq 1}$ are assumed to be independent and identically distributed (i.i.d.) positive random variables with common distribution F . Each main claim induces another type of claim called a by-claim with probability q . Let $\{X_i\}_{i \geq 1}$ be the by-claim amounts, assumed to be i.i.d. positive random variables with common distribution G . The main claim amounts and by-claim amounts are independent and their means are denoted by μ_F and μ_G . Moreover, we assume the claim occurrence process to be of the following type: there will be a main claim Y_i at every epoch T_i of the Poisson process. It is assumed that each main claim induces a by-claim with probability q and the main claim does not induce a by-claim with probability $1 - q$. Moreover, if the main claim induces a by-claim and the main claim amount Y_i is less than a threshold B_i , the by-claim X_i and its associated main claim Y_i occur simultaneously; if the main claim induces a by-claim but the main claim amount Y_i is larger than or equal to the threshold B_i , the occurrence of the by-claim X_i is delayed to T_{i+1} . If the occurrence of the by-claim X_i is delayed to T_{i+1} , then the delayed by-claim X_i and main claim Y_{i+1} occur simultaneously. The quantities $\{B_i\}_{i \geq 1}$ are assumed to be i.i.d. random variables with common distribution B .

In this setup, the surplus process $U(t)$ of this risk model is defined as

$$U(t) = u + ct - \sum_{i=1}^{N(t)} Y_i - R(t), \quad (1.1)$$

where u is the initial capital, c is the constant rate per unit time at which the premiums are received, and $R(t)$ is the sum of all by-claims X_i that occurred before time t .

One of the key quantities in the risk model is the survival probability, or non-ruin probability, denoted by $\Phi(u)$ as a function of $u \geq 0$, which is the probability that the surplus of the insurer is always above zero, namely,

$$\Phi(u) = \Pr(U(t) \geq 0; \text{ for all } t \geq 0).$$



The corresponding ruin probability is then $\Psi(u) = 1 - \Phi(u)$, which is the probability that the surplus of the insurer is below zero at some time. The survival probability (ruin probability) can be used to provide an early warning system for the guidance of an insurance project (see [6]).

The expectation of the aggregate claims at time t is given by

$$E \left[\sum_{i=1}^{N(t)} Y_i + R(t) \right] = \lambda t \mu_F + \lambda t q \mu_G - q \Pr(Y_1 \geq B_1) \mu_G (1 - e^{-\lambda t}).$$

Thus in order to ensure that the premium rate exceeds the net claim rate and guarantee that ruin does not occur almost surely, we assume the following positive safely loading condition holds, i.e.,

$$\lambda(\mu_F + q\mu_G) < c. \tag{1.2}$$

We point out that when $q = 1$, our model reduces to the continuous time risk model proposed by [17]. Moreover, when B is the constant, our model reduces to the delayed claims risk model of [12]. Therefore, the work of this paper can also be seen as a complement to the work of [12], [17] and extends their results by taking the randomness of delayed claim into account.

The rest of this paper is structured as follows. In Sect. 2, we derive an integro-differential equations system for survival probabilities. Then, by employing Rouché’s theorem, both of the survival probability with zero initial surplus and the Laplace transform of the survival probability are obtained. By using Laplace transforms, we derive a defective renewal equation for the survival probability in Sect. 3. An exact representation for the solution of this equation is also derived through an associated compound geometric distribution. For exponential claim size distribution, in Sect. 4 we present the explicit formulas for survival probabilities. In Sect. 5, we illustrate the influence of the model parameters in the dependent risk model on the survival probability by numerical examples.

2 System of integro-differential equations and Laplace transforms

In this section, we first derive the integro-differential equations for the survival probabilities. We then study the Laplace transforms for the survival probability $\Phi(u)$, based on the integro-differential equations.

With other things being the same, we consider a slight change in the risk model (1.1). Instead of having one main claim Y_1 and a by-claim X_1 with probability $q\Pr(Y_1 < B_1)$ at the first epoch T_1 , another by-claim X is added at the first epoch T_1 , i.e., by-claim X and main claim Y_1 occur at T_1 simultaneously. Hence, the corresponding surplus process $U_1(t)$ of this auxiliary risk model is defined as

$$U_1(t) = u + ct - \sum_{i=1}^{N(t)} Y_i - R(t) - X, \tag{2.1}$$

where X denotes the extra by-claim amount, $U_1(0) = u$. Assume that X and $\{X_i\}_{i \geq 1}$ are i.i.d. positive random variables.

This surplus process is similar to (1.1) except for the subtraction of the by-claim random variable X . Denote the corresponding survival probability for this auxiliary model by $\Phi_1(u)$ which is very useful in the derivation of $\Phi(u)$.

Theorem 2.1 For any $u \geq 0$, $\Phi(u)$ and $\Phi_1(u)$ satisfy the following system of integro-differential equations:

$$\begin{aligned} &c \frac{d\Phi(u)}{du} - \lambda\Phi(u) + (1 - q)\lambda \int_0^u \Phi(u - y) dF(y) \\ &+ q\lambda \int \int_{0 < x+y < u} \Pr(y < B_1) \Phi(u - x - y) dF(y) dG(x) \\ &+ q\lambda \int_0^u \Pr(y \geq B_1) \Phi_1(u - y) dF(y) = 0, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
 & c \frac{d\Phi_1(u)}{du} - \lambda \Phi_1(u) + (1-q)\lambda \int \int_{0 < x+y < u} \Phi(u-x-y) dF(y) dG(x) \\
 & + q\lambda \int \int_{0 < x+y < u} \Pr(y < B_1) \Phi(u-x-y) dF(y) dG * G(x) \\
 & + q\lambda \int \int_{0 < x+y < u} \Pr(y \geq B_1) \Phi_1(u-x-y) dF(y) dG(x) = 0. \tag{2.3}
 \end{aligned}$$

Proof Consider what will happen at the first epoch T_1 . We need to study the claim occurrences in three scenarios.

- (I) Obviously there will be a main claim Y_1 at the first epoch T_1 . The main claim does not induce a by-claim, then the surplus process gets renewed expect for the initial value.
- (II) There will be a main claim Y_1 at the first epoch T_1 . The main claim will induce a by-claim X_1 and the main claim size $Y_1 < B_1$, then the by-claim X_1 also occurs at the first epoch T_1 , the surplus process $U(t)$ will renew itself with different initial reserve. The probability of this event is $q\Pr(Y_1 < B_1)$.
- (III) There will be a main claim Y_1 at the first epoch T_1 . The main claim will induce a by-claim X_1 and the main claim size $Y_1 \geq B_1$, the occurrence of the by-claim X_1 will be delayed to T_2 , i.e., the delayed by-claim X_1 and main claim Y_2 occur simultaneously. In this case, $U(t)$ will not renew itself but transfer to the auxiliary model described in the paragraph above. The probability of this event is $q\Pr(Y_1 \geq B_1)$.

Taking what happened at the first epoch T_1 into account, we can set up the following equation for $\Phi(u)$:

$$\begin{aligned}
 \Phi(u) &= (1-q) \int_0^\infty \lambda e^{-\lambda t} \int_0^{u+ct} \Phi(u+ct-y) dF(y) dt \\
 &+ q \int_0^\infty \lambda e^{-\lambda t} \int \int_{0 < x+y < u+ct} \Pr(y < B_1) \Phi(u+ct-x-y) dF(y) dG(x) dt \\
 &+ q \int_0^\infty \lambda e^{-\lambda t} \int_0^{u+ct} \Pr(y \geq B_1) \Phi_1(u+ct-y) dF(y) dt. \tag{2.4}
 \end{aligned}$$

With the auxiliary model, similar analysis gives

$$\begin{aligned}
 \Phi_1(u) &= (1-q) \int_0^\infty \lambda e^{-\lambda t} \int \int_{0 < x+y < u+ct} \Phi(u+ct-x-y) dF(y) dG(x) dt \\
 &+ q \int_0^\infty \lambda e^{-\lambda t} \int \int_{0 < x+y < u+ct} \Pr(y < B_1) \Phi(u+ct-x-y) dF(y) dG * G(x) dt \\
 &+ q \int_0^\infty \lambda e^{-\lambda t} \int \int_{0 < x+y < u+ct} \Pr(y \geq B_1) \Phi_1(u+ct-y) dF(y) dG(x) dt. \tag{2.5}
 \end{aligned}$$

where $*$ denotes the operation of convolution.

Setting $s = u + ct$ in (2.4), (2.5) and differentiating with respect to u , we get the system of integro-differential equations (2.2) and (2.3). This completes the proof of this theorem. \square

In the following, we will give the Laplace transforms of the survival probabilities $\Phi(u)$ and $\Phi_1(u)$ from the integro-differential equations.



Let $\chi_1(y) = B(y)F'(y)$ and $\chi_2(y) = (1 - B(y))F'(y)$. For $\text{Re } s \geq 0$, we define

$$\begin{aligned} \tilde{\chi}_1(s) &= \int_{y=0}^{\infty} \exp(-sy)\chi_1(y)dy = E[\exp(-sF)1_{(F \geq B)}] = \int_{y=0}^{\infty} \exp(-sy)B(y)dF(y); \\ \tilde{\chi}_2(s) &= \int_{y=0}^{\infty} \exp(-sy)\chi_2(y)dy = E[\exp(-sF)1_{(F < B)}] = \int_{y=0}^{\infty} \exp(-sy)(1 - B(y))dF(y); \\ \tilde{b}(s) &= \int_0^{\infty} \exp(-sy)dF(y); \\ \tilde{b}_1(s) &= \int_0^{\infty} \exp(-sx)dG(x); \\ \tilde{b}_2(s) &= \int_0^{\infty} \exp(-sx)dG * G(x). \end{aligned}$$

Note that $\tilde{b}(s) = \tilde{\chi}_1(s) + \tilde{\chi}_2(s)$ and $\tilde{b}_2(s) = (\tilde{b}_1(s))^2$.

We also define the Laplace transforms of $\Phi(u)$ and $\Phi_1(u)$ as

$$\tilde{\Phi}(s) = \int_0^{\infty} \exp(-su)\Phi(u)du; \quad \tilde{\Phi}_1(s) = \int_0^{\infty} \exp(-su)\Phi_1(u)du.$$

Taking Laplace transforms of (3.3) and (3.4) and making some simplifications, we obtain

$$\begin{aligned} c(-\Phi(0) + s\tilde{\Phi}(s)) - \lambda\tilde{\Phi}(s) + (1 - q)\lambda\tilde{\Phi}(s)\tilde{b}(s) + q\lambda\tilde{\Phi}(s)\tilde{\chi}_2(s)\tilde{b}_1(s) + q\lambda\tilde{\Phi}_1(s)\tilde{\chi}_1(s) &= 0, \\ c(-\Phi_1(0) + s\tilde{\Phi}_1(s)) - \lambda\tilde{\Phi}_1(s) + (1 - q)\lambda\tilde{\Phi}(s)\tilde{b}(s)\tilde{b}_1(s) + q\lambda\tilde{\Phi}(s)\tilde{\chi}_2(s)\tilde{b}_2(s) + q\lambda\tilde{\Phi}_1(s)\tilde{\chi}_1(s)\tilde{b}_1(s) &= 0, \end{aligned}$$

which can further be simplified to

$$\tilde{\Phi}(s) = \frac{c\Phi(0) \left(cs - \lambda + q\lambda\tilde{b}_1(s)\tilde{\chi}_1(s) \right) - q\lambda c\Phi_1(0)\tilde{\chi}_1(s)}{(cs - \lambda)\{cs - \lambda + q\lambda\tilde{b}(s)\tilde{b}_1(s) + (1 - q)\lambda\tilde{b}(s)\}}. \tag{2.6}$$

In order to obtain $\tilde{\Phi}(s)$, for the further sake of deriving $\Phi(u)$, we only need to find $\Phi(0)$ and $\Phi_1(0)$. Denoting the denominator on the right-hand side of (2.6) as

$$\Delta(s) = (cs - \lambda)\{cs - \lambda + q\lambda\tilde{b}(s)\tilde{b}_1(s) + (1 - q)\lambda\tilde{b}(s)\},$$

we have

$$\Delta(s) = cs \left\{ (cs - 2\lambda) + (1 - q)\lambda\tilde{b}(s) + q\lambda\tilde{b}(s)\tilde{b}_1(s) + \frac{\lambda^2 - (1 - q)\lambda^2\tilde{b}(s) - q\lambda^2\tilde{b}(s)\tilde{b}_1(s)}{cs} \right\}.$$

Since $\lim_{u \rightarrow +\infty} \Phi(u) = 1$, by the Final-value theorem of Laplace transforms, we have $\lim_{s \rightarrow 0} s \tilde{\Phi}(s) = 1$, so

$$\begin{aligned} 1 &= \lim_{s \rightarrow 0} s \frac{c\Phi(0) \left(cs - \lambda + q\lambda \tilde{b}_1(s) \tilde{\chi}_1(s) \right) - q\lambda c \Phi_1(0+) \tilde{\chi}_1(s)}{(cs - \lambda) \{ cs - \lambda + q\lambda \tilde{b}(s) \tilde{b}_1(s) + (1 - q)\lambda \tilde{b}(s) \}} \\ &= \lim_{s \rightarrow 0} s \frac{c\Phi(0) \left(cs - \lambda + q\lambda \tilde{b}_1(s) \tilde{\chi}_1(s) \right) - q\lambda c \Phi_1(0+) \tilde{\chi}_1(s)}{\Delta(s)} \\ &= \frac{\Phi(0) \left(q\tilde{b}_1(0) \tilde{\chi}_1(0) - 1 \right) - q\Phi_1(0) \tilde{\chi}_1(0)}{-2 + (1 - q)\tilde{b}(0) + q\tilde{b}(0)\tilde{b}_1(0) - \frac{(1-q)\lambda \left(\tilde{b}(s) \right)' \Big|_{s=0} + q\lambda \left(\tilde{b}(s)\tilde{b}_1(s) \right)' \Big|_{s=0}}{c}} \\ &= \frac{\Phi(0)(1 - q\tilde{\chi}_1(0)) + q\Phi_1(0)\tilde{\chi}_1(0)}{1 - \frac{\lambda}{c}(\mu_F + q\mu_G)}, \end{aligned}$$

which leads to the following equation for $\Phi(0)$ and $\Phi_1(0)$:

$$\Phi(0)(1 - q\tilde{\chi}_1(0)) + q\Phi_1(0)\tilde{\chi}_1(0) = 1 - \frac{\lambda}{c}(\mu_F + q\mu_G). \tag{2.7}$$

In order to derive the explicit result for $\tilde{\Phi}(s)$, we must obtain a second equation for $\Phi(0)$ and $\Phi_1(0)$. Using Rouché’s theorem, we have:

Lemma 2.2 Equation $\Delta(s) = 0$ has exactly one positive real root, say $\sigma = \lambda/c$, on the right half complex plane.

Proof Define $l(s) = cs - \lambda + (1 - q)\lambda \tilde{b}(s) + q\lambda \tilde{b}(s)\tilde{b}_1(s)$. Then $\Delta(s) = (cs - \lambda)l(s)$. It is easy to check that $l(0) = 0$ and $\lim_{s \rightarrow +\infty} l(s) = +\infty$. Also, for $s \geq 0$,

$$l'(s) = c + (1 - q)\lambda \tilde{b}'(s) + q\lambda \left[\tilde{b}'(s)\tilde{b}_1(s) + \lambda \tilde{b}(s)\tilde{b}'_1(s) \right] > c - \lambda(\mu_F + q\mu_G) > 0,$$

so $l(s)$ is an increasing function of s . Hence, $l(s) = 0$ has no positive real root. Noting that λ/c is one positive real root of equation $\Delta(s) = 0$, we conclude that equation $\Delta(s) = 0$ has exactly one positive real root, say, $\sigma = \lambda/c$.

Now, we prove that $\sigma = \lambda/c$ is the unique positive real root of equation $\Delta(s) = 0$ on the right half complex plane. In order to prove this result, we only need to prove that $l(s) = 0$ has no positive real root on the right half complex plane. For $\delta > 0$, assume that $l_\delta(s) = cs - \lambda - \delta + (1 - q)\lambda \tilde{b}(s) + q\lambda \tilde{b}(s)\tilde{b}_1(s)$. If s is on the half circle: $|z| = r (r > 0)$ and $\mathbf{Re}(z) \geq 0$ on the complex plane, $|cs - \lambda - \delta| > \lambda = (1 - q)\lambda \tilde{b}(0) + q\lambda \tilde{b}(0)\tilde{b}_1(0) > |(1 - q)\lambda \tilde{b}(s) + q\lambda \tilde{b}(s)\tilde{b}_1(s)|$ for r is sufficiently large, while if s is on the imaginary axis, $\mathbf{Re}(s) = 0$, $|cs - \lambda - \delta| > \lambda \geq |(1 - q)\lambda \tilde{b}(s) + q\lambda \tilde{b}(s)\tilde{b}_1(s)|$. This implies on the boundary of the contour enclosed by the half circle and the imaginary axis, that $|cs - \lambda - \delta| > |(1 - q)\lambda \tilde{b}(s) + q\lambda \tilde{b}(s)\tilde{b}_1(s)|$. We conclude, by Rouché’s theorem, that on the right half complex plane, the number of roots of the equation $l_\delta(s) = 0$ equals the number of roots of the equation $cs - \lambda - \delta = 0$. Furthermore, the latter has exactly one root on the right half complex plane. It follows that $l_\delta(s) = 0$ has exactly one positive real root, say, $\rho(\delta)$, on the right half complex plane.

Finally, it is easy to see that $\rho(\delta) \rightarrow 0$ as $\delta \rightarrow 0^+$ from the expression of $l_\delta(s)$. Since $\lim_{\delta \rightarrow 0^+} l_\delta(s) = l(s)$, we can conclude that $l(s) = 0$ has no root with positive real part on the right half plane.

It follows from everything above that equation $\Delta(s) = 0$ has exactly one positive real root $\sigma = \lambda/c$ on the right half complex plane. This completes the proof. \square

Since $\tilde{\Phi}(s)$ is analytic function for $\mathbf{Re} s \geq 0$, σ must also be a zero of the numerator of (2.6). This yields the relation between $\Phi(0)$ and $\Phi_1(0)$, namely

$$\Phi(0) = \frac{\Phi_1(0)}{\tilde{b}_1(\sigma)}. \tag{2.8}$$

Combining (2.7) and (2.8), we can obtain the solutions for $\Phi(0)$ and $\Phi_1(0)$ in the following theorem.

Theorem 2.3 *The survival probabilities with zero initial surplus are given by*

$$\Phi(0) = \frac{c - \lambda\mu_F - \lambda q\mu_G}{c \left(1 - q\tilde{\chi}_1(0) + q\tilde{\chi}_1(0)\tilde{b}_1(\sigma)\right)}, \tag{2.9}$$

$$\Phi_1(0) = \frac{\tilde{b}_1(\sigma)(c - \lambda\mu_F - \lambda q\mu_G)}{c \left(1 - q\tilde{\chi}_1(0) + q\tilde{\chi}_1(0)\tilde{b}_1(\sigma)\right)}. \tag{2.10}$$

Remark 2.4 1. Let $q = 1$ and $B = +\infty$, i.e., $\Pr(Y < B) = 1$, then $\tilde{\chi}_1(s) = 0$ and $\tilde{\chi}_2(s) = \tilde{b}(s)$. In this case, each main claim and its associated by-claim occur simultaneously. Actually, the risk model given by (1.1) is the classic compound Poisson risk model and the claim amounts are $\{Y_i + X_i\}_{i \geq 1}$. Then Eq. (2.6) can be simplified as

$$\tilde{\Phi}(s) = \frac{c\Phi(0)}{cs - \lambda + \lambda\tilde{b}(s)\tilde{b}_1(s)}, \tag{2.11}$$

which is the Laplace transform of the non-ruin probability in the classic compound Poisson risk model. Similarly, since $\lim_{u \rightarrow +\infty} \Phi(u) = 1$, by the Final-value theorem of Laplace transforms, we have $\lim_{s \rightarrow 0} s\tilde{\Phi}(s) = 1$, i.e.,

$$1 = \lim_{s \rightarrow 0} s \frac{c\Phi(0)}{s(c + \frac{\lambda(\tilde{b}(s)\tilde{b}_1(s)-1)}{s})} = \frac{c\Phi(0)}{c - \lambda(\mu_F + \mu_G)},$$

so we obtain

$$\Phi(0) = \frac{c - \lambda(\mu_F + \mu_G)}{c}, \tag{2.12}$$

which is the well-known formula for the non-ruin probability with zero initial capital in the classic compound Poisson risk model.

2. Let $q = 1$ and B be a constant, i.e., $\Pr(Y \geq B) = \theta$, then $\tilde{\chi}_1(s) = \theta\tilde{b}(s)$ and $\tilde{\chi}_2(s) = (1 - \theta)\tilde{b}(s)$. In this case, each main claim and its associated by-claim occur simultaneously with probability $1 - \theta$, or the occurrence of the by-claim may be delayed with probability θ . Actually, the risk model given by (1.1) is the compound Poisson risk model with delayed claims studied by [12]. Then Eq. (2.6) can be simplified as

$$\tilde{\Phi}(s) = \frac{c\Phi(0) \left(cs - \lambda + \lambda\theta\tilde{b}(s) \left(\tilde{b}_1(s) - \tilde{b}_1\left(\frac{\lambda}{c}\right) \right) \right)}{(cs - \lambda)(cs - \lambda + \lambda\tilde{b}(s)\tilde{b}_1(s))}. \tag{2.13}$$

This equation is consistent with Eq. 4.1 in [12]. Similarly, since $\lim_{u \rightarrow +\infty} \Phi(u) = 1$, by the Final-value theorem of Laplace transforms, we have $\lim_{s \rightarrow 0} s\tilde{\Phi}(s) = 1$, i.e.,

$$\begin{aligned} 1 &= \lim_{s \rightarrow 0} s \frac{c\Phi(0) \left(cs - \lambda + \lambda\theta\tilde{b}(s) \left(\tilde{b}_1(s) - \tilde{b}_1\left(\frac{\lambda}{c}\right) \right) \right)}{(cs - \lambda)(cs - \lambda + \lambda\tilde{b}(s)\tilde{b}_1(s))} \\ &= \lim_{s \rightarrow 0} s \frac{c\Phi(0) \left(cs - \lambda + \lambda\theta\tilde{b}(s) \left(\tilde{b}_1(s) - \tilde{b}_1\left(\frac{\lambda}{c}\right) \right) \right)}{cs \left\{ (cs - 2\lambda) + \lambda\tilde{b}(s)\tilde{b}_1(s) + \frac{\lambda^2 - \lambda^2\tilde{b}(s)\tilde{b}_1(s)}{cs} \right\}} \\ &= \frac{\Phi(0) \left(-1 + \theta\tilde{b}(0) \left(\tilde{b}_1(0) - \tilde{b}_1\left(\frac{\lambda}{c}\right) \right) \right)}{\left\{ -2 + \tilde{b}(0)\tilde{b}_1(0) - \frac{\lambda \left(\tilde{b}(s)\tilde{b}_1(s) \right)' \Big|_{s=0}}{c} \right\}} = \frac{\Phi(0) \left(-1 + \theta \left(1 - \tilde{b}_1\left(\frac{\lambda}{c}\right) \right) \right)}{-1 + \frac{\lambda}{c} (\mu_F + \mu_G)}, \end{aligned}$$

so we obtain

$$\Phi(0) = \frac{1 - \frac{\lambda}{c} (\mu_F + \mu_G)}{1 - \theta \left(1 - \tilde{b}_1\left(\frac{\lambda}{c}\right)\right)}, \quad (2.14)$$

which is the formula for the survival probability with zero initial capital in the compound Poisson risk model with delayed claims.

3 Defective renewal equations for survival probabilities

In this section, our goal is to show that the survival probabilities also satisfy a defective renewal equation in the dependent risk model with delayed claims. To identify the form of this defective renewal equation, we first analyse the Laplace transform of $\Phi(u)$.

As in [5], we define an operator Γ_r of a real-valued function f , with respect to a complex number r , to be

$$\Gamma_r f(x) = \int_x^{\infty} e^{-r(y-x)} f(y) dy, \quad x \geq 0.$$

It is clear that the Laplace transform of f , $\tilde{f}(s)$, can be expressed as $\Gamma_s f(0)$, and that for distinct r_1 and r_2 ,

$$\Gamma_{r_1} \Gamma_{r_2} f(x) = \Gamma_{r_2} \Gamma_{r_1} f(x) = \frac{\Gamma_{r_1} f(x) - \Gamma_{r_2} f(x)}{r_2 - r_1}, \quad x \geq 0.$$

If $r_1 = r_2 = r$,

$$\Gamma_{r_1} \Gamma_{r_2} f(x) = \int_x^{\infty} (y-x) e^{-r(y-x)} f(y) dy, \quad x \geq 0.$$

The properties for this operator can be found in [5].

Lemma 3.1 *The Laplace transform $\tilde{\Phi}(s)$ of the survival probability satisfies*

$$\tilde{\Phi}(s) = \frac{(1-q)\lambda\Gamma_s\Gamma_0 b(0) + q\lambda\Gamma_s\Gamma_0 b * b_1(0)}{c} \tilde{\Phi}(s) + \Phi(0) \left(\frac{1}{s} - \frac{q\lambda\tilde{\chi}_1(s)}{cs} \Gamma_s \Gamma_{\sigma} b_1(0) \right). \quad (3.1)$$

Proof Substituting (2.8) into (2.6), we can get

$$\begin{aligned} \tilde{\Phi}(s) &= \frac{c\Phi(0) \left(cs - \lambda + q\lambda\tilde{\chi}_1(s)(\tilde{b}_1(s) - \tilde{b}_1(\sigma)) \right)}{(cs - \lambda)\{cs - \lambda + (1-q)\lambda\tilde{b}(s) + q\lambda\tilde{b}(s)\tilde{b}_1(s)\}} \\ &= \frac{cs(s - \sigma)\Phi(0) \left(\frac{c}{s} + \frac{q\lambda\tilde{\chi}_1(s)}{s} \frac{\tilde{b}_1(s) - \tilde{b}_1(\sigma)}{s - \sigma} \right)}{cs(s - \sigma) \left(c - \frac{(1-q)(\lambda - \lambda\tilde{b}(s))}{s} - \frac{q(\lambda - \lambda\tilde{b}(s)\tilde{b}_1(s))}{s} \right)} \\ &= \frac{\Phi(0) \left(\frac{c}{s} - \frac{q\lambda\tilde{\chi}_1(s)}{s} \Gamma_s \Gamma_{\sigma} b_1(0) \right)}{c - (1-q)\lambda\Gamma_s\Gamma_0 b(0) - q\lambda\Gamma_s\Gamma_0 b * b_1(0)}, \end{aligned}$$

which leads to (3.1). This completes the proof. \square

Using Lemma 3.1, we are now in a position to derive the defective renewal equation for $\Phi(u)$.



Theorem 3.2 $\Phi(u)$ satisfies the following defective renewal equation

$$\Phi(u) = \frac{\lambda(\mu_F + q\mu_G)}{c} \int_0^u \Phi(u - y)\vartheta(y)dy + z(u), \tag{3.2}$$

where

$$\vartheta(y) = \frac{(1 - q)\Gamma_0 b(y) + q\Gamma_0 b * b_1(y)}{\mu_F + q\mu_G}, \quad z(u) = \Phi(0) \left(1 - \frac{q\lambda}{c} * \chi_1 * \Gamma_\sigma b_1(u) \right).$$

Proof Inverting the Laplace transform in (3.1), one finds

$$\begin{aligned} \Phi(u) &= \frac{\lambda}{c} \int_0^u \Phi(u - y)[(1 - q)\Gamma_0 b(y) + q\Gamma_0 b * b_1(y)]dy + \Phi(0) \left(1 - \frac{q\lambda}{c} * \chi_1 * \Gamma_\sigma b_1(u) \right) \\ &= \frac{\lambda(\mu_F + q\mu_G)}{c} \int_0^u \Phi(u - y) \frac{(1 - q)\Gamma_0 b(y) + q\Gamma_0 b * b_1(y)}{\mu_F + q\mu_G} dy + z(u), \end{aligned}$$

which corresponds to (3.2).

For (3.2) to be a defective renewal equation, it remains to show that $\lambda(\mu_F + q\mu_G) < c$. The inequality is the positive safely loading condition (1.2). Thus, we complete the proof. \square

Now, we define an associated compound geometric distribution function $K(u) = 1 - \bar{K}(u)$ by

$$\bar{K}(u) = \frac{\zeta}{1 + \zeta} \sum_{n=1}^{\infty} \left(\frac{1}{1 + \zeta} \right)^n \bar{Z}^{*n}(u), \quad u \geq 0,$$

where $\zeta = (c - \lambda(\mu_F + q\mu_G))/[\lambda(\mu_F + q\mu_G)]$, $\bar{Z}^{*n}(u)$ is the tail of the n -fold convolution of $Z(u) = 1 - \bar{Z}(u) = \int_0^u \vartheta(y)dy$. Explicit solutions of the defective renewal Eq.(3.2) can be derived directly by applying Theorem 2.1 of [7].

Theorem 3.3 The survival probability $\Phi(u)$ satisfying the defective renewal equation (3.2) can be expressed as

$$\Phi(u) = \frac{1}{\zeta} \int_0^u [1 - \bar{K}(u - y)]dH(y) + \frac{H(0)}{\zeta} [1 - \bar{K}(u)], \tag{3.3}$$

or

$$\Phi(u) = \frac{1}{\zeta} \int_0^u H(u - y)dK(y) + \frac{1}{1 + \zeta} H(u), \tag{3.4}$$

where $H(u) = cz(u)/[\lambda(\mu_F + q\mu_G)]$.

Proof The proof is straightforward using Theorem 2.1 of [7] and Eq. (3.2). \square

Remark 3.4 We point out that when $q = 1$, the compound Poisson risk model with potentially delayed claims is equivalent to the dependent risk model studied by [17]. To see that, letting $q = 1$ in (3.2), we have

$$\Phi(u) = \frac{\lambda(\mu_F + \mu_G)}{c} \int_0^u \Phi(u - y)\vartheta(y)dy + z(u),$$

which is constant with defective renewal equation (5.2) derived by [17].

4 Explicit results for exponential claim size distributions

We now consider the case where both the claim sizes are exponentially distributed, with distribution functions $F \sim \text{Exp}(\nu)$ and $G \sim \text{Exp}(\omega)$, respectively, where $\nu = \frac{1}{\mu_F}$ and $\omega = \frac{1}{\mu_G}$. Then we have

$$\tilde{b}(s) = \frac{\nu}{\nu + s}, \quad \tilde{b}_1(s) = \frac{\omega}{\omega + s}, \quad \tilde{b}_2(s) = \frac{\omega^2}{(\omega + s)^2}.$$

For the special case $B \sim \text{Exp}(\mu)$ we obtain

$$\tilde{\chi}_2(s) = \int_{y=0}^{\infty} \exp(-sy) \exp(-\mu y) dF(y) = b(s + \mu), \quad \tilde{\chi}_1(s) = b(s) - b(s + \mu).$$

So we can derive

$$\tilde{\chi}_2(s) = \frac{\nu}{\nu + s + \mu}; \quad \tilde{\chi}_1(s) = \frac{\nu}{\nu + s} - \frac{\nu}{\nu + s + \mu};$$

and σ in (2.8) is the unique solution s with $\text{Re } s > 0$ of

$$(cs - \lambda) \left(cs - \lambda + (1 - q)\lambda \left(\frac{\nu}{\nu + s} \right) + q\lambda \left(\frac{\omega}{\omega + s} \right) \left(\frac{\nu}{\nu + s} \right) \right) = 0.$$

By solving this equation, we obtain four roots:

$$\begin{aligned} \sigma_1 &= 0, & \sigma_2 &= \frac{\lambda}{c}, \\ \sigma_3 &= \frac{\lambda - c\nu - c\omega - \sqrt{(-\lambda + c\nu + c\omega)^2 - 4c(-q\lambda\nu - \lambda\omega + c\nu\omega)}}{2c}, \\ \sigma_4 &= \frac{\lambda - c\nu - c\omega + \sqrt{(-\lambda + c\nu + c\omega)^2 - 4c(-q\lambda\nu - \lambda\omega + c\nu\omega)}}{2c}. \end{aligned}$$

The positive relative security loading condition (1.2) implies that only σ_2 has a positive real part. Hence, $\sigma = \lambda/c$ is the only zero with positive real part on the right half plane. This proves the correctness of Lemma 2.2.

From (2.9) and (2.10), it is easy to obtain the survival probabilities with zero initial surplus:

$$\Phi(0) = \frac{(\mu + \nu)(\lambda + c\omega)[(c\nu - \lambda)\omega - q\lambda\nu]}{c\nu\omega[\lambda(\nu + (1 - q)\mu) + c(\mu + \nu)\omega]},$$

and

$$\Phi_1(0) = \frac{(\mu + \nu)[(c\nu - \lambda)\omega - q\lambda\nu]}{\nu[\lambda(\nu + (1 - q)\mu) + c(\mu + \nu)\omega]}.$$

Substituting the expressions of $\tilde{b}_1(s)$, $\tilde{b}_2(s)$, $\tilde{\chi}_1(s)$, $\tilde{\chi}_2(s)$, $\Phi(0)$ and $\Phi_1(0)$ into (2.6) and taking inverse Laplace transform, we can derive explicit expression for survival probability $\Phi(u)$,

$$\begin{aligned} \Phi(u) &= 1 - \frac{e^{-u(\mu+\nu)} q\lambda\mu(q\lambda\nu + (\lambda - c\nu)\omega)}{[c\mu(\mu + \nu - \omega) + \lambda(\mu + (1 - q)\nu - \omega)](\lambda(\mu(1 - q) + \nu) + c(\mu + \nu)\omega)} \\ &- \frac{e^{-\frac{u(-\lambda+c(\nu+\omega)+\rho_q)}{2c}} 2\lambda(\mu + \nu)(q\lambda\nu + (\lambda - c\nu)\omega)[- \lambda^3 + \lambda^2(\rho_q - \vartheta_1^q) + c^2\omega(\rho_q(\mu + \nu) + \vartheta_2^q) + c\lambda(\rho_q(\mu + \nu) + \vartheta_3^q)]}{c\nu\omega(\lambda(\mu(1 - q) + \nu) + c(\mu + \nu)\omega)\rho_q(\rho_q - \lambda - c(2\mu + \nu - \omega))(\rho_q + c(\nu + \omega) - \lambda)} \\ &- \frac{e^{-\frac{u(\lambda-c(\nu+\omega)+\rho_q)}{2c}} 2\lambda(\mu + \nu)(q\lambda\nu + (\lambda - c\nu)\omega)[\lambda^3 + \lambda^2(\rho_q + \vartheta_1^q) + c^2\omega(\rho_q(\mu + \nu) - \vartheta_2^q) + c\lambda(\rho_q(\mu + \nu) + \vartheta_3^q)]}{c\nu\omega(\lambda(\mu(1 - q) + \nu) + c(\mu + \nu)\omega)\rho_q(\rho_q + \lambda + c(2\mu + \nu - \omega))(\rho_q - c(\nu + \omega) + \lambda)}, \end{aligned} \quad (4.1)$$

where $\rho_q = \sqrt{\lambda^2 + c^2(\nu - \omega)^2 + 2c\lambda(\nu(2q - 1) + \omega)}$, $\vartheta_1^q = c(\mu - \nu + 3q\nu + 2\omega)$, $\vartheta_2^q = c(\mu - q\nu)(\nu - \omega)$, $\vartheta_3^q = c[\omega(\omega - \nu) + q\nu(\nu + 2\omega) + \mu(2\omega - (1 - 2q)\nu)]$.



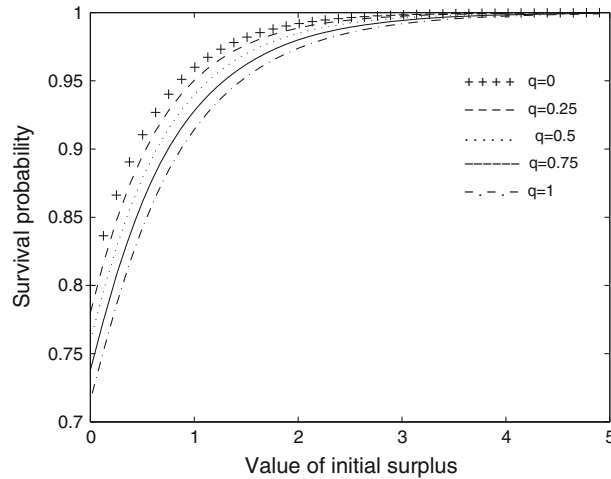


Fig. 1 Influence of the probability q on survival probabilities

Remark 4.1 1. Letting $q = 1$ in (4.1), we have

$$\begin{aligned} \Phi(u) = & 1 - \frac{e^{-u(\mu+v)}\lambda\mu(\lambda v + (\lambda - cv)\omega)}{[c\mu(\mu + v - \omega) + \lambda(\mu - \omega)](\lambda v + c(\mu + v)\omega)} \\ & - \frac{e^{-\frac{u(-\lambda+c(v+\omega)+\rho_1)}{2c}} 2\lambda(\mu + v)(\lambda v + (\lambda - cv)\omega)[- \lambda^3 + \lambda^2(\rho_1 - \vartheta_1^1) + c^2\omega(\rho_1(\mu + v) + \vartheta_2^1) + c\lambda(\rho_1(\mu + v + \omega) - \vartheta_3^1)]}{cv\omega(\lambda v + c(\mu + v)\omega)\rho_1(\rho_1 - \lambda - c(2\mu + v - \omega))(\rho_1 + c(v + \omega) - \lambda)} \\ & - \frac{e^{\frac{u(\lambda-c(v+\omega)+\rho_1)}{2c}} 2\lambda(\mu + v)(\lambda v + (\lambda - cv)\omega)[\lambda^3 + \lambda^2(\rho_1 + \vartheta_1^1) + c^2\omega(\rho_1(\mu + v) - \vartheta_2^1) + c\lambda(\rho_1(\mu + v + \omega) + \vartheta_3^1)]}{cv\omega(\lambda v + c(\mu + v)\omega)\rho_1(\rho_1 + \lambda + c(2\mu + v - \omega))(\rho_1 - c(v + \omega) + \lambda)}, \end{aligned} \tag{4.2}$$

which is constant with the expression (6.1) obtained in [17].

2. When $q = 0$, each main claim does not induce a by-claim. In this case, the survival probability is given by

$$\begin{aligned} \Phi(u) = & 1 - \frac{e^{-\frac{u(-\lambda+c(v+\omega)+\rho_0)}{2c}} 2\lambda(\mu + v)(\lambda - cv)\omega[- \lambda^3 + \lambda^2(\rho_0 - \vartheta_1^0) + c^2\omega(\rho_0\mu + \vartheta_2^0) + c\lambda(\rho_0(\mu + \omega) - \vartheta_3^0)]}{cv\omega(\lambda(\mu + v) + c(\mu + v)\omega)\rho_0(\rho_0 - \lambda - c(2\mu + v - \omega))(\rho_0 + c(v + \omega) - \lambda)} \\ & - \frac{e^{\frac{u(\lambda-c(v+\omega)+\rho_0)}{2c}} 2\lambda(\mu + v)(\lambda - cv)\omega[\lambda^3 + \lambda^2(\rho_0 + \vartheta_1^0) + c^2\omega(\rho_0\mu + -\vartheta_2^0) + c\lambda(\rho_0(\mu + \omega) + \vartheta_3^0)]}{cv\omega(\lambda(\mu + v) + c(\mu + v)\omega)\rho_0(\rho_0 + \lambda + c(2\mu + v - \omega))(\rho_0 - c(v + \omega) + \lambda)}. \end{aligned} \tag{4.3}$$

5 Numerical examples

In this section, we illustrate the applications of the results in the previous sections and the influence of the model parameters in the dependent risk model on survival probabilities by numerical examples.

Example 5.1 Let $\lambda = 1, c = 2.5, B \sim \text{Exp}(2.8), F \sim \text{Exp}(2), G \sim \text{Exp}(3)$. The positive relative security loading condition (1.2) is obviously fulfilled.

Figure 1 shows the survival probabilities in Example 5.1, for $u \in [0, 5]$ and $q = 0, 0.25, 0.5, 0.75, 1$. One can see from the graph that with fixed u , these survival probabilities decrease as q increases.

Example 5.2 In this example, we show the influence of the threshold B on the survival probabilities. Let $q = 0.8, \lambda = 1, c = 2, F \sim \text{Exp}(1.5), G \sim \text{Exp}(1.3), B \sim \text{Exp}(\mu)$. The positive relative security loading condition (1.2) is obviously fulfilled.

Figure 2 shows the survival probabilities in Example 5.2, for $u \in [0, 4]$ and $\mu = 0.5, 1, 1.5, 2, 2.5$. We see from Fig. 2 that the coefficient μ of threshold B has a significant effect on the non-ruin probabilities. Figure 2 also shows the fact that the non-ruin probability is an increasing function of μ .

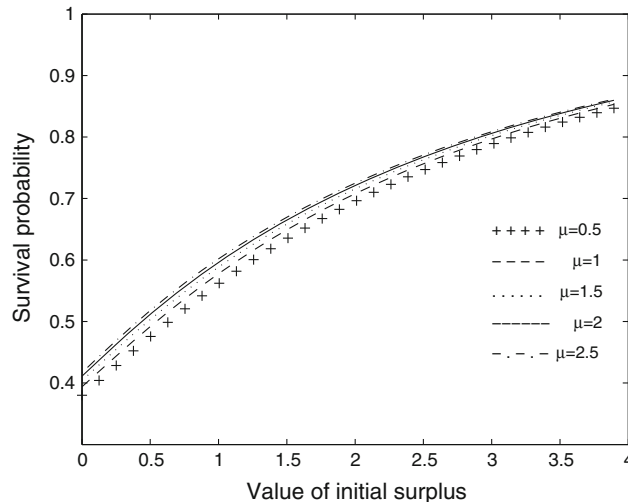


Fig. 2 Influence of the threshold B on survival probabilities

6 Concluding remarks

In this paper, we study the compound Poisson risk model with potentially delayed claims. In this risk model, there will be a main claim Y_i at every epoch T_i of the Poisson process and the main claim Y_i will induce a by-claim X_i with probability q . Moreover, the occurrence of the by-claim X_i may be delayed to T_{i+1} depending on main claim size Y_i and the threshold B_i . The quantities $\{B_i\}_{i \geq 1}$ are assumed to be i.i.d. non-negative random variables.

We show how to apply the Laplace transform to this dependent risk model. A defective renewal equation for the survival probability is obtained and an exact representation for the solution of this equation is derived through an associated compound geometric distribution. The explicit results for survival probabilities are derived when the claims are exponentially distributed. Some examples and numerical illustrations are also given.

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