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Control problems for semilinear second order equations with cosine families

Jin-Mun Jeong*, Eun-Young Ju and Seong-Ho Cho

*Correspondence:
jmjeong@pknu.ac.kr
Department of Applied Mathematics, Pukyong National University, Busan, 608-737, Korea

Abstract
This paper aims to obtain the approximate controllability for the second order nonlinear control systems with a strongly cosine family and the associated with sine family.

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1 Introduction

The first part of this paper gives some basic results on the regularity of solutions of the abstract semilinear second order initial value problem

$$\begin{cases} \frac{d^2 w(t)}{dt^2} = Aw(t) + F(t,w) + f(t), & 0 < t \leq T, \\ w(0) = x_0, & \frac{d}{dt} w(0) = y_0, \end{cases} \tag{1.1}$$

in a Banach space X. Here, the nonlinear part is given by

$$F(t,w) = \int_0^t k(t-s) g\bigl(s, w(s)\bigr)\, ds,$$

where k belongs to $L^2(0,T)$ and $g : [0,T] \times X \longrightarrow X$ is a nonlinear mapping such that $w \mapsto g(t,w)$ satisfies Lipschitz continuity. In (1.1) A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$. Let E be a subspace of all $x \in X$ for which $C(t)x$ is a once continuously differentiable function of t.

In [1], when $f : \mathbb{R} \to X$ is continuously differentiable, $x_0 \in D(A)$, $y_0 \in E$, and $k \in W^{1,2}(0,T)$, the existence of a solution $w \in L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$ of (1.1) for each $T > 0$ is given. Moreover, one established a variation of constant formula for solutions of the second order nonlinear system (1.1).

The work presented in this paper, based on the regularity for solution of (1.1), investigates necessary and sufficient conditions for the approximate controllability for (1.1) with the strict range condition on B even though the system (1.1) contains unbounded principal operators and the convolution nonlinear term, which is a more flexible necessary assumption than the one in [2].

We will make use of some of the basic ideas from cosine family referred to [3, 4] and the regular properties for solutions in [1, 5] for a discussion of the control results. In [6, 7] a one-dimensional nonlinear hyperbolic equation of convolution type which is nonlinear in the partial differential equation part and linear in the hereditary part is treated.

As a second part in this paper, we consider the approximate controllability for the nonlinear second order control system

$$\begin{cases} \frac{d^2w(t)}{dt^2} = Aw(t) + F(t,w) + Bu(t), & 0 < t \le T, \\ w(0) = x_0, & \frac{d}{dt}w(0) = y_0, \end{cases} \tag{1.2}$$

in a Banach space X where the controller B is a bounded linear operator from some Banach space U to X. In [2, 8, 9] the approximate controllability for (1.2) was studied under the particular range conditions of the controller B depending on the time T.

In Section 3 we establish to the approximate controllability for the second order nonlinear system (1.2) under a condition for the range of the controller B without the inequality condition independent to the time T, and we see that the necessary assumption is more flexible than the one in [2, 9]. Finally, we give a simple example to which our main result can be applied.

2 Preliminaries

Let X be a Banach space. The norm of X is denoted by $\|\cdot\|$. We start by introducing a strongly continuous cosine family and sine family in X.

Definition 2.1 [1] Let $C(t)$ for each $t \in \mathbb{R}$ be a bounded linear operators in X. $C(t)$ is called a strongly continuous cosine family if the following conditions are satisfied:

c(1) $C(s+t) + C(s-t) = 2C(s)C(t)$, for all $s, t \in \mathbb{R}$, and $C(0) = I$;

c(2) $C(t)x$ is continuous in t on $\mathbb{R}$ for each fixed $x \in X$.

Let A be the infinitesimal generator of a one parameter cosine family $C(t)$ defined by

$$Ax = \frac{d^2}{dt^2}C(0)x.$$

Then we endow it with the domain $D(A) = \{x \in X : \frac{d^2}{dt^2}C(t)x \text{ is continuous}\}$ endowed with the norm

$$\|x\|_{D(A)} = \|x\| + \sup\left\{\left\|\frac{d}{dt}C(t)x\right\| : t \in \mathbb{R}\right\} + \|Ax\|.$$

We shall also make use of the set

$$E = \left\{x \in X : \frac{d}{dt}C(t)x \text{ is continuous}\right\}$$

with the norm

$$\|x\|_E = \|x\| + \sup\left\{\left\|\frac{d}{dt}C(t)x\right\| : t \in \mathbb{R}\right\}.$$

It is well known that $D(A)$ and E with the given norms are Banach spaces.

Let $S(t)$, $t \in \mathbb{R}$, be the one parameter family of operators in X defined by

$$S(t)x = \int_0^t C(s)x\,ds, \qquad x \in X, t \in \mathbb{R}. \tag{2.1}$$

The following basic results on cosine and sine families are from Propositions 2.1 and 2.2 of [1].

Lemma 2.1 *Let $C(t)$ ($t \in \mathbb{R}$) be a strongly continuous cosine family in X. The following are true*:

c(3) $C(-t) = C(t)$ *for all* $t \in \mathbb{R}$;

c(4) $C(s)$ *and* $S(t)$ *commute for all* $s, t \in \mathbb{R}$;

c(5) $S(t)x$ *is continuous in* t *on* $\mathbb{R}$ *for each fixed* $x \in X$;

c(6) *there exist constants* $K \geq 1$ *and* $\omega \geq 0$ *such that*

$$\|C(t)\| \leq Ke^{\omega|t|} \quad \text{for all } t \in \mathbb{R},$$

and

$$\|S(t_1) - S(t_2)\| \leq K\left|\int_{t_2}^{t_1} e^{\omega|s|}\,ds\right| \quad \text{for all } t_1, t_2 \in \mathbb{R};$$

c(7) *if* $x \in E$, *then* $S(t)x \in D(A)$ *and*

$$\frac{d}{dt}C(t)x = S(t)Ax = AS(t)x = \frac{d^2}{dt^2}S(t)x,$$

moreover, if $x \in D(A)$, *then* $C(t)x \in D(A)$ *and*

$$\frac{d^2}{dt^2}C(t)x = AC(t)x = C(t)Ax;$$

c(9) *if* $x \in X$ *and* $r, s \in \mathbb{R}$, *then*

$$\int_r^s S(\tau)x\,d\tau \in D(A) \quad \text{and} \quad A\left(\int_r^s S(\tau)x\,d\tau\right) = C(s)x - C(r)x.$$

First, we consider the following linear equation:

$$\begin{cases} \frac{d^2w(t)}{dt^2} = Aw(t) + f(t), & t \geq 0, \\ w(0) = x_0, \quad \dot{w}(0) = y_0. \end{cases} \tag{2.2}$$

The following results are crucial in discussing regular problem for the linear case (for a proof see [1]).

Proposition 2.1 *Let $f : R \to X$ be continuously differentiable, $x_0 \in D(A)$, $y_0 \in E$. Then the mild solution $w(t)$ of* (2.2) *represented by*

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)\,ds, \quad t \in \mathbb{R},$$

belongs to $L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$, *and we see that there exists a positive constant* C_1 *such that, for any* $T > 0$,

$$\|w\|_{L^2(0,T;D(A))} \le C_1\bigl(1 + \|x_0\|_{D(A)} + \|y_0\|_E + \|f\|_{W^{1,2}(0,T;X)}\bigr). \tag{2.3}$$

If f is continuously differentiable and $(x_0, y_0) \in D(A) \times E$, it is easily shown that w is continuously differentiable and satisfies

$$\dot{w}(t) = AS(t)x_0 + C(t)y_0 + \int_0^t C(t-s)f(s)\,ds, \quad t \in \mathbb{R}.$$

Here, we note that if w is a solution of (2.2) in an interval $[0, t_1 + t_2]$ with $t_1, t_2 > 0$. Then for $t \in [0, t_1 + t_2]$, we have

$$\begin{aligned} w(t) &= C(t-t_1)w(t_1) + S(t-t_1)\dot{w}(t_1) + \int_{t_1}^t S(t-s)f(s)\,ds \\ &= C(t-t_1)\left\{C(t_1)x_0 + S(t_1)y_0 + \int_0^{t_1} S(t_1-\tau)f(\tau)\,d\tau\right\} \\ &\quad + S(t-t_1)\left\{AS(t_1)x_0 + C(t_1)y_0 + \int_0^{t_1} C(t_1-\tau)f(\tau)\,d\tau\right\} \\ &\quad + \int_{t_1}^t S(t-s)f(s)\,ds \\ &= C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)f(s)\,ds; \end{aligned}$$

here, we used the following relations, for all $s, t \in \mathbb{R}$:

$$\begin{aligned} &C(s+t) + C(s-t) = 2C(s)C(t), \\ &S(s+t) = S(s)C(t) + S(t)C(s), \\ &C(s+t) = C(t)C(s) - S(t)S(s), \\ &C(s+t) - C(t-s) = 2AS(t)S(s), \end{aligned}$$

and

$$S(t)AS(s) = AS(t)S(s) = \frac{1}{2}C(t+s) - \frac{1}{2}C(t-s) = C(t+s) - C(t)C(s).$$

This means the mapping $t \mapsto w(t_1 + t)$ is a solution of (2.2) in $[0, t_1 + t_2]$ with initial data $(w(t_1), \dot{w}(t_1)) \in D(A) \times E$.

From now on, we introduce the regularity of solutions of the abstract semilinear second order initial value problem (1.1) in a Banach space X. We make the following assumptions.

The nonlinear mapping g from $[0,T] \times D(A)$ to X is such that $t \mapsto g(t,w)$ is measurable and

$$\begin{cases} \|g(t,w_1) - g(t,w_2)\|_{D(A)} \le L\|w_1 - w_2\|, \\ g(t,0) = 0, \end{cases} \tag{2.4}$$

for a positive constant L.

For $w \in L^2(0,T;D(A))$, we set

$$F(t,w) = \int_0^t k(t-s)g\big(s,w(s)\big)\,ds,$$

where k belongs to $L^2(0,T)$. We will seek a mild solution of (1.1), that is, a solution of the integral equation

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\big\{F(s,w) + f(s)\big\}\,ds.$$

Lemma 2.2 *If $w \in L^2(0,T;D(A))$ for any $T > 0$, then $F(\cdot,w) \in L^2(0,T;X)$ and*

$$\big\|F(\cdot,w)\big\|_{L^2(0,T;X)} \le L\|k\|_{L^2(0,T)}\sqrt{T}\|w\|_{L^2(0,T;D(A))}.$$

Moreover, let $w_1, w_2 \in L^2(0,T;D(A))$. Then we have

$$\big\|F(\cdot,w_1) - F(\cdot,w_2)\big\|_{L^2(0,T;X)} \le L\|k\|_{L^2(0,T)}\sqrt{T}\|w_1 - w_2\|_{L^2(0,T;D(A))}.$$

Proof By using the Hölder inequality and (2.4), it is easily shown that

$$\begin{aligned}\big\|F(\cdot,w)\big\|^2_{L^2(0,T;X)} &\le \int_0^T \left\|\int_0^t k(t-s)g\big(s,w(s)\big)\,ds\right\|^2 dt \\ &\le \|k\|^2_{L^2(0,T)}\int_0^T\int_0^t L^2\big\|w(s)\big\|^2\,ds\,dt \\ &\le L^2\|k\|^2_{L^2(0,T)}T\|w\|^2_{L^2(0,T;D(A))}.\end{aligned}$$

By a similar argument, the second paragraph is obtained. □

Now, as in Theorem 3.1 of [1], we give a norm estimation of the solution of (1.1) and establish the global existence of solutions with the aid of norm estimations.

Proposition 2.2 *Suppose that the assumption* (2.4) *are satisfied. If $f : \mathbb{R} \longrightarrow X$ is continuously differentiable, $x_0 \in D(A)$, $y_0 \in E$, and $k \in W^{1,2}(0,T)$, then the solution w of* (1.1) *exists and is unique in $L^2(0,T;D(A)) \cap W^{1,2}(0,T;E)$ for $T > 0$, and there exists a constant C_3 depending on T such that*

$$\|w\|_{L^2(0,T;D(A))} \le C_3\big(1 + \|x_0\|_{D(A)} + \|y_0\|_E + \|f\|_{W^{1,2}(0,T;X)}\big). \tag{2.5}$$

3 Approximate controllability

In this section, we deal with the approximate controllability for the semilinear second order control system

$$\begin{cases} \frac{d^2w(t)}{dt^2} = Aw(t) + F(t,w) + Bu(t), & 0 < t \le T, \\ w(0) = x_0, \qquad \frac{d}{dt}w(0) = y_0, \end{cases} \tag{3.1}$$

in a Banach space X where the controller B is a bounded linear operator from some Banach space U to X, where U is another Banach space. Assume the following.

Assumption (G) The nonlinear mapping $g : [0, T] \times X \longrightarrow X$ is such that $t \mapsto g(t, w)$ is measurable and

$$\|g(t, w_1) - g(t, w_2)\| \leq L\|w_1 - w_2\|, \quad |k(t)| \leq M, \tag{3.2}$$

for a positive constant L.

Here, we remark that since the Assumption (G) is a more general condition than (2.4), the equation of (3.1), written

$$w(t) = C(t)x_0 + S(t)y_0 + \int_0^t S(t-s)\{F(s, w) + Bu(s)\}\, ds, \tag{3.3}$$

belongs to $L^2(0, T; D(A)) \cap W^{1,2}(0, T; E)$ for $T > 0$.

Given a strongly continuous cosine family $C(t)$ $(t \in R)$, we define linear bounded operators $\hat{C}$ and $\hat{S}$ mapping $L^2(0, T; X)$ into X by

$$\hat{C}p = \int_0^T C(T - t)p(t)\, dt, \qquad \hat{S}p = \int_0^T S(T - t)p(t)\, dt,$$

for $p(\cdot) \in L^2(0, T; X)$ and $S(t)$ is the associated sine family of $C(t)$.

We define the reachable sets for the system (3.1) as follows.

Definition 3.1 Let $w(t; F, u)$ be a solution of the (3.1) associated with nonlinear term F and control u at the time t. Then

$$R_T(F) = \{(w(T; F, u), \dot{w}(T; F, u)) : u \in L^2(0, T; U)\} \subset X^2 = X \times X,$$
$$R_T(0) = \{(w(T; 0, u), \dot{w}(T; 0, u)) : u \in L^2(0, T; U)\} \subset X^2.$$

The nonempty subset $R_T(F)$ in X^2 consisting of all terminal states of (3.1) is called the reachable sets at the time T of the system (3.1). The set $R_T(0)$ is one of the linear cases where $F \equiv 0$.

Definition 3.2 The system (3.1) is said to be approximately controllable on the interval $[0, T]$ if

$$\overline{R_T(F)} = X^2,$$

where $\overline{R_T(F)}$ is the closure of $R_T(F)$ in X^2, that is, for any $\epsilon > 0$, $\bar{x} \in D(A)$ and $\bar{y} \in E$ there exists a control $u \in L^2(0, T; U)$ such that

$$\|\bar{x} - C(T)x_0 - S(T)y_0 - \hat{S}F(\cdot, w) - \hat{S}Bu\| < \epsilon,$$
$$\|\bar{y} - AS(T)x_0 - C(T)y_0 - \hat{C}F(\cdot, w) - \hat{C}Bu\| < \epsilon.$$

We introduce the following hypothesis.

Assumption (B) For any $\varepsilon > 0$ and $p \in L^2(0,T;X)$, there exists a $u \in L^2(0,T;U)$ such that

$$\begin{cases} \|\hat{C}p - \hat{C}Bu\| < \varepsilon, \\ \|Bu\|_{L^2(0,t;X)} \le q_1 \|p\|_{L^2(0,t;X)}, \quad 0 \le t \le T, \end{cases}$$

where q_1 is a constant independent of p.

We remark that from the relations between the cosine and sine families, the operator $\hat{S}$ also satisfies the condition (B), that is, for any $\varepsilon > 0$ and $p \in L^2(0,T;X)$ there exists a $u \in L^2(0,T;U)$ such that

$$\begin{cases} \|\hat{S}p - \hat{S}Bu\| < \varepsilon, \\ \|Bu\|_{L^2(0,t;X)} \le q_1 \|p\|_{L^2(0,t;X)}, \quad 0 \le t \le T. \end{cases}$$

For the sake of simplicity we assume that the sine family $S(t)$ is bounded as in c(6):

$$\|S(t)\| \le K(t), \quad t \ge 0.$$

Here, we may consider the following inequality:

$$K(t) \le \omega^{-1} K(e^{\omega t} - 1).$$

Lemma 3.1 *Let the Assumption* (G) *be satisfied. If u_1 and u_2 are in $L^2(0,T;U)$, then we have*

$$\|w(t;F,u_1) - w(t;F,u_2)\| \le K(T) e^{K(T)MLT^2} \sqrt{T} \|Bu_1 - Bu_2\|_{L^2(0,t;X)}$$

for $0 \le t \le T$.

Proof From the Assumption (G), it follows that, for $0 \le t \le T$,

$$\begin{aligned} &\|w(t;F,u_1) - w(t;F,u_2)\| \\ &\quad \le K(T)\sqrt{t} \|Bu_1(s) - Bu_2(s)\|_{L^2(0,t;X)} \\ &\qquad + K(T)MLt \int_0^t \|w(s;F,u_1) - w(s;F,u_2)\|\, ds, \end{aligned}$$

where L is the constant in the Assumption (G). Therefore, by using Gronwall's inequality this lemma follows. □

For the approximate controllability for the linear equation, we recall the following necessary lemma before proving the main theorem.

Lemma 3.2 *Let the Assumption* (G) *be satisfied. Then we have* $\overline{R_T(0)} = X^2$.

Proof Let $\bar{x} \in D(A)$, $\bar{y} \in E$. Putting

$$\eta_1 = \bar{x} - C(T)x_0 - S(T)y_0 \in D(A), \qquad \eta_2 = \bar{y} - AS(T)x_0 - C(T)y_0 \in E,$$

then there exists some $p \in C^1([0,T]:X)$ such that

$$\eta_1 = \int_0^T S(T-t)p(t)\,dt, \qquad \eta_2 = \int_0^T C(T-t)p(t)\,dt,$$

for instance, take $p(t) = \{C(t-T) + S(t-T)\}\eta_2/T$. By hypothesis (B) there exists a function $u \in L^2(0,T;U)$ such that

$$\begin{cases} \|\bar{x} - C(T)x_0 - S(T)y_0 - \hat{S}Bu\| < \epsilon, \\ \|\bar{y} - AS(T)x_0 - C(T)y_0 - \hat{C}Bu\| < \epsilon. \end{cases}$$

The denseness of the domain $D(A)$ in X implies the approximate controllability of the corresponding linear system. □

Theorem 3.1 *Let the Assumptions* (G) *and* (B) *be satisfied. Then the system* (3.1) *is approximately controllable on* $[0,T]$, $T > 0$.

Proof We will show that $D(A) \times E \subset \overline{R_T(F)}$, *i.e.*, for given $\varepsilon > 0$ and $(\xi_T, \tilde{\xi}_T) \in D(A) \times E$ there exists $u \in L^2(0,T;U)$ such that

$$\left\| \xi_T - w(T;F,u) \right\| < \varepsilon, \tag{3.4}$$

$$\left\| \tilde{\xi}_T - \dot{w}(T;F,u) \right\| < \varepsilon. \tag{3.5}$$

Since $(\xi_T, \tilde{\xi}_T) \in D(A) \times E$, there exists a $p \in L^2(0,T;X)$ such that

$$\hat{S}p = \xi_T - C(T)x_0 - S(T)y_0, \qquad \hat{C}p = \tilde{\xi}_T - AS(T)x_0 - C(T)y_0.$$

Let $u_1 \in L^2(0,T;U)$ be arbitrary fixed. Then by the Assumption (B), there exists $u_2 \in L^2(0,T;U)$ such that

$$\left\| \hat{S}\big(p - F\big(\cdot, w(\cdot;F,u_1)\big)\big) - \hat{S}Bu_2 \right\| < \frac{\varepsilon}{4},$$
$$\left\| \hat{C}\big(p - F\big(\cdot, w(\cdot;F,u_1)\big)\big) - \hat{C}Bu_2 \right\| < \frac{\varepsilon}{4}.$$

Hence, we have

$$\begin{aligned} &\left\| \xi_T - C(T)x_0 - S(T)y_0 - \hat{S}F\big(\cdot, w(\cdot;F,u_1)\big) - \hat{S}Bu_2 \right\| < \frac{\varepsilon}{4}, \\ &\left\| \tilde{\xi}_T - AS(T)x_0 - C(T)y_0 - \hat{C}F\big(\cdot, w(\cdot;F,u_1)\big) - \hat{C}Bu_2 \right\| < \frac{\varepsilon}{4}. \end{aligned} \tag{3.6}$$

Moreover, by the Assumption (B), we can also choose $v_2 \in L^2(0,T;U)$ such that

$$\begin{aligned} &\left\| \hat{S}\big(F\big(\cdot, w(\cdot;F,u_2)\big) - F\big(\cdot, w(\cdot;F,u_1)\big)\big) - \hat{S}Bv_2 \right\| < \frac{\varepsilon}{8}, \\ &\left\| \hat{C}\big(F\big(\cdot, w(\cdot;F,u_2)\big) - F\big(\cdot, w(\cdot;F,u_1)\big)\big) - \hat{C}Bv_2 \right\| < \frac{\varepsilon}{8}, \end{aligned} \tag{3.7}$$

and also

$$\|Bv_2\|_{L^2(0,t;X)} \le q_1 \big\| F\big(\cdot, w(\cdot;F,u_1)\big) - F\big(\cdot, w(\cdot;F,u_2)\big)\big\|_{L^2(0,t;X)}$$

for $0 \le t \le T$. From now, we will only prove (3.4), while the proof of (3.5) is similar. In view of Lemma 3.1 and the Assumption (B), we have

$$\begin{aligned}
\|Bv_2\|_{L^2(0,t;X)} &\le q_1 \left\{ \int_0^t \big\| F\big(\tau, w(\tau;F,u_2)\big) - F\big(\tau, w(\tau;F,u_1)\big)\big\|^2 \,d\tau \right\}^{\frac{1}{2}} \\
&\le q_1 ML \left\{ \int_0^t \int_0^\tau \big\| w(\tau;F,u_2) - w(\tau;F,u_1)\big\|^2 \,ds\,d\tau \right\}^{\frac{1}{2}} \\
&\le q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T} \left\{ \int_0^t \int_0^\tau \|Bu_2 - Bu_1\|^2_{L^2(0,s;X)} \,ds\,d\tau \right\}^{\frac{1}{2}} \\
&\le q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T} \left(\int_0^t \int_0^\tau 1 \,ds\,d\tau \right)^{\frac{1}{2}} \|Bu_2 - Bu_1\|_{L^2(0,t;X)} \\
&= q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T} \left(\frac{t^2}{2} \right)^{\frac{1}{2}} \|Bu_2 - Bu_1\|_{L^2(0,t;X)}.
\end{aligned}$$

Put $u_3 = u_2 - v_2$. We choose v_3 such that

$$\begin{aligned}
&\big\| \hat{S}\big(F\big(\cdot, w(\cdot;F,u_3)\big) - F\big(\cdot, w(\cdot;F,u_2)\big)\big) - \hat{S}Bv_3 \big\| < \frac{\varepsilon}{8}, \\
&\|Bv_3\|_{L^2(0,t;X)} \le q_1 \big\| F\big(\cdot, w(\cdot;F,u_3)\big) - F\big(\cdot, w(\cdot;F,u_2)\big)\big\|_{L^2(0,t;X)}
\end{aligned}$$

for $0 \le t \le T$. Thus, we have

$$\begin{aligned}
&\|Bv_3\|_{L^2(0,t;X)} \\
&\quad\le q_1 \left\{ \int_0^t \big\| F\big(\tau, w(\tau;F,u_3)\big) - F\big(\tau, w(\tau;F,u_2)\big)\big\|^2 \,d\tau \right\}^{\frac{1}{2}} \\
&\quad\le q_1 ML \left\{ \int_0^t \int_0^\tau \big\| w(s;F,u_3) - w(s;F,u_2)\big\|^2 \,ds\,d\tau \right\}^{\frac{1}{2}} \\
&\quad\le q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T} \left\{ \int_0^t \int_0^\tau \|Bu_3 - Bu_2\|^2_{L^2(0,s:X)} \,ds\,d\tau \right\}^{\frac{1}{2}} \\
&\quad\le q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T} \left\{ \int_0^t \int_0^\tau \|Bv_2\|^2_{L^2(0,s;X)} \,ds\,d\tau \right\}^{\frac{1}{2}} \\
&\quad\le \big(q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T}\big)^2 \left\{ \int_0^t \int_0^\tau \frac{s^2}{2} \|Bu_2 - Bu_1\|^2_{L^2(0,s;X)} \,ds\,d\tau \right\}^{\frac{1}{2}} \\
&\quad\le \big(q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T}\big)^2 \left(\int_0^t \int_0^\tau \frac{s^2}{2} \,ds\,d\tau \right)^{\frac{1}{2}} \|Bu_2 - Bu_1\|_{L^2(0,t;X)} \\
&\quad\le \big(q_1 MLK(T)e^{K(T)MLT^2}\sqrt{T}\big)^2 \left(\frac{t^4}{2 \cdot 4} \right)^{\frac{1}{2}} \|Bu_2 - Bu_1\|_{L^2(0,t;X)}.
\end{aligned}$$

By proceeding this process, and from the equality

$$\begin{aligned}
&\big\|B(u_n-u_{n+1})\big\|_{L^2(0,t;X)}\\
&\quad=\|Bv_n\|_{L^2(0,t;X)}\\
&\quad\le\big(q_1MLK(T)\sqrt{T}e^{K(T)MLT^2}\big)^{n-1}\left(\frac{t^{2n-2}}{2\cdot 4\cdots(2n-2)}\right)^{\frac{1}{2}}\|Bu_2-Bu_1\|_{L^2(0,t;X)}\\
&\quad=\left(\frac{q_1MLK(T)\sqrt{T}e^{K(T)MLT^2}t}{\sqrt{2}}\right)^{n-1}\frac{1}{\sqrt{(n-1)!}}\|Bu_2-Bu_1\|_{L^2(0,t;X)},
\end{aligned}$$

we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty}\|Bu_{n+1}-Bu_n\|_{L^2(0,T;X)}\\
&\quad\le\sum_{n=0}^{\infty}\left(\frac{q_1MLK(T)\sqrt{T}e^{K(T)MLT^2}t}{\sqrt{2}}\right)^{n}\\
&\qquad\times\frac{1}{\sqrt{n!}}\|Bu_2-Bu_1\|_{L^2(0,T;X)}<\infty.
\end{aligned}$$

Thus, there exists $u^*\in L^2(0,T;X)$ such that

$$\lim_{n\to\infty}Bu_n=u^*\quad\text{in } L^2(0,T;X). \tag{3.8}$$

Combining (3.6) and (3.7), we have

$$\begin{aligned}
&\big\|\xi_T-C(T)x_0-S(T)y_0-\hat{S}F\big(\cdot,w(\cdot F,u_2)\big)-\hat{S}Bu_3\big\|\\
&\quad=\big\|\xi_T-C(T)x_0-S(T)y_0-\hat{S}F\big(\cdot,w(\cdot F,u_1)\big)-\hat{S}Bu_2+\hat{S}Bv_2\\
&\qquad-\hat{S}\big[F\big(\cdot,w(\cdot;F,u_2)\big)-F\big(\cdot,w(\cdot;F,u_1)\big)\big]\big\|\\
&\quad<\left(\frac{1}{2^2}+\frac{1}{2^3}\right)\varepsilon.
\end{aligned}$$

If we determine $v_n\in L^2(0,T;U)$ such that

$$\big\|\hat{S}\big(F\big(\cdot w(\cdot;F,u_n)\big)-F\big(\cdot w(\cdot;F,u_{n-1})\big)\big)-\hat{S}Bv_n\big\|<\frac{\varepsilon}{2^{n+1}},$$

then putting $u_{n+1}=u_n-v_n$, we have

$$\begin{aligned}
&\big\|\xi_T-S(T)g-\hat{S}F\big(\cdot,z(\cdot;g,f,u_n)\big)-\hat{S}\Phi u_{n+1}\big\|\\
&\quad<\left(\frac{1}{2^2}+\cdots+\frac{1}{2^{n+1}}\right)\varepsilon,\quad n=1,2,\ldots.
\end{aligned}$$

Therefore, for any $\varepsilon>0$ there exists an integer N such that

$$\|\hat{S}Bu_{N+1}-\hat{S}Bu_N\|<\frac{\varepsilon}{2},$$

and hence

$$\begin{aligned}
&\left\| \xi_T - C(T)x_0 - S(T)y_0 - \hat{S}F\big(\cdot, w(\cdot; F, u_N)\big) - \hat{S}Bu_N \right\| \\
&\quad \le \left\| \xi_T - C(T)x_0 - S(T)y_0 - \hat{S}F\big(\cdot, w(\cdot; F, u_N)\big) - \hat{S}Bu_{N+1} \right\| \\
&\qquad + \|\hat{S}Bu_{N+1} - \hat{S}Bu_N\| \\
&\quad < \left(\frac{1}{2^2} + \cdots + \frac{1}{2^{N+1}} \right)\varepsilon + \frac{\varepsilon}{2} \le \varepsilon.
\end{aligned}$$

By a similar method, we also obtain

$$\left\| \tilde{\xi}_T - AS(T)x_0 - C(T)y_0 - \hat{C}F\big(\cdot, w(\cdot; F, u_N)\big) - \hat{C}Bu_N \right\| \le \varepsilon.$$

Thus, as N tends to infinity, the system (3.1) is approximately controllable on $[0, T]$.

Example We consider the following partial differential equation:

$$\begin{cases} \frac{d^2 w(t,x)}{dt^2} = Aw(t,x) + F(t,w) + Bu(t), & 0 < t, 0 < x < \pi, \\ w(t,0) = w(t,\pi) = 0, & t \in \mathbb{R}, \\ w(0,x) = x_0(x), & \frac{d}{dt} w(0,x) = y_0(x), \quad 0 < x < \pi. \end{cases} \tag{3.9}$$

Let $X = L^2([0,\pi]; \mathbb{R})$. If $e_n(x) = \sqrt{\frac{2}{\pi}} \sin nx$, then $\{e_n : n = 1, \ldots\}$ is an orthonormal base for X. The operator $A : X \to X$ is defined by

$$Aw(x) = w''(x),$$

where $D(A) = \{w \in X : w, \dot{w}$ are absolutely continuous, $\ddot{w} \in X, w(0) = w(\pi) = 0\}$. Then

$$Aw = \sum_{n=1}^{\infty} -n^2 (w, e_n) e_n, \quad w \in D(A),$$

and A is the infinitesimal generator of a strongly continuous cosine family $C(t)$, $t \in \mathbb{R}$, in X given by

$$C(t)w = \sum_{n=1}^{\infty} \cos nt (w, e_n) e_n, \quad w \in X.$$

Let us for $g_1(t,x,w,p)$, $p \in \mathbb{R}^m$, assume that there is a continuous $\rho(t,r) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ and a real constant $1 \le \gamma$ such that

(g1) $g_1(t,x,0,0) = 0$,

(g2) $|g_1(t,x,w,p) - g_1(t,x,w,q)| \le \rho(t,|w|)|p - q|$,

(g3) $|g_1(t,x,w_1,p) - g_1(t,x,w_2,p)| \le \rho(t,|w_1| + |w_2|)|w_1 - w_2|$.

Let

$$g(t,w)x = g_1(t,x,w,Dw).$$

Then noting that

$$\begin{aligned}
&\big\|g(t,w_1)-g(t,w_2)\big\|_{0,2}^2 \\
&\quad\le 2\int_\Omega \big|g_1(t,x,w_1,Dw_1)-g_1(t,x,w_2,Dw_2)\big|^2\,du \\
&\qquad + 2\int_\Omega \big|g_1(t,x,w_1,Dw_1)-g_1(t,x,w_2,Dw_2)\big|^2\,du,
\end{aligned}$$

it follows from (g1)-(g3) that

$$\big\|g(t,w_1)-g(t,w_2)\big\|_{0,2}^2 \le L\big(\|w_1\|_{D(A)},\|y\|_{D(A)}\big)\|w_1-w_2\|_{D(A)},$$

where $L(\|w_1\|_{D(A)},\|w_2\|_{D(A)})$ is a constant depending on $\|w_1\|_{D(A)}$ and $\|w_2\|_{D(A)}$. We set

$$F(t,w)=\int_0^t k(t-s)g\big(s,w(s)\big)\,ds,$$

where k belongs to $L^2(0,T)$.

Let $U=X$, $0<\alpha<T$, and define the intercept controller operator B_α on $L^2(0,T;X)$ by

$$B_\alpha u(t)=\begin{cases}0, & 0\le t<\alpha,\\ u(t), & \alpha\le t\le T,\end{cases}$$

for $u\in L^2(0,T;X)$ (see [10]). For a given $p\in L^2(0,T;X)$ let us choose a control function u satisfying

$$u(t)=\begin{cases}0, \quad 0\le t<\alpha,\\ p(t)+\frac{\alpha}{T-\alpha}C(t-\frac{\alpha}{T-\alpha}(t-\alpha))p(\frac{\alpha}{T-\alpha}(t-\alpha)), \quad \alpha\le t\le T.\end{cases}$$

Then $u\in L^2(0,T;X)$ and $\hat{S}p=\hat{S}B_\alpha u$. From

$$\begin{aligned}
\|B_\alpha u\|_{L^2(0,T;X)} &= \|u\|_{L^2(\alpha,T;X)} \\
&\le \|p\|_{L^2(\alpha,T;X)}+Ke^{\omega T}\left\|p\left(\frac{\alpha}{T-\alpha}(\cdot-\alpha)\right)\right\|_{L^2(\alpha,T;X)} \\
&\le \left(1+Ke^{\omega T}\sqrt{\frac{T-\alpha}{\alpha}}\right)\|p\|_{L^2(0,T;X)},
\end{aligned}$$

it follows that the controller B_α satisfies Assumption (B). Therefore, from Theorem 3.1, we see that the nonlinear system given by (3.9) is approximately controllable on $[0,T]$. □

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors have made equal contributions. All authors read and approved the final manuscript.

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