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Research Article

Higher-Order Stationary Dispersive Equations on Bounded Intervals

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A boundary value problem for a stationary nonlinear dispersive equation of $2l + 1$ order on an interval $(0, L)$ was considered. The existence, uniqueness, and continuous dependence of a regular solution have been established.

1. Introduction

This work concerns the existence, uniqueness, and continuous dependence of regular solutions to a boundary value problem for one class of nonlinear stationary dispersive equations posed on bounded intervals,

$$au + \sum_{j=1}^{l} (-1)^{j+1} D_x^{2j+1} u + uu_x = f(x), \quad l \in \mathbb{N}, \tag{1}$$

where a is a positive constant. This class of stationary equations appears naturally while one wants to solve a corresponding evolution equation

$$u_t + \sum_{j=1}^{l} (-1)^{j+1} D_x^{2j+1} u + uu_x = 0, \quad l \in \mathbb{N}, \tag{2}$$

making use of an implicit semidiscretization scheme:

$$\frac{u^n - u^{n-1}}{h} + \sum_{j=1}^{l} (-1)^{j+1} D_x^{2j+1} u^n + u^n u_x^n = 0, \quad l \in \mathbb{N}, \tag{3}$$

where $h > 0$, [1]. Comparing (3) with (1), it is clear that $a = 1/h > 0$ and $f(x) = u^{n-1}/h$.

For $l = 1$, we have the well-known generalized KdV equation and for $l = 2$ the Kawahara equation. Initial value problems for the Kawahara equation, which had been derived in [2] as a perturbation of the Korteweg-de Vries (KdV) equation, have been considered in [3–12] and attracted attention due to various applications of those results in mechanics and physics such as dynamics of long small-amplitude waves in various media [13–15]. On the other hand, last years appeared publications on solvability of initial-boundary value problems for dispersive equations (which included the KdV and Kawahara equations) in bounded and unbounded domains [16–23]. In spite of the fact that there is not some clear physical interpretation for the problems on bounded intervals, their study is motivated by numerics [24]. The KdV and Kawahara equations have been developed for unbounded regions of wave propagations; however, if one is interested in implementing numerical schemes to calculate solutions in these regions, there arises the issue of cutting off a spatial domain approximating unbounded domains by bounded ones. In this case, some boundary conditions are needed to specify a solution. Therefore, precise mathematical analysis of mixed problems in bounded domains for dispersive equations is welcome and attracts attention of specialists in this area [16–19, 21, 25].

As a rule, simple boundary conditions at $x = 0$ and $x = 1$ such as $u = u_x = 0|_{x=0}$, $u = u_x = u_{xx} = 0|_{x=1}$ for the Kawahara equation were imposed. Different kind of boundary conditions was considered in [19, 26]. Obviously, boundary conditions for (1) are the same as for (2). Because of that, study of boundary value problems for (1) helps to understand solvability of initial-boundary value problems for (2).

Last years, publications on dispersive equations of higher orders appeared [7, 9, 10, 21, 27]. Here, we propose (1) as a stationary analog of (2) because the last equation includes classical models such as the KdV and Kawahara equations.

The goal of our work is to formulate a correct boundary value problem for (1) and to prove the existence, uniqueness, and continuous dependence on perturbations of $f(x)$ for regular solutions.

The paper has the following structure. Section 1 is Introduction. Section 2 contains formulation of the problem and main results of the article. In Section 3 we give some useful facts. Section 4 is devoted to the boundary value problem for a complete linear equation, necessary to prove in Section 5 the existence of regular solutions for the original problem. Finally, in Section 6 uniqueness is proved which provided certain restriction on f as well as continuous dependence of solutions.

2. Formulation of the Problem and Main Results

For real $a > 0$, consider the following one-dimensional stationary higher-order equation:

$$au + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1}u + uDu = f \quad \text{in } (0, L) \tag{4}$$

subject to boundary conditions

$$D^k u(0) = D^k u(L) = D^l u(L) = 0, \quad k = 0, \dots, l-1, \tag{5}$$

where $0 < L < \infty$, $l \in \mathbb{N}$, $D^i = d^i/dx^i$, $D^1 \equiv D$ are the derivatives of order $i \in \mathbb{N}$, and $f \in L^2(0, L)$ is the given function.

Throughout this paper we adopt the usual notation $(\cdot,\cdot)$, $\|\cdot\|$, and $\|\cdot\|_{H^i}$, $i \in \mathbb{N}$, for the inner product and the norm in $L^2(0, L)$ and the norm in $H^i(0, L)$, respectively [28]. Symbols C_0, C_i, $i \in \mathbb{N}$, mean positive constants appearing during the text.

The main results of the article are the following theorem.

Theorem 1. *Let $f \in L^2(0, L)$. Then for fixed $a > 0$, problem (4)-(5) admits at least one regular solution $u = u(x) \in H^{2l+1}(0, L)$ such that*

$$\|u\|_{H^{2l+1}} \le C_6 \left((1 + x), f^2\right)^{1/2} \tag{6}$$

with C_6 depending only on L, l, a, and $((1 + x), f^2)$. Moreover, if $l \ge 2$ and $((1 + x), f^2)^{1/2} < (2a/3)\sqrt{a\beta/2}$ with $\beta = \min\{a/2, 1\}$, then the solution is uniquely defined and depends continuously on f. For $l = 1$ the uniqueness and continuous dependence are satisfied if $((1 + x), f^2)^{1/2}$ is sufficiently small.

3. Preliminary Results

Lemma 2. *For all $u \in H^1(0, L)$, such that $u(x_0) = 0$ for some $x_0 \in [0, L]$, one has*

$$\sup_{x\in(0,L)} |u(x)| \le \sqrt{2}\, \|u\|^{1/2} \|Du\|^{1/2}. \tag{7}$$

Proof. Let $x_0 \in [0, L]$, such that $u(x_0) = 0$. Then for any $x \in (0, L)$

$$\begin{aligned} u^2(x) &= \int_{x_0}^{x} D\left[u^2(\xi)\right] d\xi \le 2\int_{x_0}^{x} \left|u(\xi)\right| \left|D(\xi)\right| d\xi \\ &\le 2\int_0^L |u(x)|\,|Du(x)|\,dx \le 2\|u\|\,\|Du\|. \end{aligned} \tag{8}$$

From this, the result follows immediately. □

We will use the following version of the Gagliardo-Nirenberg's inequality [29–31].

Theorem 3. *For $1 \le q, r \le \infty$ suppose u belongs to $L^q(0, L)$ and its derivatives of order m belong to $L^r(0, L)$. Then for the derivatives $D^i u$, $0 \le i < m$ the following inequalities hold:*

$$\left\|D^i u\right\|_{L^p} \le K_1 \left\|D^m u\right\|_{L^r}^{\theta} \|u\|_{L^q}^{1-\theta} + K_2 \|u\|_{L^q}, \tag{9}$$

where

$$\frac{1}{p} = i + \theta\left(\frac{1}{r} - m\right) + (1 - \theta)\frac{1}{q}, \tag{10}$$

for all $\theta \in [i/m, 1]$ (the constants K_1, K_2 depending only on L, m, i, q, and r).

We will use the following fixed point theorem [32].

Theorem 4 (Schaefer's fixed point theorem). *Let X be a real Banach Space. Suppose $B : X \to X$ is a compact and continuous mapping. Assume further that the set*

$$\{u \in X \mid u = \lambda Bu \text{ for some } 0 \le \lambda \le 1\} \tag{11}$$

is bounded. Then B has a fixed point.

We start with the linearized version of (4), (5).

4. Linear Problem

Consider the linear equation

$$Au \equiv au + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1}u = f \quad \text{in } (0, L) \tag{12}$$

subject to boundary conditions (5).

Theorem 5. *Let $f \in L^2(0, L)$. Then problem (12)-(5) admits a unique regular solution $u = u(x) \in H^{2l+1}(0, L)$ such that*

$$\|u\|_{H^{2l+1}} \le C_3 \|f\| \tag{13}$$

with C_3 depending only on L and a.

Proof. Denote

$$U(u) \equiv \mathrm{Id}_{2l+1} \begin{pmatrix} u(0) \\ \vdots \\ D^{l-1}u(0) \\ u(L) \\ \vdots \\ D^{l}u(L) \end{pmatrix}, \quad (14)$$

where Id_{2l+1} is the identity matrix of order $2l+1$. Suppose $f \in C([0,L])$ and consider the following problem:

$$\begin{aligned} Au &= f, \\ U(u) &= 0 \end{aligned} \quad (15)$$

as well as the associated homogeneous problem

$$Au = 0, \quad (16)$$

$$U(u) = 0. \quad (17)$$

It is known, [33, 34], that (15) has a unique classical solution if and only if (16)-(17) has only the trivial solution.

Let $u \in C^{2l+1}([0,L])$ be a nontrivial solution of (16)-(17). Multiplying (16) by u and integrating over $(0,L)$, we have

$$a\|u\|^2 + \sum_{j=1}^{l}(-1)^{j+1}\left(D^{2j+1}u, u\right) = 0. \quad (18)$$

By integration by parts and the principle of finite induction, we calculate

$$\begin{aligned} &\left(D^{2j+1}u, u\right) \\ &= \left[\sum_{k=1}^{j}(-1)^{k+1} D^{k-1}u(x)\, D^{(2j+1)-k}u(x)\right]_{x=0}^{x=L} \\ &+ \left[(-1)^{j}\frac{1}{2}\left(D^{j}u(x)\right)^2\right]_{x=0}^{x=L} \end{aligned} \quad (19)$$

for all $j \in \mathbb{N}$. Fixing $l \in \mathbb{N}$ and making use of (5), we find that

$$\begin{aligned} &\left(D^{2j+1}u, u\right) = 0 \quad \text{for } j \in \{1, \ldots, l-1\}, \\ &\left(D^{2l+1}u, u\right) = (-1)^{l+1}\frac{1}{2}\left(D^{l}u(0)\right)^2. \end{aligned} \quad (20)$$

Thus

$$\begin{aligned} \sum_{j=1}^{l}(-1)^{j+1}\left(D^{2j+1}u, u\right) &= (-1)^{l+1}(-1)^{l+1}\frac{1}{2}\left(D^{l}u(0)\right)^2 \\ &= \frac{1}{2}\left(D^{l}u(0)\right)^2; \end{aligned} \quad (21)$$

therefore

$$a\|u\|^2 + \frac{1}{2}\left(D^{l}u(0)\right)^2 = 0 \quad (22)$$

which implies $a\|u\|^2 \leq 0$. Since $a > 0$, it follows that $u \equiv 0$.

Therefore, (15) has a unique classical solution $u \in C^{2l+1}([0,L])$ given by

$$u(x) = \int_0^L G(x,\xi) f(\xi)\, d\xi, \quad (23)$$

where $G : [0,L] \times [0,L] \to \mathbb{R}$ is Green's function associated with problem (16)-(17), [33, 34]. That is,

$$G(x,\xi) = \begin{cases} v(x-\xi) + \sum_{k=0}^{2l} u_k(x)\, d_k(\xi), & 0 \leq \xi \leq x \leq L \\ \sum_{k=0}^{2l} u_k(x)\, d_k(\xi), & 0 \leq x < \xi \leq L, \end{cases} \quad (24)$$

with

$$u_k(x) = v^{2l-k}(x) + \sum_{s=k+1}^{2l} b_{2l+1-s} v^{s-k-1}(x), \qquad k = 0, \ldots, 2l, \quad (25)$$

where b_{2l+1-s} are the coefficients of (12). The function v is a unique solution to the following initial value problem:

$$\begin{aligned} &au + \sum_{j=1}^{l}(-1)^{j+1} D^{2j+1}u = 0 \quad \text{in } \mathbb{R} \\ &D^{2l}u(0) = 1, \\ &D^{i}u(0) = 0, \\ &\qquad i = 0, \ldots, 2l-1, \end{aligned} \quad (26)$$

and the continuous real functions d_k are determined by u_k, v, and (5).

We prove the following estimates.

Estimate I. Multiplying (12) by u, we obtain

$$a\|u\|^2 + \frac{1}{2}\left(D^{l}u(0)\right)^2 = (f, u). \quad (27)$$

By Cauchy-Schwarz's inequality, we get

$$\|u\| \leq \frac{1}{a}\|f\|. \quad (28)$$

Estimate II. Multiplying (12) by $(1+x)u$ and integrating over $(0,L)$ we have

$$\begin{aligned} &a(u, (1+x)u) + \sum_{j=1}^{l}(-1)^{j+1}\left(D^{2j+1}u, (1+x)u\right) \\ &= (f, (1+x)u). \end{aligned} \quad (29)$$

Integration by parts and the principle of finite induction give

$$\begin{aligned}&\left(D^{2j+1}u,(1+x)u\right)\\&=\left[\sum_{k=1}^{j}(-1)^{k+1}(1+x)D^{k-1}u(x)D^{(2j+1)-k}u(x)\right]_{x=0}^{x=L}\\&\quad+\left[\sum_{k=1}^{j}(-1)^{k}kD^{k-1}u(x)D^{2j-k}u(x)\right]_{x=0}^{x=L}\\&\quad+\left[(-1)^{j}\frac{(1+x)}{2}\left(D^{j}u(x)\right)^{2}\right]_{x=0}^{x=L}\\&\quad+(-1)^{j+1}\frac{(2j+1)}{2}\left\|D^{j}u\right\|^{2}\end{aligned}\tag{30}$$

for all $j\in\mathbb{N}$. Fixing $l\in\mathbb{N}$ and making use of (5), we get

$$\begin{aligned}&\left(D^{2j+1}u,(1+x)u\right)=(-1)^{j+1}\frac{(2j+1)}{2}\left\|D^{j}u\right\|^{2}\quad\text{for } j\in\{1,\dots,l-1\},\\&\left(D^{2l+1}u,(1+x)u\right)=(-1)^{l+1}\frac{1}{2}\left(D^{l}u(0)\right)^{2}+(-1)^{l+1}\frac{(2l+1)}{2}\left\|D^{l}u\right\|^{2}.\end{aligned}\tag{31}$$

Therefore

$$\sum_{j=1}^{l}(-1)^{j+1}\left(D^{2j+1}u,(1+x)u\right)=\sum_{j=1}^{l}\left(\frac{2j+1}{2}\right)\left\|D^{j}u\right\|^{2}+\frac{1}{2}\left(D^{l}u(0)\right)^{2}.\tag{32}$$

Applying Schwarz's inequality on the right-hand side of (29), we conclude

$$\|u\|_{H_0^l}\le C_0\|f\|\tag{33}$$

with C_0 depending only on L and a.

Estimate III. Rewriting (12) in the form

$$(-1)^{l+1}D^{2l+1}u=f-au-\sum_{j=1}^{l-1}(-1)^{j+1}D^{2j+1}u,\tag{34}$$

we estimate

$$\left\|D^{2l+1}u\right\|\le\|f\|+a\|u\|+\sum_{j=1}^{l-1}\left\|D^{2j+1}u\right\|.\tag{35}$$

For $l=1$ we have $\sum_{j=1}^{l-1}(-1)^{j+1}D^{2j+1}u=0$ and for $l\ge 2$ denote $J=\{1,\dots,l-1\}$ and

$$\begin{aligned}I_1&=\{j\in J\mid 2j+1\le l\},\\I_2&=\{j\in J\mid l<2j+1<2l+1\}.\end{aligned}\tag{36}$$

Hence we can write

$$\sum_{j=1}^{l-1}\left\|D^{2j+1}u\right\|=\sum_{j\in I_1}\left\|D^{2j+1}u\right\|+\sum_{j\in I_2}\left\|D^{2j+1}u\right\|.\tag{37}$$

Then (35) becomes

$$\left\|D^{2l+1}u\right\|\le\|f\|+a\|u\|+\sum_{j\in I_1}\left\|D^{2j+1}u\right\|+\sum_{j\in I_2}\left\|D^{2j+1}u\right\|.\tag{38}$$

Making use of (33), we get

$$a\|u\|+\sum_{j\in I_1}\left\|D^{2j+1}u\right\|\le(a+l)C_0\|f\|.\tag{39}$$

On the other hand, $l<2j+1<2l+1$ for all $j\in I_2$. Hence, by Theorem 3, there are K_1^j, K_2^j, depending only on L and l, such that

$$\left\|D^{2j+1}u\right\|\le K_1^j\left\|D^{2l+1}u\right\|^{\theta^j}\|u\|^{1-\theta^j}+K_2^j\|u\|,\quad\text{with }\theta^j=\frac{2j+1}{2l+1}.\tag{40}$$

Making use of Young's inequality with $p^j=1/\theta^j$, $q^j=1/(1-\theta^j)$, and arbitrary $\epsilon>0$, we get

$$\left\|D^{2j+1}u\right\|\le\epsilon\left\|D^{2l+1}u\right\|+C_1^j(\epsilon)\|u\|+K_2^j\|u\|,\tag{41}$$

where $C_1^j(\epsilon)=[q^j(p^j\epsilon/(K_1^j)^{p^j})^{q^j/p^j}]^{-1}$. Summing over $j\in I_2$ and making use of (28), we find

$$\sum_{j\in I_2}\left\|D^{2j+1}u\right\|\le l\epsilon\left\|D^{2l+1}u\right\|+\left(\frac{1}{a}\sum_{j\in I_2}\left(C_1^j(\epsilon)+K_2^j\right)\right)\|f\|.\tag{42}$$

Substituting (39), (42) into (38), we obtain

$$\left\|D^{2l+1}u\right\|\le l\epsilon\left\|D^{2l+1}u\right\|+\left(1+(a+l)C_0+\frac{1}{a}\sum_{j\in I_2}\left(C_1^j(\epsilon)+K_2^j\right)\right)\|f\|.\tag{43}$$

Taking $\epsilon=1/2l$, we conclude

$$\left\|D^{2l+1}u\right\|\le C_2\|f\|,\tag{44}$$

where C_2 depends only on L, l, and a.

Again by Theorem 3, for all $i=l+1,\dots,2l$, there are K_1^i, K_2^i depending only on L and l such that

$$\left\|D^{i}u\right\|\le K_1^i\left\|D^{2l+1}u\right\|^{\theta^i}\|u\|^{1-\theta^i}+K_2^i\|u\|,\quad\text{with }\theta^i=\frac{i}{2l+1}.\tag{45}$$

Making use of (28), (44), we obtain

$$\left\|D^i u\right\| \le \left(\frac{K_1^i C_2^{\theta^i}}{a^{1-\theta^i}} + \frac{K_2^i}{a}\right)\|f\|, \quad \text{for } i = l+1, \dots, 2l. \quad (46)$$

Taking into account (33), (44), and (46), we conclude that $u \in H^{2l+1}(0, \mathrm{L})$ and

$$\|u\|_{H^{2l+1}} \le C_3 \|f\| \quad (47)$$

with C_3 depending only on L, l, and a. Uniqueness of u follows from (28). In fact, such calculations must be performed for smooth solutions and the general case can be obtained via density arguments. Therefore, the proof of the Theorem 5 is complete. □

5. Nonlinear Case

Given $u \in H_0^1(0, L)$, set $F := f - uDu$. Clearly, $F \in L^2(0, L)$ and by Lemma 2,

$$\begin{aligned}\|F\| &\le \|f\| + \|uDu\| \\ &\le \|f\| + \left(\sup_{x\in(0,L)} |u(x)|^2\right)^{1/2} \|Du\| \\ &\le \|f\| + \sqrt{2}\|u\|^{1/2}\|Du\|^{3/2}.\end{aligned} \quad (48)$$

By the Young inequality with $p = 4$, $q = 4/3$, and $\epsilon = 1$, we obtain

$$\|F\| \le \|f\| + \|u\|_{H_0^1}^2. \quad (49)$$

Let $w \in H^{2l+1}(0, L)$ be a unique solution of the linear equation

$$aw + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1} w = F \quad \text{in } (0, L) \quad (50)$$

subject to boundary conditions (5). By (47), we know additionally that

$$\|w\|_{H^{2l+1}} \le C_3 \|F\| \le C_3 \left(\|f\| + \|u\|_{H_0^1}^2\right). \quad (51)$$

Let us henceforth write $Bu = w$ whenever w is derived from u via (50), (5). We assert that $B : H_0^1(0, L) \to H_0^1(0, L)$ is compact and continuous.

Indeed, if $\{u_k\}$ is a bounded sequence in $H_0^1(0, L)$, then, in view of estimate (51), we have that sequence $\{w_k\}$ is bounded in $H^{2l+1}(0, L)$. Since $H^{2l+1}(0, L)$ is compactly embedded in $H_0^1(0, L)$, then there exists a convergent in $H_0^1(0, L)$ subsequence $\{Bu_{k_s}\}_{s=1}^{\infty}$; therefore B is compact.

Similarly, let $u_k \to u$ in $H_0^1(0, L)$, then there are a subsequence $\{w_{k_s}\}_{s=1}^{\infty}$ and a function $w \in H_0^1(0, L)$ such that $w_{k_s} \to w$ in $H_0^1(0, L)$. Write (50) in the form

$$aw_{k_s} + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1} w_{k_s} = f - u_{k_s} Du_{k_s} \quad (52)$$

for all $s \in \mathbb{N}$. Consequently by (51), passing to the limit as $s \to \infty$, we find

$$aw + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1} w = f - uDu. \quad (53)$$

Thus $w = Bu$. Hence, $u_k \to u$ in $H_0^1(0, L)$ implies $Bu_k \to Bu$ in $H_0^1(0, L)$. This proves that B is continuous.

Finally, we must show that the set

$$\left\{u \in H_0^1(0, L) \mid u = \lambda Bu \text{ for some } 0 \le \lambda \le 1\right\} \quad (54)$$

is bounded in $H_0^1(0, L)$. Assume $u \in H_0^1(0, L)$ such that

$$u = \lambda Bu \quad \text{for some } 0 < \lambda \le 1, \quad (55)$$

then

$$\begin{gathered} a\left(\frac{u}{\lambda}\right) + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1}\left(\frac{u}{\lambda}\right) = f - uDu \\ \text{in } (0, L); \\ D^k\left(\frac{u}{\lambda}\right)(0) = D^k\left(\frac{u}{\lambda}\right)(L) = D^l\left(\frac{u}{\lambda}\right)(L) = 0, \\ k = 0, \dots, l-1, \end{gathered} \quad (56)$$

that is,

$$au + \sum_{j=1}^{l} (-1)^{j+1} D^{2j+1} u + \lambda uDu = \lambda f \quad \text{in } (0, L) \quad (57)$$

and u satisfies boundary conditions (5).

5.1. A Priori Estimates

Estimate IV. Multiplying (57) by u and integrating over $(0, L)$, we have

$$\begin{aligned} a\|u\|^2 &+ \sum_{j=1}^{l} (-1)^{j+1} \left(D^{2j+1}u, u\right) + \lambda (uDu, u) \\ &= (\lambda f, u). \end{aligned} \quad (58)$$

Integrating by parts and using boundary conditions (5), we get

$$\lambda (uDu, u) = 0. \quad (59)$$

Hence, similar to (28), we obtain

$$\|u\| \le \frac{1}{a}\|f\|. \quad (60)$$

Estimate V. Multiplying (57) by $(1 + x)u$ and integrating over $(0, L)$, we have

$$\begin{aligned} a(u, (1+x)u) &+ \sum_{j=1}^{l} (-1)^{j+1} \left(D^{2j+1}u, (1+x)u\right) \\ &+ \lambda (uDu, (1+x)u) = (\lambda f, (1+x)u). \end{aligned} \quad (61)$$

It is easy to verify that

$$\sum_{j=1}^{l}(-1)^{j+1}\left(D^{2j+1}u,(1+x)u\right) = \sum_{j=1}^{l}\left(\frac{2j+1}{2}\right)\left\|D^{j}u\right\|^{2}+\frac{1}{2}\left(D^{l}u(0)\right)^{2}. \quad (62)$$

Integrating by parts, using boundary conditions (5) and Lemma 2, we get

$$\begin{aligned}-\lambda(uDu,(1+x)u) &= \frac{\lambda}{3}\int_0^L u^3(x)\,dx \\ &\le \frac{1}{3}\int_0^L |u(x)|\,|u(x)|^2\,dx \\ &\le \frac{1}{3}\sup_{x\in(0,L)}|u(x)|\int_0^L |u(x)|^2\,dx \\ &\le \frac{\sqrt{2}}{3}\|u\|^{5/2}\|Du\|^{1/2}.\end{aligned} \quad (63)$$

By the Young inequality, with $p=4$, $q=4/3$, and $\epsilon=2^{1/4}$, we obtain

$$\lambda(uDu,(1+x)u) \ge -\frac{1}{2}\|Du\|^2 - b\|u\|^{10/3}, \quad (64)$$

where $b=2^{-5/3}3^{-1/2}$.

Moreover, by the Young inequality with arbitrary $\epsilon>0$, we get

$$(f,(1+x)u) \le \frac{\epsilon}{2}\left((1+x),u^2\right)+\frac{1}{2\epsilon}\left((1+x),f^2\right); \quad (65)$$

therefore

$$\begin{aligned}&\left(a-\frac{\epsilon}{2}\right)\left((1+x),u^2\right)+\|Du\|^2 \\ &\quad+\sum_{j=2}^{l}\left(\frac{2j+1}{2}\right)\left\|D^{j}u\right\|^{2}+\frac{1}{2}\left(D^{l}u(0)\right)^{2} \\ &\le b\|u\|^{10/3}+\frac{1}{2\epsilon}\left((1+x),f^2\right).\end{aligned} \quad (66)$$

Since

$$\int_0^L (1+x)f^2dx = \|f\|^2+\int_0^L xf^2dx \ge \|f\|^2, \quad (67)$$

it follows from (28) that

$$\begin{aligned}&\left(a-\frac{\epsilon}{2}\right)\left((1+x),u^2\right)+\|Du\|^2 \\ &\quad+\sum_{j=2}^{l}\left(\frac{2j+1}{2}\right)\left\|D^{j}u\right\|^{2}+\frac{1}{2}\left(D^{l}u(0)\right)^{2} \\ &\le \left(\frac{1}{2\epsilon}+\frac{b}{a^{10/3}}\left((1+x),f^2\right)^{2/3}\right)\left((1+x),f^2\right).\end{aligned} \quad (68)$$

Taking $\epsilon=a>0$, we conclude

$$\|u\|_{H_0^l} \le C_4\left((1+x),f^2\right)^{1/2}, \quad (69)$$

where

$$C_4=\frac{1}{\sqrt{\beta}}\left(\frac{1}{2a}+\frac{b}{a^{10/3}}\left((1+x),f^2\right)^{2/3}\right)^{1/2}, \qquad \beta=\min\left\{\frac{a}{2},1\right\}. \quad (70)$$

Remark 6. Note that estimate (69) does not depend on $L\in(0,\infty)$. This estimate may be used to prove the existence of a weak solution, $u\in H_0^l(0,L)$.

Estimate VI. Rewriting (57) in the form

$$(-1)^{l+1}D^{2l+1}u=\lambda f-au-\sum_{j=1}^{l-1}(-1)^{j+1}D^{2j+1}u - \lambda uDu, \quad (71)$$

we estimate

$$\left\|D^{2l+1}u\right\| \le \|f\|+a\|u\|+\sum_{j\in I_1}\left\|D^{2j+1}u\right\| + \sum_{j\in I_2}\left\|D^{2j+1}u\right\|+\|uDu\|. \quad (72)$$

By (69),

$$\begin{aligned}a\|u\|+\sum_{j\in I_1}\left\|D^{2j+1}u\right\| &\le (a+l)C_4\left((1+x),f^2\right)^{1/2}, \\ \|uDu\| \le \sqrt{2}\|u\|^{1/2}\|Du\|^{3/2} &\le \|u\|_{H_0^1}^2 \\ &\le C_4^2\left((1+x),f^2\right).\end{aligned} \quad (73)$$

Acting in the same way as we have proved (42),

$$\begin{aligned}&\sum_{j\in I_2}\left\|D^{2j+1}u\right\| \\ &\quad\le l\epsilon\left\|D^{2l+1}u\right\| \\ &\quad+\left(\frac{1}{a}\sum_{j\in I_2}\left(C_1^j(\epsilon)+K_2^j\right)\right)\left((1+x),f^2\right)^{1/2}.\end{aligned} \quad (74)$$

Substituting (73), (74) into (72), we obtain

$$\begin{aligned}&\left\|D^{2l+1}u\right\| \\ &\quad\le l\epsilon\left\|D^{2l+1}u\right\| \\ &\quad+\left(\frac{1}{a}\sum_{j\in I_2}\left(C_1^j(\epsilon)+K_2^j\right)\right)\left((1+x),f^2\right)^{1/2} \\ &\quad+\left(1+(a+l)C_4\right)\left((1+x),f^2\right)^{1/2}.\end{aligned} \quad (75)$$

Setting $\epsilon = 1/2l$, we conclude

$$\left\|D^{2l+1}u\right\| \leq C_5\left(\left(1+x\right), f^2\right)^{1/2}, \quad (76)$$

where C_5 depends only on L, l, a, and $((1+x), f^2)$.

Making use of (60), (76), and Theorem 3, we obtain

$$\left\|D^i u\right\| \leq \left(\frac{K_1^i C_5^{\theta^i}}{a^{1-\theta^i}} + \frac{K_2^i}{a}\right)\left(\left(1+x\right), f^2\right)^{1/2}, \quad \text{for } i = l+1, \ldots, 2l. \quad (77)$$

Taking into account (69), (76), and (77), we finally conclude

$$\|u\|_{H^{2l+1}} \leq C_6\left(\left(1+x\right), f^2\right)^{1/2} \quad (78)$$

with C_6 depending only on L, l, a, and $((1+x), f^2)$.

Applying Theorem 4, we complete the proof of the existence part of Theorem 1.

6. Uniqueness and Continuous Dependence

We separated two cases.

(1) Case $l \geq 2$. Let u_1 and u_2 be two distinct solutions of (4)-(5). Then the difference $u = u_1 - u_2$ satisfies the equation

$$au + \sum_{j=1}^{l}(-1)^{j+1} D^{2j+1}u + u_1 Du + u Du_2 = 0 \quad (79)$$

and boundary conditions (5).

Multiplying (79) by u and integrating over $(0, L)$, we have

$$a\|u\|^2 + \sum_{j=1}^{l}(-1)^{j+1}\left(D^{2j+1}u, u\right) + \left(u_1 Du, u\right) + \left(u Du_2, u\right) = 0. \quad (80)$$

Integrating by parts and using boundary conditions (5), we get

$$\left(u_1 Du, u\right) \geq -\frac{1}{2}\sup_{x\in(0,L)}\left|Du_1(x)\right|\|u\|^2. \quad (81)$$

Similarly,

$$\left(u Du_2, u\right) \geq -\sup_{x\in(0,L)}\left|Du_2(x)\right|\|u\|^2. \quad (82)$$

We reduce (80) to the inequality

$$\left(a - \frac{1}{2}\sup_{x\in(0,L)}\left|Du_1(x)\right| - \sup_{x\in(0,L)}\left|Du_2(x)\right|\right)\|u\|^2 \leq 0. \quad (83)$$

For $i = 1, 2$, we have $u_i \in H^{2l+1}(0, L)$ and $Du_i(L) = Du_i(0) = 0$. By Lemma 2 and estimate (69), we obtain

$$\begin{aligned}\sup_{x\in(0,L)}\left|Du_i(x)\right| &\leq \sqrt{2}\left\|Du_i\right\|^{1/2}\left\|D^2 u_i\right\|^{1/2} \\ &\leq \frac{\sqrt{2}}{2}\left(\left\|Du_i\right\| + \left\|D^2 u_i\right\|\right) \leq \sqrt{2}\left\|u_i\right\|_{H_0^l} \\ &\leq \sqrt{2}C_4\left(\left(1+x\right), f^2\right)^{1/2}\end{aligned} \quad (84)$$

with

$$C_4 = \frac{1}{\sqrt{\beta}}\left(\frac{1}{2a} + \frac{2^{-5/3}3^{-1/3}}{a^{10/3}}\left(\left(1+x\right), f^2\right)^{2/3}\right)^{1/2}. \quad (85)$$

Therefore (83) can be rewritten as

$$\left(a - \frac{3\sqrt{2}}{2}C_4\left(\left(1+x\right), f^2\right)^{1/2}\right)\|u\|^2 \leq 0. \quad (86)$$

Since

$$\frac{2a}{3}\sqrt{\frac{a\beta}{2}} < \sqrt{2}\sqrt[4]{3}a^{7/4} \quad \forall a > 0, \quad (87)$$

it follows that if $((1+x), f^2)^{1/2} < (2a/3)\sqrt{a\beta/2}$, for fixed $a > 0$, then

$$C_4 < \left(\frac{1}{a\beta}\right)^{1/2} \quad (88)$$

and consequently

$$a - \frac{3\sqrt{2}}{2}C_4\left(\left(1+x\right), f^2\right)^{1/2} > 0. \quad (89)$$

Hence, (86) implies $\|u\| = 0$ and uniqueness is proved for $l \geq 2$.

To show continuous dependence of solutions of perturbations of $f(x)$, let $f_i \in L^2(0, L)$ such that

$$\left(\left(1+x\right), f_i^2\right)^{1/2} < \frac{2a}{3}\sqrt{\frac{a\beta}{2}}, \quad i = 1, 2. \quad (90)$$

Consider u_1 and u_2 solutions of (4)-(5) with the right-hand sides f_1 and f_2, respectively. Then, similar to (83), $u_1 - u_2$ satisfies the following inequality:

$$\left(a - \frac{1}{2}\sup_{x\in(0,L)}\left|Du_1(x)\right| - \sup_{x\in(0,L)}\left|Du_2(x)\right|\right)\left\|u_1 - u_2\right\| \leq \left\|f_1 - f_2\right\| \quad (91)$$

which can be rewritten as

$$\left(a - \frac{3\sqrt{2}}{2}C_4^M M\right)\left\|u_1 - u_2\right\| \leq \left\|f_1 - f_2\right\|, \quad (92)$$

where

$$M = \max\left\{\left(\left(1+x\right), f_1^2\right)^{1/2}, \left(\left(1+x\right), f_2^2\right)^{1/2}\right\}, \quad C_4^M = \frac{1}{\sqrt{\beta}}\left(\frac{1}{2a} + \frac{b}{a^{10/3}}M^{4/3}\right)^{1/2}. \quad (93)$$

Making use of (90), we obtain

$$\left\|u_1 - u_2\right\| \leq C_7\left\|f_1 - f_2\right\| \quad (94)$$

with $C_7 = (a - (3\sqrt{2}/2)C_4^M M)^{-1} > 0$. Hence $\|f_1 - f_2\| \to 0$ implies $\|u_1 - u_2\| \to 0$. This proves the continuous dependence for $l \geq 2$.

(2) Case $l = 1$. For $l = 1$, problem (4)-(5) becomes

$$au + D^3 u + uDu = f \quad \text{in } (0, L) \tag{95}$$

$$u(0) = u(L) = Du(L) = 0. \tag{96}$$

Let u_1 and u_2 be two distinct solutions of (95)-(96). Then the difference $u = u_1 - u_2$ satisfies the equation

$$au + D^3 u + \frac{1}{2} D\left(u_1^2 - u_2^2\right) = 0 \tag{97}$$

and boundary conditions (96).

Multiplying (97) by u and integrating over $(0, L)$, we have

$$a\|u\|^2 + \frac{1}{2}(Du(0))^2 + \frac{1}{2}\left(D\left(u_1^2 - u_2^2\right), u\right) = 0. \tag{98}$$

Integrating by parts and using the boundary conditions (96), we get

$$\begin{aligned}\frac{1}{2}\left(D\left(u_1^2 - u_2^2\right), u\right) &= \frac{1}{2}\left(D\left[(u_1 + u_2)u\right], u\right)\\ &= -\frac{1}{2}\left((u_1 + u_2)u, Du\right)\\ &= -\frac{1}{4}\left((u_1 + u_2), D\left[u^2\right]\right)\\ &= \frac{1}{4}\left(D(u_1 + u_2), u^2\right)\\ &\leq \frac{1}{4}\sup_{x\in(0,L)}\left|D(u_1 + u_2)(x)\right| \|u\|^2\end{aligned} \tag{99}$$

and (98) becomes

$$\left[a - \frac{1}{4}\left(\sup_{x\in(0,L)}\left|Du_1(x)\right| + \sup_{x\in(0,L)}\left|Du_2(x)\right|\right)\right]\|u\|^2 \leq 0. \tag{100}$$

By (60), (69), it follows that

$$\left\|D^3 u_i\right\| \leq 2\|f\| + C_4^2\left((1 + x), f^2\right), \quad i = 1, 2. \tag{101}$$

According to Theorem 3 and (60), (101), we estimate for $i = 1, 2$

$$\begin{aligned}\sup_{x\in(0,L)}\left|Du_i(x)\right| &\leq K_1\left\|D^3 u_i\right\|^{1/2}\|u_i\|^{1/2} + K_2\|u_i\|\\ &\leq \frac{K_1}{2}\left\|D^3 u_i\right\| + \left(\frac{K_1}{2} + K_2\right)\|u_i\|\\ &\leq \frac{K_1}{2}C_4^2\left((1 + x), f^2\right)\\ &\quad + \left(K_1 + \frac{K_1}{2a} + \frac{K_2}{a}\right)\|f\|.\end{aligned} \tag{102}$$

Suppose $((1 + x), f^2)^{1/2} < 1$, then $((1 + x), f^2) < ((1 + x), f^2)^{1/2}$; therefore

$$\sup_{x\in(0,L)}\left|Du_i(x)\right| \leq \left[\frac{K_1}{2}C_4^2 + \left(K_1 + \frac{K_1}{2a} + \frac{K_2}{a}\right)\right]\left((1 + x), f^2\right)^{1/2}, \quad i = 1, 2. \tag{103}$$

Hence we can rewrite (100) as follows:

$$\left[a - \left(\frac{K_1}{4}C_4^2 + \left(\frac{K_1}{2} + \frac{K_1}{4a} + \frac{K_2}{2a}\right)\right)\left((1 + x), f^2\right)^{1/2}\right] \cdot \|u\|^2 \leq 0. \tag{104}$$

For a fixed $a > 0$ assume that

$$\left((1 + x), f^2\right)^{1/2} < \min\left\{\sqrt{2}\sqrt[4]{3}a^{7/4}, \frac{4a^2\beta}{(1 + 2a\beta + \beta)K_1 + 2\beta K_2}\right\}. \tag{105}$$

Then $C_4^2 < 1/a\beta$ and

$$\left[a - \left(\frac{K_1}{4}C_4^2 + \left(\frac{K_1}{2} + \frac{K_1}{4a} + \frac{K_2}{2a}\right)\right)\left((1 + x), f^2\right)^{1/2}\right] > 0; \tag{106}$$

hence (104) implies $\|u\| = 0$. Assuming that, for $l = 1$,

$$\left((1 + x), f^2\right)^{1/2} < \min\left\{1, \sqrt{2}\sqrt[4]{3}a^{7/4}, \frac{4a^2\beta}{(1 + 2a\beta + \beta)K_1 + 2\beta K_2}\right\}, \tag{107}$$

we complete the proof of uniqueness. The continuous dependence for this case follows in the same manner as it has been done for the case $l \geq 2$ provided $((1 + x), f^2)^{1/2}$ is sufficiently small.

This completes the proof of the uniqueness and continuous dependence part of Theorem 1.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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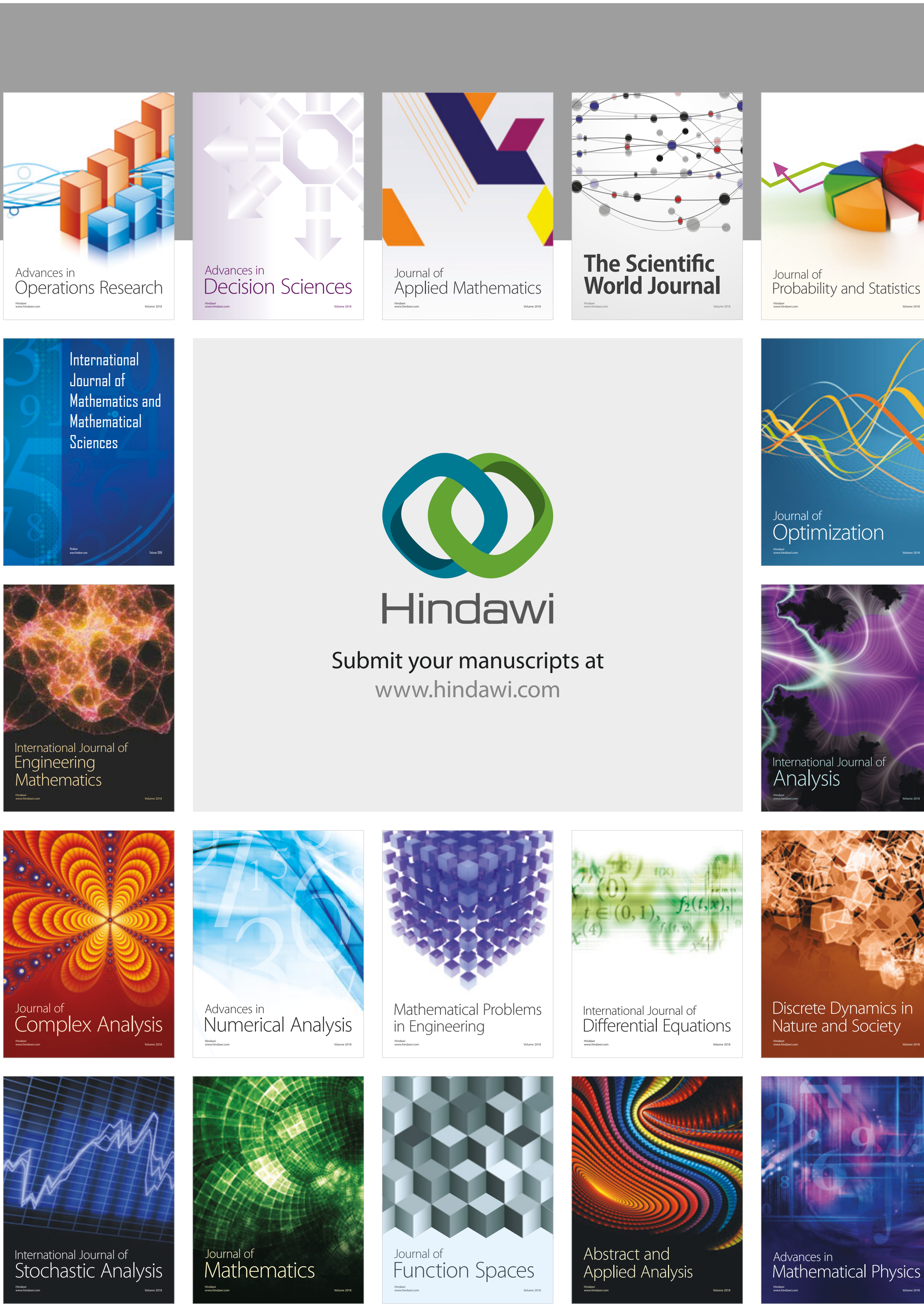
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