

ARGUMENT ESTIMATES OF CERTAIN MULTIVALENT FUNCTIONS INVOLVING A LINEAR OPERATOR

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The purpose of this paper is to derive some argument properties of certain multivalent functions in the open unit disk involving a linear operator. We also investigate their integral preserving property in a sector.

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1. Introduction. Let \mathcal{A}_p denote the class of functions of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{n+p} z^{n+p} \quad (p \in \mathbb{N} = \{1, 2, \dots\}) \quad (1.1)$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. A function $f \in \mathcal{A}_p$ is said to be p -valently starlike of order α in \mathcal{U} , if it satisfies

$$\operatorname{Re} \left\{ \frac{z f'(z)}{f(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in \mathcal{U}). \quad (1.2)$$

We denote this class by $\mathcal{S}_p^*(\alpha)$. A function $f \in \mathcal{A}_p$ is said to be p -valently convex of order α in \mathcal{U} , if it satisfies

$$\operatorname{Re} \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > \alpha \quad (0 \leq \alpha < p; z \in \mathcal{U}). \quad (1.3)$$

The class of p -valently convex functions of order α is denoted by $\mathcal{K}_p(\alpha)$. It follows from (1.2) and (1.3) that

$$f \in \mathcal{K}_p(\alpha) \iff \frac{z f'}{p} \in \mathcal{S}_p(\alpha). \quad (1.4)$$

Further, a function $f \in \mathcal{A}_p$ is said to be p -valently close-to-convex of order β and type α , if there exists a function $g \in \mathcal{S}_p^*(\alpha)$ such that

$$\operatorname{Re} \left\{ \frac{z f'(z)}{g(z)} \right\} > \beta \quad (0 \leq \alpha, \beta < p; z \in \mathcal{U}). \quad (1.5)$$

It is well known (see [10]) that every p -valently close-to-convex function is p -valent in \mathcal{U} .

For arbitrary fixed real numbers A and B ($-1 \leq B < A \leq 1$), let $\mathcal{P}(A, B)$ denote the class of functions of the form

$$\phi(z) = 1 + c_1 z + c_2 z^2 + \dots \quad (1.6)$$

which are analytic in \mathcal{U} and satisfies the condition

$$\phi(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathcal{U}), \tag{1.7}$$

where the symbol \prec stands for subordination. The class $\mathcal{P}(A, B)$ was introduced and studied by Janowski [8].

We note that a function $\phi \in \mathcal{P}(A, B)$, if and only if

$$\begin{aligned} \left| \phi(z) - \frac{1 - AB}{1 - B^2} \right| &< \frac{A - B}{1 - B^2} \quad (B \neq -1, z \in \mathcal{U}), \\ \operatorname{Re} \{ \phi(z) \} &> \frac{1 - A}{2} \quad (B = -1, z \in \mathcal{U}). \end{aligned} \tag{1.8}$$

For a function $f \in \mathcal{A}$, given by (1.1), the generalized Bernardi-Libera-Livingston integral operator F [1] is defined by

$$\begin{aligned} F(z) &= \frac{\gamma + p}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt \\ &= z^p + \sum_{n=1}^\infty \frac{\gamma + p}{\gamma + p + n} a_{n+p} z^{n+p} \quad (\gamma > -p; z \in \mathcal{U}). \end{aligned} \tag{1.9}$$

It readily follows from (1.9) that

$$f \in \mathcal{A}_p \implies F \in \mathcal{A}_p. \tag{1.10}$$

Let

$$\phi_p(a, c; z) = \sum_{n=0}^\infty \frac{(a)_n}{(c)_n} z^{n+p} \quad (c \neq 0, -1, -2, \dots; z \in \mathcal{U}), \tag{1.11}$$

and we define a linear operator $L_p(a, c)$ on \mathcal{A}_p by

$$L_p(a, c)f(z) = \phi_p(a, c; z) * f(z) \quad (z \in \mathcal{U}), \tag{1.12}$$

where $(x)_n = \Gamma(n + x)/\Gamma(x)$ and the symbol $*$ is the Hadamard product or convolution. Clearly, $L_p(a, c)$ maps \mathcal{A}_p into itself. Further, $L_p(a, a)$ is the identity operator and

$$L_p(a, c) = L_p(a, b)L_p(b, c) = L_p(b, c)L_p(a, b) \quad (b, c \neq 0, -1, -2, \dots). \tag{1.13}$$

Thus, if $a \neq 0, -1, -2, \dots$, then $L_p(a, c)$ has an inverse $L_p(c, a)$. We also observe that for $f \in \mathcal{A}_p$,

$$L_p(p + 1, p)f(z) = \frac{zf'(z)}{p}, \quad L_p(\mu + p, 1)f(z) = D^{\mu+p-1}f(z), \tag{1.14}$$

where μ ($\mu > -p$) is any real number. In case of $p = 1$ and $\mu \in \mathbb{N}$, $D^\mu f(z)$ is the Ruscheweyh derivative [14]. The operator $L_p(a, c)$ was introduced and studied by Saitoh and Nunokawa [15]. This operator is a generalization of the linear operator

$L(a, c)$ introduced by Carlson and Shaffer [3] in their systemic investigation of certain classes of starlike, convex, and prestarlike hypergeometric functions.

In the present paper, we give some argument properties of certain class of analytic functions in \mathcal{A}_p involving the linear operator $L_p(a, c)$. An application of a certain integral operator is also considered. The results obtained here, besides extending the works of Bulboacă [2], Chichra [4], Cho et al. [5], Fukui et al. [6], Libera [9], Nunokawa [13], and Sakaguchi [16], it yields a number of new results.

2. Main results. To establish our main results, we need the following lemmas.

LEMMA 2.1 [11]. *Let $h(z)$ be convex (univalent) in \mathcal{U} and let $\psi(z)$ be analytic in \mathcal{U} with $\text{Re}\{\psi(z)\} \geq 0$. If $\phi(z)$ is analytic in \mathcal{U} and $\phi(0) = \psi(0)$, then*

$$\phi(z) + \psi(z)z\phi'(z) \prec h(z) \quad (z \in \mathcal{U}) \tag{2.1}$$

implies

$$\phi(z) \prec h(z) \quad (z \in \mathcal{U}). \tag{2.2}$$

LEMMA 2.2 [12]. *Let $\phi(z)$ be analytic in \mathcal{U} , $\phi(0) = 1$, $\phi(z) \neq 0$ in \mathcal{U} and suppose that there exists a point $z_0 \in \mathcal{U}$ such that*

$$\begin{aligned} |\arg \phi(z)| &< \frac{\pi}{2}\eta \quad (|z| < |z_0|), \\ |\arg \phi(z_0)| &= \frac{\pi}{2}\eta, \end{aligned} \tag{2.3}$$

where $\eta > 0$. Then

$$\frac{z_0\phi'(z_0)}{\phi(z_0)} = ik\eta, \tag{2.4}$$

where

$$\begin{aligned} k &\geq \frac{1}{2}\left(d + \frac{1}{d}\right) \quad \text{when } \arg \phi(z_0) = \frac{\pi}{2}\eta, \\ k &\leq -\frac{1}{2}\left(d + \frac{1}{d}\right) \quad \text{when } \arg \phi(z_0) = -\frac{\pi}{2}\eta, \end{aligned} \tag{2.5}$$

where

$$\phi(z_0)^{1/\eta} = \pm id \quad (d > 0). \tag{2.6}$$

We now derive the following theorem.

THEOREM 2.3. *Let $a > 0$, $-1 \leq B < A \leq 1$, $f \in \mathcal{A}_p$, and suppose that $g \in \mathcal{A}_p$ satisfies*

$$\frac{L_p(a+1, c)g(z)}{L_p(a, c)g(z)} \prec \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}). \tag{2.7}$$

If

$$\begin{aligned} &\left| \arg \left\{ (1-\lambda) \frac{L_p(a, c)f(z)}{L_p(a, c)g(z)} + \lambda \frac{L_p(a+1, c)f(z)}{L_p(a+1, c)g(z)} - \beta \right\} \right| \\ &< \frac{\pi}{2}\delta \quad (\lambda \geq 0; 0 \leq \beta < 1; 0 < \delta \leq 1; z \in \mathcal{U}), \end{aligned} \tag{2.8}$$

then

$$\left| \arg \left\{ \frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} - \beta \right\} \right| < \frac{\pi}{2} \eta \quad (z \in \mathcal{U}), \tag{2.9}$$

where η ($0 < \eta \leq 1$) is the solution of the equation

$$\delta = \begin{cases} \eta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\lambda \eta \sin(\pi/2)(1-t(A,B))}{a(1+A)/(1+B) + \lambda \eta \cos(\pi/2)(1-t(A,B))} \right\}, & \text{for } B \neq -1, \\ \eta, & \text{for } B = -1, \end{cases} \tag{2.10}$$

when

$$t(A,B) = \frac{2}{\pi} \sin^{-1} \left(\frac{A-B}{1-AB} \right). \tag{2.11}$$

PROOF. Let

$$\frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} = \beta + (1-\beta)\phi(z). \tag{2.12}$$

Then $\phi(z)$ is analytic in \mathcal{U} with $\phi(0) = 1$. On differentiating both sides of (2.12) and using the identity

$$z(L_p(a,c)f(z))' = aL_p(a+1,c)f(z) - (a-p)L_p(a,c)f(z) \tag{2.13}$$

in the resulting equation, we deduce that

$$(1-\lambda) \frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} + \lambda \frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)} - \beta = (1-\beta) \left\{ \phi(z) + \frac{\lambda z \phi'(z)}{ar(z)} \right\}, \tag{2.14}$$

where

$$r(z) = \frac{L_p(a+1,c)g(z)}{L_p(a,c)g(z)}. \tag{2.15}$$

If we let

$$r(z) = \rho e^{(\pi\theta/2)i}, \tag{2.16}$$

then from (2.7) followed by (1.8), it follows that

$$\frac{1-A}{1-B} < \rho < \frac{1+A}{1+B}, \tag{2.17}$$

$$-t(A,B) < \theta < t(A,B) \quad \text{for } B \neq -1,$$

when $t(A,B)$ is given by (2.11), and

$$\frac{1-A}{2} < \rho < \infty, \tag{2.18}$$

$$-1 < \theta < 1 \quad \text{for } B = -1.$$

Let $h(z)$ be the function which maps onto the angular domain $\{w : |\arg\{w\}| < (\pi/2)\delta\}$ with $h(0) = 1$. Applying Lemma 2.1 for this $h(z)$ with $\psi(z) = \lambda/(ar(z))$, we see that $\operatorname{Re} \phi(z) > 0$ in \mathcal{U} and hence $\phi(z) \neq 0$ in \mathcal{U} .

If there exists a point $z_0 \in \mathcal{U}$ such that conditions (2.3) are satisfied, then by Lemma 2.2 we obtain (2.4) under restrictions (2.5) and (2.6).

At first, suppose that $p(z_0)^{1/\eta} = id$ ($d > 0$). For the case $B \neq -1$, we obtain

$$\begin{aligned} & \arg \left\{ (1-\lambda) \frac{L_p(a,c)f(z_0)}{L_p(a,c)g(z_0)} + \lambda \frac{L_p(a+1,c)f(z_0)}{L_p(a+1,c)g(z_0)} - \beta \right\} \\ &= \arg \phi(z_0) + \arg \left\{ 1 + \frac{\lambda}{ar(z_0)} \frac{z_0 \phi'(z_0)}{\phi(z_0)} \right\} \\ &= \frac{\pi}{2} \eta + \arg \left\{ 1 + i\eta k \lambda \frac{e^{-(\pi\theta/2)i}}{\rho a} \right\} \tag{2.19} \\ &= \frac{\pi}{2} \eta + \tan^{-1} \left\{ \frac{\lambda \eta k \sin(\pi/2)(1-\theta)}{\rho a + \lambda \eta k \cos(\pi/2)(1-\theta)} \right\} \\ &\geq \frac{\pi}{2} \eta + \tan^{-1} \left\{ \frac{\lambda \eta \sin(\pi/2)(1-t(A,B))}{a(1+A)/(1+B) + \lambda \eta \cos(\pi/2)(1-t(A,B))} \right\} \\ &\geq \frac{\pi}{2} \delta, \end{aligned}$$

where δ and $t(A,B)$ are given by (2.10) and (2.11), respectively. Similarly, for the case $B = -1$, we have

$$\arg \left\{ (1-\lambda) \frac{L_p(a,c)f(z_0)}{L_p(a,c)g(z_0)} + \lambda \frac{L_p(a+1,c)f(z_0)}{L_p(a+1,c)g(z_0)} - \beta \right\} \geq \frac{\pi}{2} \eta. \tag{2.20}$$

This is a contradiction to the assumption of our theorem.

Next, suppose that $\phi(z_0)^{1/\eta} = -id$ ($d > 0$). For the case $B \neq -1$, applying the same method as above, we have

$$\begin{aligned} & \arg \left\{ (1-\lambda) \frac{L_p(a,c)f(z_0)}{L_p(a,c)g(z_0)} + \lambda \frac{L_p(a+1,c)f(z_0)}{L_p(a+1,c)g(z_0)} - \beta \right\} \\ &\leq -\frac{\pi}{2} \eta - \tan^{-1} \left\{ \frac{\lambda \eta \sin(\pi/2)(1-t(A,B))}{a(1+A)/(1+B) + \lambda \eta \cos(\pi/2)(1-t(A,B))} \right\} \tag{2.21} \\ &\leq -\frac{\pi}{2} \delta, \end{aligned}$$

where δ and $t(A,B)$ are given by (2.10) and (2.11), respectively and for the case $B = -1$, we have

$$\arg \left((1-\lambda) \frac{L_p(a,c)f(z_0)}{L_p(a,c)g(z_0)} + \lambda \frac{L_p(a+1,c)f(z_0)}{L_p(a+1,c)g(z_0)} - \beta \right) \leq -\frac{\pi}{2} \eta \tag{2.22}$$

which contradicts the assumption. Therefore we complete the proof of the theorem. \square

REMARK 2.4. For $a = c = p$, $A = 1$, $B = -1$, and $\lambda = 1$, [Theorem 2.3](#) is the recent result obtained by Nunokawa [[13](#)].

Taking $a = \mu + p$ ($\mu > -p$), $c = 1$, $A = 1$, and $B = 0$ in [Theorem 2.3](#), we have the following corollary.

COROLLARY 2.5. *If $f \in \mathcal{A}_p$ satisfies*

$$\left| \arg \left\{ (1-\lambda) \frac{D^{\mu+p-1}f(z)}{D^{\mu+p-1}g(z)} + \lambda \frac{D^{\mu+p}f(z)}{D^{\mu+p}g(z)} - \beta \right\} \right| < \frac{\pi}{2} \delta \quad (\lambda \geq 0; 0 < \delta \leq 1; 0 \leq \beta < 1; z \in \mathcal{U}) \tag{2.23}$$

for some $g \in \mathcal{A}_p$ satisfying the condition

$$\left| \frac{D^{\mu+p}g(z)}{D^{\mu+p-1}g(z)} - 1 \right| < \alpha \quad (0 < \alpha \leq 1; z \in \mathcal{U}), \tag{2.24}$$

then

$$\left| \arg \left\{ \frac{D^{\mu+p-1}f(z)}{D^{\mu+p-1}g(z)} \right\} \right| < \frac{\pi}{2} \eta \quad (z \in \mathcal{U}), \tag{2.25}$$

where η ($0 < \eta \leq 1$) is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\lambda \eta \sin(\pi/2 - \sin^{-1} \alpha)}{(\mu + p)(1 + \alpha) + \lambda \eta \cos(\pi/2 - \sin^{-1} \alpha)} \right\}. \tag{2.26}$$

Letting $B \rightarrow A$ ($A < 1$) and $g(z) = z^p$ in [Theorem 2.3](#), we get the following corollary.

COROLLARY 2.6. *If $f \in \mathcal{A}_p$ satisfies*

$$\left| \arg \left\{ (1-\lambda) \frac{L_p(a,c)f(z)}{z^p} + \lambda \frac{L_p(a+1,c)f(z)}{z^p} - \beta \right\} \right| < \frac{\pi}{2} \delta \quad (a > 0; \lambda \geq 0; 0 \leq \beta < 1; 0 < \delta \leq 1; z \in \mathcal{U}), \tag{2.27}$$

then

$$\left| \arg \left\{ \frac{L_p(a,c)f(z)}{z^p} - \beta \right\} \right| < \frac{\pi}{2} \eta \quad (z \in \mathcal{U}), \tag{2.28}$$

where η ($0 < \eta \leq 1$) is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\lambda \eta}{a} \right\}. \tag{2.29}$$

COROLLARY 2.7. *Under the hypothesis of [Corollary 2.6](#), we have*

$$| \arg \{H'(z) - \beta\} | < \frac{\pi}{2} \eta \quad (z \in \mathcal{U}), \tag{2.30}$$

where the function $H(z)$ is defined in \mathcal{U} by

$$H(z) = \int_0^z \frac{L_p(a,c)f(t)}{t^p} dt \tag{2.31}$$

and η ($0 < \eta \leq 1$) is the solution of (2.29).

REMARK 2.8. Taking $a = c = p$, $\lambda = 1$, and $\beta = 0$ in Corollary 2.6, $a = c = p$ and $\beta = 0$ in Corollary 2.7, we get the corresponding results obtained by Cho et al. [5].

Setting $A = 1 - (2\alpha/p)$ ($0 \leq \alpha < p$), $B = -1$, and $\delta = 1$ in Theorem 2.3, we have the following corollary.

COROLLARY 2.9. Let $a > 0$, $f \in \mathcal{A}_p$, and $g \in \mathcal{S}_p^*(\alpha)$. If

$$\operatorname{Re} \left\{ (1-\lambda) \frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} + \lambda \frac{L_p(a+1,c)f(z)}{L_p(a+1,c)g(z)} \right\} > \beta \quad (\lambda \geq 0; 0 \leq \beta < 1; z \in \mathcal{U}), \tag{2.32}$$

then

$$\operatorname{Re} \left\{ \frac{L_p(a,c)f(z)}{L_p(a,c)g(z)} \right\} > \beta \quad (z \in \mathcal{U}). \tag{2.33}$$

REMARK 2.10. For $a = c = p = 1$ and $\alpha = 0$, Corollary 2.9 is the result by Bulboacă [2]. If we put $a = c = p = 1$, $\beta = 0$, and $g(z) = z$ in Corollary 2.9, then we have the result due to Chichra [4]. Further, taking $a = c = p$, $\lambda = 1$, and $\alpha = \beta = 0$ in Corollary 2.9, we get the corresponding results of Libera [9] and Sakaguchi [16].

THEOREM 2.11. If $f \in \mathcal{A}_p$ satisfies

$$\left| \arg \left\{ \frac{L_p(a,c)f(z)}{z^p} - \beta \right\} \right| < \frac{\pi}{2} \delta \quad (0 \leq \beta < 1; 0 < \delta \leq 1; z \in \mathcal{U}), \tag{2.34}$$

then

$$\left| \arg \left\{ \frac{(\gamma+p) \int_0^z t^{\gamma-1} L_p(a,c)f(t) dt}{z^{\gamma+p}} - \beta \right\} \right| < \frac{\pi}{2} \eta \quad (0 < \gamma+p; z \in \mathcal{U}), \tag{2.35}$$

where η ($0 < \eta \leq 1$) is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\eta}{\gamma+p} \right\}. \tag{2.36}$$

PROOF. Consider the function $\phi(z)$ defined in \mathcal{U} by

$$\frac{(\gamma+p) \int_0^z t^{\gamma-1} L_p(a,c)f(t) dt}{z^{\gamma+p}} = \beta + (1-\beta)\phi(z). \tag{2.37}$$

Then $\phi(z)$ is analytic in \mathcal{U} with $\phi(0) = 1$. Differentiating both sides of (2.37) and simplifying, we get

$$\frac{L_p(a,c)f(z)}{z^p} - \beta = (1-\beta) \left\{ \phi(z) + \frac{z\phi'(z)}{\gamma+p} \right\}. \tag{2.38}$$

Now, by using Lemma 2.1 and a similar method in the proof of Theorem 2.3, we get (2.35). □

Taking $a = p + 1$, $c = p$, $\beta = \rho/p$, and $\delta = 1$ in [Theorem 2.11](#), we have the following corollary.

COROLLARY 2.12. *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re} \left\{ \frac{f'(z)}{z^{p-1}} \right\} > \rho \quad (0 \leq \rho < p; z \in \mathcal{U}), \quad (2.39)$$

then

$$\left| \arg \left\{ \frac{(y+p) \int_0^z t^{y-1} f'(t) dt}{z^{y+p}} - \rho \right\} \right| < \frac{\pi}{2} \eta \quad (z \in \mathcal{U}), \quad (2.40)$$

where η ($0 < \eta \leq 1$) is the solution of the equation

$$\eta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\eta}{y+p} \right\} = 1. \quad (2.41)$$

THEOREM 2.13. *If $f \in \mathcal{A}_p$ satisfies*

$$\left| \arg \left\{ \frac{L_p(a+1, c)f(z)}{L_p(a, c)f(z)} - \frac{a-p-y}{a} \right\} \right| < \frac{\pi}{2} \delta \quad (a > 0; p+y > 0; 0 < \delta \leq 1; z \in \mathcal{U}), \quad (2.42)$$

then

$$\left| \arg \left\{ \frac{z^y L_p(a, c)f(z)}{\int_0^z t^{y-1} L_p(a, c)f(t) dt} \right\} \right| < \frac{\pi}{2} \eta \quad (z \in \mathcal{U}), \quad (2.43)$$

where η ($0 < \eta \leq 1$) is the solution of [\(2.36\)](#).

PROOF. Our proof of [Theorem 2.13](#) is much akin to that of [Theorem 2.3](#). Indeed, in place of [\(2.37\)](#), we define the function $\phi(z)$ by

$$\phi(z) = \frac{z^y L_p(a, c)f(z)}{(y+p) \int_0^z t^{y-1} L_p(a, c)f(t) dt} \quad (z \in \mathcal{U}), \quad (2.44)$$

and apply [Lemma 2.1](#) (with $\psi(z) = 1/(y+p)$) as before. We choose to skip the details involved. \square

Setting $a = c = p$ and $\delta = 1$ in [Theorem 2.13](#), we obtain the following corollary.

COROLLARY 2.14. *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > -y \quad (y+p > 0; z \in \mathcal{U}), \quad (2.45)$$

then

$$\left| \arg \left\{ \frac{z^y f(z)}{\int_0^z t^{y-1} f(t) dt} \right\} \right| < \frac{\pi}{2} \eta \quad (z \in \mathcal{U}), \quad (2.46)$$

where η ($0 < \eta \leq 1$) is the solution of [\(2.41\)](#).

Replacing $f(z)$ by $zf'(z)/p$ in [Corollary 2.14](#), we deduce the following corollary.

COROLLARY 2.15. *If $f \in \mathcal{A}_p$ satisfies*

$$\operatorname{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > -\gamma \quad (\gamma + p > 0; z \in \mathcal{U}), \tag{2.47}$$

then

$$\left| \arg \left\{ \frac{zf'(z)}{f(z) - (\gamma/z^\gamma) \int_0^z t^{\gamma-1} f(t) dt} \right\} \right| < \frac{\pi}{2} \eta \quad (z \in \mathcal{U}), \tag{2.48}$$

where η ($0 < \eta \leq 1$) is the solution of (2.41).

By setting $\gamma = 0$ in Corollary 2.15, we have the following corollary.

COROLLARY 2.16. *If $f \in \mathcal{K}_p(0)$, then*

$$\left| \arg \left\{ \frac{zf'(z)}{f(z)} \right\} \right| < \frac{\pi}{2} \eta \quad (z \in \mathcal{U}), \tag{2.49}$$

where η ($0 < \eta \leq 1$) is the solution of the equation:

$$\eta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\eta}{p} \right\} = 1. \tag{2.50}$$

Similarly, we have the following theorem.

THEOREM 2.17. *If $f \in \mathcal{A}_p$ satisfies*

$$\left| \arg \left\{ \frac{L_p(a+1, c)f(z)}{L_p(a, c)f(z)} - \beta \right\} \right| < \frac{\pi}{2} \delta \quad (a > 0; 0 \leq \beta < 1; 0 < \delta \leq 1; z \in \mathcal{U}), \tag{2.51}$$

then

$$\left| \arg \left\{ \frac{L_p(a, c)f(z)}{z^p} \right\} \right| < \frac{\pi}{2} \eta \quad (z \in \mathcal{U}), \tag{2.52}$$

where η ($0 < \eta \leq 1$) is the solution of the equation

$$\delta = \frac{2}{\pi} \tan^{-1} \left\{ \frac{\eta}{(1-\beta)a} \right\}. \tag{2.53}$$

THEOREM 2.18. *Let $f \in \mathcal{A}_p$ and suppose that*

$$B < A \leq B + \frac{p(1-B)}{a} \quad (a > 0; -1 \leq B < A \leq 1). \tag{2.54}$$

If

$$\left| \arg \left\{ (1-\lambda) \frac{L_p(a+1, c)f(z)}{L_p(a, c)g(z)} + \lambda \frac{(L_p(a+1, c)f(z))'}{(L_p(a, c)g(z))'} - \beta \right\} \right| < \frac{\pi}{2} \delta \quad (\lambda \geq 0; 0 \leq \beta < 1; 0 < \delta \leq 1; z \in \mathcal{U}), \tag{2.55}$$

for some $g \in \mathcal{A}_p$ satisfying

$$\frac{L_p(a+1, c)g(z)}{L_p(a, c)g(z)} < \frac{1+Az}{1+Bz} \quad (z \in \mathcal{U}), \tag{2.56}$$

then

$$\left| \arg \left\{ \frac{L_p(a+1, c)f(z)}{L_p(a, c)g(z)} - \beta \right\} \right| < \frac{\pi}{2} \eta \quad (z \in \mathcal{U}), \tag{2.57}$$

where η ($0 < \eta \leq 1$) is the solution of the equation

$$\delta = \begin{cases} \eta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\lambda \eta \sin(\pi/2)(1-t(A, B))}{(p(1+B)+a(A-B))/(1+B)+\lambda \eta \cos(\pi/2)(1-t(A, B))} \right\}, & \text{for } B \neq -1, \\ \eta, & \text{for } B = -1, \end{cases} \tag{2.58}$$

when

$$t(A, B) = \frac{2}{\pi} \sin^{-1} \left(\frac{a(A-B)}{p(1-B^2) - aB(A-B)} \right). \tag{2.59}$$

PROOF. Let

$$\frac{L_p(a+1, c)f(z)}{L_p(a, c)g(z)} = \beta + (1-\beta)\phi(z), \quad r(z) = \frac{L_p(a+1, c)g(z)}{L_p(a, c)g(z)}, \tag{2.60}$$

we have

$$(1-\lambda) \frac{L_p(a+1, c)f(z)}{L_p(a, c)g(z)} + \lambda \frac{(L_p(a+1, c)f(z))'}{(L_p(a+1, c)g(z))'} - \beta = (1-\beta) \left\{ \phi(z) + \frac{\lambda z \phi'(z)}{ar(z) + p - a} \right\}. \tag{2.61}$$

The remaining part of the proof of [Theorem 2.18](#) is similar to that of [Theorem 2.3](#). So we omit the details. □

Put $a = c = p$, $\lambda = 1$, $A = \alpha/p$, and $B = 0$ in [Theorem 2.18](#), we have the following corollary.

COROLLARY 2.19. *If $f \in \mathcal{A}_p$ satisfies*

$$\left| \arg \left\{ \frac{(zf'(z))'}{g'(z)} - \beta \right\} \right| < \frac{\pi}{2} \delta \quad (0 \leq \beta < p; 0 < \delta \leq 1; z \in \mathcal{U}), \tag{2.62}$$

for some $g \in \mathcal{A}_p$ satisfying the condition

$$\left| \frac{zg'(z)}{g(z)} - p \right| < \alpha \quad (0 < \alpha \leq p; z \in \mathcal{U}), \tag{2.63}$$

then

$$\left| \arg \left\{ \frac{zf'(z)}{g(z)} - \beta \right\} \right| < \frac{\pi}{2} \eta \quad (z \in \mathcal{U}), \tag{2.64}$$

where η ($0 < \eta \leq 1$) is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\eta \sin(\pi/2 - \sin^{-1}(\alpha/p))}{p + \alpha + \eta \cos(\pi/2 - \sin^{-1}(\alpha/p))} \right\}. \tag{2.65}$$

LEMMA 2.20. Let

$$\alpha = \xi + \frac{\xi}{\gamma + p + a\xi} \quad (0 \leq (a-1)/a < \xi < \alpha < 1) \tag{2.66}$$

and the function $G(z)$ be defined by

$$G(z) = \frac{\gamma + p}{z^\gamma} \int_0^z t^{\gamma-1} g(t) dt \quad (g \in \mathcal{A}_p) \tag{2.67}$$

for $\gamma > (a\xi^2 + (p+1-a)\xi - p)/(1-\xi)$. If $g \in \mathcal{A}_p$ satisfies

$$\left| \frac{L_p(a+1, c)g(z)}{L_p(a, c)g(z)} - 1 \right| < \alpha \quad (z \in \mathcal{U}), \tag{2.68}$$

then

$$\left| \frac{L_p(a+1, c)G(z)}{L_p(a, c)G(z)} - 1 \right| < \xi \quad (z \in \mathcal{U}). \tag{2.69}$$

PROOF. Defining the function $w(z)$ by

$$\frac{L_p(a+1, c)G(z)}{L_p(a, c)G(z)} = 1 + \xi w(z), \tag{2.70}$$

we see that $w(z)$ is analytic in \mathcal{U} with $w(0) = 0$. Now, using the identities

$$z(L_p(a, c)G(z))' = aL_p(a+1, c)G(z) - (a-p)L_p(a, c)G(z), \tag{2.71}$$

$$z(L_p(a, c)g(z))' = (\gamma+p)L_p(a, c)g(z) - \gamma L_p(a, c)G(z) \tag{2.72}$$

in (2.70), we get

$$\frac{L_p(a, c)G(z)}{L_p(a, c)g(z)} = \frac{\gamma + p}{\gamma + p + a\xi w(z)}. \tag{2.73}$$

Making use of the logarithmic differentiation of both sides of (2.73) and using identity (2.71) for both $g(z)$ and $f(z)$ in the resulting equation, we deduce that

$$\left| \frac{L_p(a+1, c)g(z)}{L_p(a, c)g(z)} - 1 \right| = \xi \left| w(z) + \frac{zw'(z)}{\gamma + p + a\xi w(z)} \right|. \tag{2.74}$$

We assume that there exists a point $z_0 \in \mathcal{U}$ such that $\max_{|z| < |z_0|} |w(z)| = |w(z_0)| = 1$. Then by Jack's lemma [7], we have $z_0 w'(z_0) = k w(z_0)$ ($k \geq 1$). Let $w(z_0) = e^{i\theta}$, and apply this result to $w(z)$ at $z_0 \in \mathcal{U}$, we get

$$\begin{aligned} \left| \frac{L_p(a+1, c)g(z_0)}{L_p(a, c)g(z_0)} - 1 \right| &= \xi \left| 1 + \frac{k}{y+p+a\xi e^{i\theta}} \right| \\ &= \xi \left[\frac{(y+p+k)^2 + 2a\xi(y+p+k)\cos\theta + (a\xi)^2}{(y+p)^2 + 2a\xi(y+p)\cos\theta + (a\xi)^2} \right]^{1/2}. \end{aligned} \tag{2.75}$$

Since the right side of (2.75) is decreasing for $0 \leq \theta < 2\pi$ and $y > \{a\xi^2 + (p+1-a)\xi - p\} / (1-\xi)$, we obtain

$$\left| \frac{L_p(a+1, c)g(z_0)}{L_p(a, c)g(z_0)} - 1 \right| \leq \frac{\xi(y+p+1+a\xi)}{y+p+a\xi}, \tag{2.76}$$

which contradicts our hypothesis and hence we get

$$|w(z)| = \frac{1}{\xi} \left| \frac{L_p(a+1, c)G(z)}{L_p(a, c)G(z)} - 1 \right| < 1 \quad (z \in \mathcal{U}). \tag{2.77}$$

This completes the proof of Lemma 2.20. □

REMARK 2.21. We note that for $a = c = p = 1$, Lemma 2.20 yields the corresponding result obtained by Fukui et al. [6].

THEOREM 2.22. Let α be as given in (2.66) and $y^* > \max\{(a\xi^2 + (p+1-a)\xi - p) / (1-\xi), a\xi - p\}$. If $f \in \mathcal{A}_p$ satisfies

$$\left| \arg \left\{ \frac{L_p(a+1, c)f(z)}{L_p(a, c)g(z)} - \beta \right\} \right| < \frac{\pi}{2} \delta \quad (0 \leq \beta < 1; 0 < \delta \leq 1; z \in \mathcal{U}), \tag{2.78}$$

for some $f \in \mathcal{A}_p$ satisfying condition (2.68), then

$$\left| \arg \left\{ \frac{L_p(a+1, c)F(z)}{L_p(a, c)G(z)} - \beta \right\} \right| < \frac{\pi}{2} \eta \quad (z \in \mathcal{U}), \tag{2.79}$$

where the function $F(z)$ and $G(z)$ are defined for y^* by (1.9) and (2.67), respectively and η ($0 < \eta \leq 1$) is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\eta \sin(\pi/2 - \sin^{-1}(a\xi/(y^*+p)))}{y^*+p+a\xi+\eta \cos(\pi/2 - \sin^{-1}(a\xi/(y^*+p)))} \right\}. \tag{2.80}$$

PROOF. Consider the function $\phi(z)$ defined in \mathcal{U} by

$$\frac{L_p(a+1, c)F(z)}{L_p(a, c)G(z)} = \beta + (1-\beta)\phi(z). \tag{2.81}$$

Then $\phi(z)$ is analytic in \mathcal{U} with $\phi(0) = 1$. Taking logarithmic differentiation on both sides of (2.81) and using identity (2.71) in the resulting equation, we get

$$\frac{z(L_p(a+1,c)F(z))'}{L_p(a+1,c)F(z)} = p - a + a \frac{L_p(a+1,c)G(z)}{L_p(a,c)G(z)} + (1-\beta) \frac{z\phi'(z)}{\beta + (1-\beta)\phi(z)}. \tag{2.82}$$

From the definition of $F(z)$, we have

$$(y^* + p)L_p(a,c)f(z) = a(L_p(a+1,c)F(z))' + y^*L_p(a+1,c)F(z). \tag{2.83}$$

Again, from (2.71) and (2.72), it follows that

$$(y^* + p)L_p(a+1,c)g(z) = zL_p(a+1,c)G(z) + (p + y^* - a)L_p(a,c)G(z). \tag{2.84}$$

Thus, by using (2.83) and (2.84) followed by (2.82), we obtain

$$\frac{L_p(a+1,c)f(z)}{L_p(a,c)g(z)} - \beta = (1-\beta) \left\{ \phi(z) + \frac{z\phi'(z)}{ar(z) + y^* + p - a} \right\}, \tag{2.85}$$

where $r(z) = L_p(a+1,c)G(z)/L_p(a,c)G(z)$. By using Lemma 2.20, we have

$$r(z) < 1 + \xi z \quad (z \in \mathcal{U}), \tag{2.86}$$

where ξ is given by (2.66). Letting

$$ar(z) + y^* + p - a = \rho e^{i\pi\theta/2} \tag{2.87}$$

and using the techniques of Theorem 2.3, the remaining part of the proof of Theorem 2.22 follows. □

REMARK 2.23. We easily find the following:

$$y > \begin{cases} a\xi - p, & \text{if } \frac{a-1}{a} < \xi < \frac{2a-1}{2a}, \\ \frac{2(a-p)-1}{2}, & \text{if } \xi = \frac{2a-1}{2a}, \\ \frac{a\xi^2 + (p+1-a)\xi - p}{1-\xi}, & \text{if } \frac{2a-1}{2a} < \xi < 1. \end{cases} \tag{2.88}$$

Taking $a = c = p$ in Theorem 2.22, we get the following corollary.

COROLLARY 2.24. Let

$$\alpha = \xi + \frac{\xi}{y^* + p(1+\xi)} \quad ((p-1)/p < \xi < \alpha < 1), \tag{2.89}$$

where $\gamma^* > \max\{(p\xi^2 + \xi - p)/(1 - \xi), p(\xi - 1)\}$. If $f \in \mathcal{A}_p$ satisfies

$$\left| \arg \left\{ \frac{zf'(z)}{g(z)} - \beta \right\} \right| < \frac{\pi}{2} \delta \quad (0 \leq \beta < p; 0 < \delta \leq 1; z \in \mathcal{U}) \quad (2.90)$$

for some $g \in \mathcal{A}_p$ satisfying the condition

$$\left| \frac{zg'(z)}{g(z)} - p \right| < p\alpha \quad (z \in \mathcal{U}), \quad (2.91)$$

then

$$\left| \arg \left\{ \frac{zF'(z)}{G(z)} - \beta \right\} \right| < \frac{\pi}{2} \quad (z \in \mathcal{U}), \quad (2.92)$$

where η ($0 < \eta \leq 1$) is the solution of the equation

$$\delta = \eta + \frac{2}{\pi} \tan^{-1} \left\{ \frac{\eta \sin(\pi/2 - \sin^{-1}(p\xi/(\gamma^* + p)))}{\gamma^* + p(1 + \xi) + \eta \cos(\pi/2 - \sin^{-1}(p\xi/(\gamma^* + p)))} \right\}. \quad (2.93)$$

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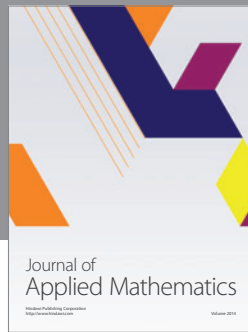

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